

# Four-loop photon quark form factor and cusp anomalous dimension in the large- $N_c$ limit of QCD

Roman N. Lee<sup>a,d</sup>, Alexander V. Smirnov<sup>b</sup>,  
Vladimir A. Smirnov<sup>c,d</sup>, Matthias Steinhauser<sup>d</sup>,

*(a) Budker Institute of Nuclear Physics,  
630090 Novosibirsk, Russia*

*(b) Research Computing Center, Moscow State University  
119991, Moscow, Russia*

*(c) Skobeltsyn Institute of Nuclear Physics of Moscow State University  
119991, Moscow, Russia*

*(d) Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT)  
76128 Karlsruhe, Germany*

## Abstract

We compute the four-loop QCD corrections to the massless quark-anti-quark-photon form factor  $F_q$  in the large- $N_c$  limit. From the pole part we extract analytic expressions for the corresponding cusp and collinear anomalous dimensions.

PACS numbers: 11.15.Bt, 12.38.Bx, 12.38.Cy

# 1 Introduction

Perturbation theory is a powerful tool to obtain reliable predictions for physical observables within the Standard Model of particle physics or its extensions. Due to the high precision of experimental results, e.g., at the CERN Large Hadron Collider (LHC) or at the  $B$  factories, it is on the one hand mandatory to advance the development of tools, which can be used for higher order calculations. On the other hand it is necessary to improve the understanding of the perturbative structure of quantum field theories. Form factors are ideal objects to obtain deeper insight into the latter. Especially in the context of QCD they are indispensable tools to investigate the infrared structure of scattering amplitudes to high orders in perturbation theory. Moreover, from the pole part it is possible to extract universal process-independent quantities [1–7] like the cusp anomalous dimension which can be extracted from the  $1/\epsilon^2$  pole of the form factor. The finite parts of the form factors serve as building blocks for a variety of physical processes. For example, the quark-anti-quark-photon form factor enters the virtual corrections of the Drell-Yan process for the production of lepton pairs at hadron colliders.

In this paper we consider the quark-anti-quark-photon form factor which is conveniently obtained from the photon-quark vertex function  $\Gamma_q^\mu$  by applying an appropriate projector. In  $D = 4 - 2\epsilon$  space-time dimensions we have

$$F_q(q^2) = -\frac{1}{4(1-\epsilon)q^2} \text{Tr}(\not{p}_2 \Gamma_q^\mu \not{p}_1 \gamma_\mu), \quad (1)$$

with  $q = p_1 + p_2$  where  $p_1$  and  $p_2$  are the incoming quark and anti-quark momenta and  $q$  is the momentum of the photon. We perform our calculation in the framework of QCD keeping the number of colours,  $N_c$ , generic. In the limit of large  $N_c$  the calculation of  $F_q$  is simplified since only planar Feynman diagrams contribute. This is the limit we consider in this paper. Besides  $N_c$  we also keep the number of active quark flavours,  $n_f$  as a parameter and thus have at four-loop order the colour structures  $N_c^4$ ,  $N_c^3 n_f$ ,  $N_c^2 n_f^2$ ,  $N_c n_f^3$  where each factor of  $n_f$  counts the number of closed fermion loops.

Two- and three-loop corrections to  $F_q$  have been computed in Refs. [8–15]. To obtain the four-loop corrections two obstacles need to be overcome: (i) the reduction to a set of basis integrals and (ii) the (if possible) analytic calculation of the latter. Recently, the first steps towards four loops have been initiated by computing the fermionic contributions to  $F_q$  in the planar limit [16]. The  $n_f^3$  terms have been confirmed in Ref. [17] using different methods both for the reduction and the computation of the master integrals. Let us mention that the four-loop corrections to the cusp anomalous dimension with two and three closed fermion loops have also been obtained in Ref. [18]. In Ref. [19] one finds a discussion of non-planar master integrals relevant for the four-loop form factor in a  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory.

In the present paper, we evaluate the  $n_f^0$  contribution and therefore complete the evaluation of the form factors and anomalous dimensions in the limit of large  $N_c$ .

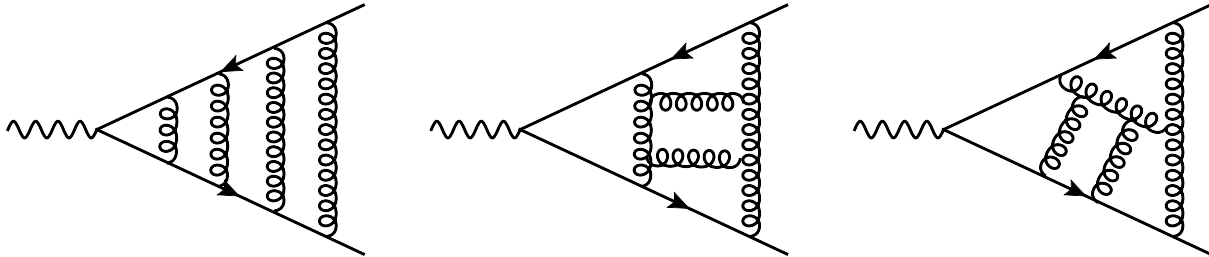


Figure 1: Sample four-loop Feynman diagrams contributing to  $F_q$ . Solid, curly and wavy lines represent quarks, gluons and photons, respectively. All particles are massless.

In the next Section we provide some technical details, in particular to the calculation of the most complicated master integral, and we discuss our results in Section 3. We provide explicit expressions for the four-loop cusp and collinear anomalous dimensions in the planar limit. Furthermore, we provide results for the finite part of  $\log(F_q)$ .

## 2 Technical details

For the calculation of  $F_q$  we use a well-tested automated chain of programs which work hand-in-hand. The Feynman amplitudes are generated with `qgraf` [20]. Since there is no possibility to select already at this point the planar diagrams also non-planar amplitudes are generated and we obtain in total 1, 15, 337 and 9784 diagrams at one, two, three and four loops. Sample Feynman diagrams at four loops can be found in Fig. 1. Next, we transform the output to FORM [21] notation using `q2e` and `exp` [22, 23]. The program `exp` furthermore maps each Feynman diagram to predefined integral families for massless four-loop vertices with two different non-vanishing external momenta; 68 of them are of planar type. At this point we perform the Dirac algebra and decompose the numerator into terms which appear in the denominator. This allows us to express each Feynman integral as a linear combination of scalar functions which belong to the corresponding family. After exploiting the symmetries connected to the exchange of the external momenta we can reduce the number of families, for which integral tables have to be generated, from 68 to 38. Note that for the fermionic contributions, which have been considered in Ref. [16], only 24 families are needed.

For the reduction to master integrals we use the program `FIRE` [24–26] which we apply in combination with `LiteRed` [27, 28]. We observe that the non-fermionic diagrams lead to more complex integrals for which the reduction time significantly increases. Let us remark that we have adopted Feynman gauge for the calculation of the non-fermionic parts.

Once reduction tables for each family are available we apply `tsort`, which is part of the latest `FIRE` version [26]. It is based on ideas presented in Ref. [25] to establish relations between primary master integrals and thus minimize their number. In this way

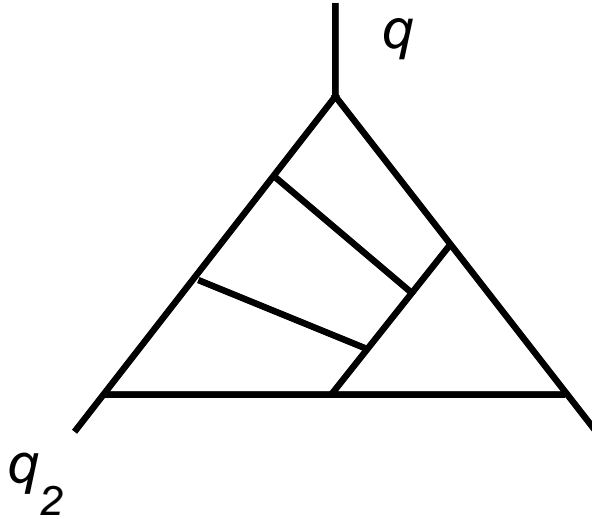


Figure 2: Massless four-loop vertex diagram with two external (incoming) momenta  $q$  and  $q_2$  with  $q^2 \neq 0 \neq q_2^2$  and  $(q + q_2)^2 = 0$ . The master integral  $I_{99}$  corresponds to  $q_2^2 = 0$ .

we arrive at 99 master integrals. It turned out that within the strategy described in Ref. [16] it was possible to evaluate all these master integrals but one. This was enough to perform the evaluation of the fermionic ( $n_f^1$ ) part where 89 of 99 master integrals were already present. The evaluation of the last missing master integral,  $I_{99}$ , corresponding to the graph of Fig. 2, has met certain difficulties in the framework which was outlined in [16] and will be described in detail in Ref. [29]. It turned out that the current version of FIRE was unable to perform an IBP reduction of integrals present in expressions which appeared as candidates for elements of the canonical basis in the top sectors within the recipes of Refs. [30, 31].

In the remainder of this section we describe the strategy used for the calculation of the last integral,  $I_{99}$  (see Fig. 2), needed to obtain the non-fermionic result for the four-loop form factor in the planar limit.

To have the possibility to apply differential equations for the evaluation of the form factor master integral corresponding to the graph of Fig. 2 we introduce a second mass scale by considering  $q_2^2 = xq^2$  and derive differential equations with respect to the ratio of the two scales,  $x$ . This strategy was advocated in [32] and used for the four-loop form-factor integrals in [16, 29]. We derive differential equations for the corresponding 332 master integrals using LiteRed [27, 28].

To solve our differential equations we use an important observation made in Ref. [30]. It has been suggested to turn from the basis of primary master integrals to a so-called canonical basis where the corresponding integrals satisfy a system differential equations which has a particular structure: the dependence on  $\epsilon$  appears as a linear prefactor and the matrix in front of the vector of master integrals has only simple poles in  $x$ , i.e. has

only so-called Fuchsian singularities. Such a system can then easily be solved in terms of iterated integrals.

Recipes to construct canonical bases were formulated in [30, 31] and in [33, 34]. The first algorithm to convert a given differential system to a canonical form has been provided in the case of one variable<sup>1</sup> in Ref. [36] (therein called  $\epsilon$ -form). Recently a public implementation [37] of the algorithm of Ref. [36] in a computer code `Fuchsia` became available. In this paper we follow [36] to construct a canonical basis. The size of the system is large but, as it concerns the diagonal blocks, their size is at most  $5 \times 5$ . Consequently, the reduction of diagonal blocks within the approach of Ref. [36] is simple. Note that already after this step one might claim that the solution is expressible in terms of harmonic polylogarithms (and also construct this solution). Nevertheless, we follow the prescription of Section 7 of Ref. [36] to reduce the whole system to an  $\epsilon$ -form. Similar to Ref. [38] we find that the differential equation-based hierarchy of the set of the master integrals is too restrictive and the use of a sector-based hierarchy when factoring  $\epsilon$  out of the whole matrix is necessary.

The family of one-scale Feynman integrals (with  $q_2^2 = 0$  and with  $q^2 \neq 0$ ) corresponding to Fig. 2 contains 76 master integrals. After introducing  $q_2^2 \neq 0$  we define  $x = q_2^2/q^2$  and obtain a family of Feynman integrals with 332 master integrals. Our strategy is to turn from a primary basis to a canonical basis, solve differential equations for the canonical basis, evaluate the naive values of the elements of the canonical basis at  $x = 0$  (i.e., setting  $x = 0$  under the integral sign) from which it will be straightforward to obtain analytical results for the primary master integrals of our one-scale family, in particular,  $I_{99}$ .

Following Ref. [36] we arrive at a canonical basis  $g$  which is obtained from the primary basis  $f$  by a linear transformation with a matrix  $T$ ,

$$f = T \cdot g. \quad (2)$$

The vectors  $f$  and  $g$  have 332 entries and  $T$  is a  $332 \times 332$  matrix. The dependence of  $f$ ,  $g$  and  $T$  on  $x$  and  $\epsilon$  has been suppressed.

It is convenient [30] to normalize the canonical master integrals such that they have uniformly transcendental  $\epsilon$ -expansion which starts from  $\epsilon^0$ . In our four-loop case, one has to compute expansion terms including  $\epsilon^8$  terms and we have

$$g(x, \epsilon) = \sum_{k=0}^8 g_k(x) \epsilon^k. \quad (3)$$

The canonical basis  $g$  satisfies (by definition) a differential equation of the form

$$g'(x, \epsilon) = \epsilon A(x) \cdot g(x, \epsilon), \quad (4)$$

---

<sup>1</sup>The very recent paper [35] makes some interesting progress towards the algorithmic reduction in the multivariate case.

where

$$A(x) = \frac{a}{x} + \frac{b}{x-1}, \quad (5)$$

with constant matrices  $a$  and  $b$ . It is straightforward to construct the generic solution of (4) order-by-order in  $\epsilon$  in terms of harmonic polylogarithms (HPLs) [39] with letters 0 and 1, with  $332 \times 9$  unknown constants.

To fix these constants we use boundary conditions at the point  $x = 1$ . The primary master integrals are regular at this point where the integrals become propagator-type integrals which are well known [40]. In particular, all the corresponding 28 master integrals are known analytically [40,41] in an  $\epsilon$ -expansion up to weight 12 and have been cross checked numerically [42]. We obtain the boundary values of the elements of the canonical basis  $g$  at  $x = 1$  by inverting Eq. (2) and considering the limit  $x \rightarrow 1$ . The matrix  $T^{-1}$  involves elements which develop poles up to order  $1/(1-x)^6$ . This requires that the first seven expansion terms for  $x \rightarrow 1$  of the primary master integrals  $f$  have to be computed. The corresponding reduction tables are again generated with FIRE. Note that the resulting two-point integrals with external momentum  $q$  have scalar products in the numerator involving the momentum  $q_2$  which complicates the calculation. It is an important cross check of the calculation that all poles in  $1/(1-x)$  cancel in the combination  $T^{-1} \cdot f$  and we obtain the values for  $g$  at  $x = 1$ , in an  $\epsilon$ -expansion up to  $\epsilon^8$ . This fixes the solution of the system of differential equations in Eq. (4).

In a next step we analyze the leading asymptotic behaviour of  $g$  near  $x = 0$ . On the one hand, we obtain it from the differential equations (4) where the term  $b/(x-1)$  on the right-hand side can be neglected and the solution has the form  $g(x, \epsilon) = h(\epsilon)x^{c\epsilon}$ . The quantity  $x^{c\epsilon} = e^{\epsilon \log(x)c}$  can be evaluated using the `Mathematica` command `MatrixExp[]` which leads to a  $332 \times 332$  matrix where each element is a linear combination of terms  $x^{k\epsilon}$  with integer  $k$ . In general both non-positive  $k$  and positive  $k$  might appear. However, in the case of Feynman integrals only terms with non-positive  $k$  can be present which we use as a check. On the other hand, the leading asymptotic behaviour in the limit  $x \rightarrow 0$  can also be obtained with the help of the `Mathematica` package HPL [43] from our analytic expression for the canonical basis. Matching the two expressions provides values for the vector  $h(\epsilon)$  in an  $\epsilon$ -expansion up to  $\epsilon^8$  and terms  $x^{k\epsilon}$  with  $k = 0, -1, -2, \dots$ . Note that this step involves powers of  $\log(x)$  terms; their cancellation in the matching provides a welcome check for our calculation. Finally, the naive value for  $g(x, \epsilon)$  at  $x = 0$  is obtained by setting all terms  $x^{k\epsilon}$  with  $k \neq 0$  to zero in the expression for  $x^{c\epsilon}$ .

In the last step, using Eq. (2), we compute the naive values of the elements of the primary basis  $f$  from the naive expansion of the canonical basis  $g(x)$  near  $x \rightarrow 0$ . Note that some of the matrix elements of  $T$  involve singularities up to order  $1/x^3$ . Thus, the naive expansion of  $g(x)$  up to order  $x^3$  is needed. It is obtained following the prescription outlined in Ref. [44] where the expansion terms can be computed from the leading order asymptotics at  $x \rightarrow 0$  after recursively solving matrix equations with  $332 \times 332$  entries. After inserting the expansion of  $g$  in Eq. (2) the poles cancel and the naive values of the

primary master integrals at  $x = 0$  are obtained. The naive value of one of the elements of our primary basis  $f$  is nothing but the one-scale master integral  $I_{99}$  (cf. Fig. 2) for which we obtain the following analytic result

$$\begin{aligned}
I_{99} = & e^{4\epsilon\gamma_E} \left( \frac{\mu^2}{-q^2} \right)^{4\epsilon} \left\{ \frac{1}{\epsilon^7} \left[ -\frac{1}{288} \right] + \frac{1}{\epsilon^6} \left[ \frac{13}{576} \right] + \frac{1}{\epsilon^5} \left[ -\frac{101}{576} - \frac{\pi^2}{48} \right] \right. \\
& + \frac{1}{\epsilon^4} \left[ -\frac{17\zeta_3}{54} + \frac{5\pi^2}{36} + \frac{145}{96} \right] + \frac{1}{\epsilon^3} \left[ \frac{1775\zeta_3}{432} - \frac{767\pi^4}{17280} - \frac{5\pi^2}{8} - \frac{1669}{144} \right] \\
& + \frac{1}{\epsilon^2} \left[ -\frac{83}{72}\pi^2\zeta_3 - \frac{21899\zeta_3}{864} - \frac{3659\zeta_5}{360} + \frac{31333\pi^4}{103680} + \frac{659\pi^2}{288} + \frac{11243}{144} \right] \\
& + \frac{1}{\epsilon} \left[ -\frac{40231\zeta_3^2}{1296} + \frac{745\pi^2\zeta_3}{288} + \frac{18751\zeta_3}{144} + \frac{50191\zeta_5}{360} - \frac{277703\pi^6}{2177280} - \frac{14015\pi^4}{10368} \right. \\
& \left. - \frac{149\pi^2}{24} - \frac{22757}{48} \right] \\
& + \left[ \frac{39173\zeta_3^2}{324} - \frac{77399\pi^4\zeta_3}{25920} + \frac{4013\pi^2\zeta_3}{432} - \frac{259559\zeta_3}{432} - \frac{568\pi^2\zeta_5}{45} - \frac{1123223\zeta_5}{1440} \right. \\
& \left. - \frac{2778103\zeta_7}{4032} + \frac{3129533\pi^6}{4354560} + \frac{28201\pi^4}{5760} + \frac{173\pi^2}{36} + \frac{382375}{144} \right] \\
& + \epsilon \left[ \frac{4931s_{8a}}{30} + \frac{2615}{144}\pi^2\zeta_3^2 - \frac{276671\zeta_3^2}{2592} - \frac{2702413\zeta_5\zeta_3}{1080} + \frac{154037\pi^4\zeta_3}{31104} \right. \\
& - \frac{55327\pi^2\zeta_3}{432} + \frac{1100461\zeta_3}{432} + \frac{205\pi^2\zeta_5}{9} + \frac{155029\zeta_5}{48} + \frac{2732549\zeta_7}{1008} - \frac{665217829\pi^8}{1306368000} \\
& \left. - \frac{131003\pi^6}{45360} - \frac{747929\pi^4}{51840} + \frac{2995\pi^2}{36} - \frac{2005247}{144} \right] \left. \right\}, \tag{6}
\end{aligned}$$

where  $\zeta_n$  is Riemann's zeta function evaluated at  $n$  and

$$s_{8a} = \zeta_8 + \zeta_{5,3} \approx 1.0417850291827918834. \tag{7}$$

$\zeta_{m_1, \dots, m_k}$  are multiple zeta values given by

$$\zeta_{m_1, \dots, m_k} = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \prod_{j=1}^k \frac{\text{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}. \tag{8}$$

As by-product we also obtain analytic results for the remaining 75 one-scale master integrals and we find agreement with the results obtained in Ref. [29]. This constitutes a further cross check for our procedure. We want to stress that the calculation which is outlined in this section is largely independent from the one performed in Ref. [29].

### 3 Results

This section is devoted to the analytic results of the cusp and collinear anomalous dimensions and the finite part of  $F_q$ . Generic formulae where the pole part of  $F_q$  is parametrized in terms of the cusp and collinear anomalous dimensions and the QCD beta function can, e.g., be found in Refs. [13, 45]. In what follows we use Eq. (2.3) of Ref. [16] which displays the pole parts of  $\log(F_q)$  up to four-loop order. In this formula it is assumed that the one-loop coefficient of the beta function is given by

$$\beta_0 = \frac{11N_c}{3} - \frac{2n_f}{3}, \quad (9)$$

with  $n_f$  being the number of active quarks and the coefficients of the anomalous dimensions are defined through

$$\gamma_x = \sum_{n \geq 0} \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^n \gamma_x^n, \quad (10)$$

with  $x \in \{\text{cusp}, q\}$  and  $\alpha_s$  is the renormalized coupling constant with  $n_f$  active flavours. From Eq. (2.3) of Ref. [16] one observes that the four-loop corrections of  $\gamma_{\text{cusp}}$  follows from the  $1/\epsilon^2$  term of  $\log(F_q)$  and  $\gamma_q$  from the linear pole terms.

In the following we start with explicit results for the cusp and collinear anomalous dimensions. The four-loop corrections to  $\gamma_{\text{cusp}}$  reads

$$\begin{aligned} \gamma_{\text{cusp}}^3 = & \left( -\frac{32\pi^4}{135} + \frac{1280\zeta_3}{27} - \frac{304\pi^2}{243} + \frac{2119}{81} \right) N_c n_f^2 + \left( \frac{128\pi^2\zeta_3}{9} + 224\zeta_5 - \frac{44\pi^4}{27} \right. \\ & - \frac{16252\zeta_3}{27} + \frac{13346\pi^2}{243} - \frac{39883}{81} \Big) N_c^2 n_f + \left( \frac{64\zeta_3}{27} - \frac{32}{81} \right) n_f^3 + (-32\zeta_3^2 \\ & - \frac{176\pi^2\zeta_3}{9} + \frac{20992\zeta_3}{27} - 352\zeta_5 - \frac{292\pi^6}{315} + \frac{902\pi^4}{45} - \frac{44416\pi^2}{243} + \frac{84278}{81} \Big) N_c^3. \end{aligned} \quad (11)$$

Note that  $\gamma_{\text{cusp}}^3$  entering the  $1/\epsilon^2$  pole of  $\log(F_q)$  is multiplied by  $C_F$ . For this reason there is only a  $N_c^3$  factor in front of the  $n_f$ -independent term in Eq. (11). The one-, two- and three-loop corrections in the large- $N_c$  limit can be found in Eq. (2.6) of Ref. [16] where also the fermionic part of  $\gamma_{\text{cusp}}^3$  is shown. The four-loop coefficient of the collinear anomalous dimension is given by

$$\begin{aligned} \gamma_q^3 = & N_c^3 \left[ \left( -\frac{680\zeta_3^2}{9} - \frac{1567\pi^6}{20412} + \frac{83\pi^2\zeta_3}{9} + \frac{557\zeta_5}{9} + \frac{3557\pi^4}{19440} - \frac{94807\zeta_3}{972} + \frac{354343\pi^2}{17496} \right. \right. \\ & \left. \left. + \frac{145651}{1728} \right) n_f \right] + \left( -\frac{8\pi^4}{1215} - \frac{356\zeta_3}{243} - \frac{2\pi^2}{81} + \frac{18691}{13122} \right) N_c n_f^3 + \left( -\frac{2}{3}\pi^2\zeta_3 \right. \\ & \left. + \frac{166\zeta_5}{9} + \frac{331\pi^4}{2430} - \frac{2131\zeta_3}{243} - \frac{68201\pi^2}{17496} - \frac{82181}{69984} \right) N_c^2 n_f^2 \end{aligned}$$



$$\begin{aligned}
& + N_c^4 \left( \frac{1175\zeta_3^2}{9} + \frac{82\pi^4\zeta_3}{45} - \frac{377\pi^2\zeta_3}{6} + \frac{867397\zeta_3}{972} + 24\pi^2\zeta_5 - 1489\zeta_5 + 705\zeta_7 \right. \\
& \left. + \frac{114967\pi^6}{204120} - \frac{59509\pi^4}{9720} - \frac{120659\pi^2}{17496} - \frac{187905439}{839808} \right). \tag{12}
\end{aligned}$$

The one-, two- and three-loop corrections and the fermionic four-loop terms to  $\gamma_q^3$  are listed in Eq. (2.7) of Ref. [16]. The  $N_c^4$  term is new.

Finally, we also present the finite part of the form factor. We parametrize the perturbative expansion in terms of the renormalized coupling constant and set  $\mu^2 = -q^2$ . Furthermore, it is convenient to consider  $\log(F_q)$  which leads to more compact expressions. Thus, we have the following parametrization

$$\log(F_q) = \sum_{n \geq 1} \left( \frac{\alpha_s}{4\pi} \right)^n \log(F_q)^{(n)}. \tag{13}$$

In the large- $N_c$  limit the four-loop term reads

$$\begin{aligned}
& \log(F_q)^{(4)}|_{\text{large-}N_c, \text{ finite part}} = \\
& N_c^4 \left( -14s_{8a} + 10\pi^2\zeta_3^2 - \frac{86647\zeta_3^2}{54} + 766\zeta_5\zeta_3 - \frac{251\pi^4\zeta_3}{6480} - \frac{57271\pi^2\zeta_3}{1296} + \frac{173732459\zeta_3}{23328} \right. \\
& + \frac{1517\pi^2\zeta_5}{216} - \frac{881867\zeta_5}{1080} - \frac{36605\zeta_7}{288} + \frac{674057\pi^8}{5443200} - \frac{135851\pi^6}{77760} + \frac{386729\pi^4}{31104} \\
& \left. - \frac{429317557\pi^2}{839808} - \frac{54900768805}{6718464} \right) + \dots \tag{14}
\end{aligned}$$

where the ellipses refer to the fermionic contributions which are given in Eq. (2.8) of Ref. [16]. For convenience of the reader we provide the results for the form factor  $F_q$  expanded in the bare strong coupling constant in an ancillary file which can be downloaded from <https://www.ttp.kit.edu/preprints/2016/ttp16-055/>. This file also contains the lower-loop results expanded to higher order in  $\epsilon$ . Furthermore, it contains the dependence of the renormalization scale  $\mu$ .

## 4 Conclusions and outlook

We compute the photon-quark form factor to four loops up to the finite term in  $\epsilon$  in the large- $N_c$  limit which is obtained from the planar Feynman diagrams. From the pole parts we extract the cusp and collinear anomalous dimensions. We discuss in detail the calculation of the most complicated master integral (see Fig. 2) and present analytic results expanded in  $\epsilon$  up to transcendental weight eight. The remaining master integrals are known analytically from Refs. [16, 29] to the required order in  $\epsilon$ . We want to remark that the same master integrals enter the Higgs gluon form factor in the planar limit. However, the corresponding reduction is significantly more complicated. The logical next step is the calculation of the non-planar contributions. Note that also here both the reduction and the computation of the master integrals turn out to be much more complicated.

## Acknowledgments

We are grateful to Gang Yang for checking numerically the highest poles of our result for  $I_{99}$ . This work is supported by the Deutsche Forschungsgemeinschaft through the project “Infrared and threshold effects in QCD”.

## References

- [1] A. H. Mueller, Phys. Rev. D **20** (1979) 2037. doi:10.1103/PhysRevD.20.2037
- [2] J. C. Collins, Phys. Rev. D **22** (1980) 1478. doi:10.1103/PhysRevD.22.1478
- [3] A. Sen, Phys. Rev. D **24** (1981) 3281. doi:10.1103/PhysRevD.24.3281
- [4] L. Magnea and G. F. Sterman, Phys. Rev. D **42** (1990) 4222. doi:10.1103/PhysRevD.42.4222
- [5] G. P. Korchemsky and A. V. Radyushkin, Phys. Lett. B **279** (1992) 359 doi:10.1016/0370-2693(92)90405-S [hep-ph/9203222].
- [6] I. A. Korchemskaya and G. P. Korchemsky, Phys. Lett. B **287** (1992) 169. doi:10.1016/0370-2693(92)91895-G
- [7] G. F. Sterman and M. E. Tejeda-Yeomans, Phys. Lett. B **552** (2003) 48 doi:10.1016/S0370-2693(02)03100-3 [hep-ph/0210130].
- [8] G. Kramer and B. Lampe, Z. Phys. C **34** (1987) 497 [Erratum-ibid. C **42** (1989) 504].
- [9] T. Matsuura and W. L. van Neerven, Z. Phys. C **38** (1988) 623.
- [10] T. Matsuura, S. C. van der Marck and W. L. van Neerven, Nucl. Phys. B **319** (1989) 570.
- [11] T. Gehrmann, T. Huber and D. Maitre, Phys. Lett. B **622** (2005) 295 [arXiv:hep-ph/0507061].
- [12] P. A. Baikov, K. G. Chetyrkin, A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Phys. Rev. Lett. **102** (2009) 212002 doi:10.1103/PhysRevLett.102.212002 [arXiv:0902.3519 [hep-ph]].
- [13] T. Gehrmann, E. W. N. Glover, T. Huber, N. Ikizlerli and C. Studerus, JHEP **1006** (2010) 094 doi:10.1007/JHEP06(2010)094 [arXiv:1004.3653 [hep-ph]].
- [14] R. N. Lee and V. A. Smirnov, JHEP **1102** (2011) 102 doi:10.1007/JHEP02(2011)102 [arXiv:1010.1334 [hep-ph]].

- [15] T. Gehrmann, E. W. N. Glover, T. Huber, N. Ikizlerli and C. Studerus, *JHEP* **1011** (2010) 102 doi:10.1007/JHEP11(2010)102 [arXiv:1010.4478 [hep-ph]].
- [16] J. M. Henn, A. V. Smirnov, V. A. Smirnov and M. Steinhauser, *JHEP* **1605** (2016) 066 doi:10.1007/JHEP05(2016)066 [arXiv:1604.03126 [hep-ph]].
- [17] A. von Manteuffel and R. M. Schabinger, arXiv:1611.00795 [hep-ph].
- [18] J. Davies, A. Vogt, B. Ruijl, T. Ueda and J. A. M. Vermaseren, arXiv:1610.07477 [hep-ph].
- [19] R. Boels, B. A. Kniehl and G. Yang, *Nucl. Phys. B* **902** (2016) 387 doi:10.1016/j.nuclphysb.2015.11.016 [arXiv:1508.03717 [hep-th]].
- [20] P. Nogueira, *J. Comput. Phys.* **105** (1993) 279.
- [21] J. Kuipers, T. Ueda, J. A. M. Vermaseren and J. Vollinga, *Comput. Phys. Commun.* **184** (2013) 1453 doi:10.1016/j.cpc.2012.12.028 [arXiv:1203.6543 [cs.SC]].
- [22] R. Harlander, T. Seidensticker and M. Steinhauser, *Phys. Lett. B* **426** (1998) 125 [hep-ph/9712228].
- [23] T. Seidensticker, hep-ph/9905298.
- [24] A. V. Smirnov, *JHEP* **0810** (2008) 107 doi:10.1088/1126-6708/2008/10/107 [arXiv:0807.3243 [hep-ph]].
- [25] A. V. Smirnov and V. A. Smirnov, *Comput. Phys. Commun.* **184** (2013) 2820 doi:10.1016/j.cpc.2013.06.016 [arXiv:1302.5885 [hep-ph]].
- [26] A. V. Smirnov, *Comput. Phys. Commun.* **189** (2015) 182 doi:10.1016/j.cpc.2014.11.024 [arXiv:1408.2372 [hep-ph]].
- [27] R. N. Lee, arXiv:1212.2685 [hep-ph].
- [28] R. N. Lee, *J. Phys. Conf. Ser.* **523** (2014) 012059 doi:10.1088/1742-6596/523/1/012059 [arXiv:1310.1145 [hep-ph]].
- [29] J. M. Henn, A. V. Smirnov and V. A. Smirnov, to appear.
- [30] J. M. Henn, *Phys. Rev. Lett.* **110** (2013) 251601 doi:10.1103/PhysRevLett.110.251601 [arXiv:1304.1806 [hep-th]].
- [31] J. M. Henn, *J. Phys. A* **48** (2015) 153001 doi:10.1088/1751-8113/48/15/153001 [arXiv:1412.2296 [hep-ph]].
- [32] J. M. Henn, A. V. Smirnov and V. A. Smirnov, *JHEP* **1403** (2014) 088 doi:10.1007/JHEP03(2014)088 [arXiv:1312.2588 [hep-th]].

- [33] T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs, JHEP **1406** (2014) 032 doi:10.1007/JHEP06(2014)032 [arXiv:1404.4853 [hep-ph]].
- [34] P. Mastrolia, M. Argeri, S. Di Vita, E. Mirabella, J. Schlenk, U. Schubert and L. Tancredi, PoS LL **2014** (2014) 007.
- [35] C. Meyer, arXiv:1611.01087 [hep-ph].
- [36] R. N. Lee, JHEP **1504** (2015) 108 doi:10.1007/JHEP04(2015)108 [arXiv:1411.0911 [hep-ph]].
- [37] O. Gituliar and V. Magerya, PoS LL **2016** (2016) 030 [arXiv:1607.00759 [hep-ph]].
- [38] R. N. Lee and V. A. Smirnov, JHEP **1610** (2016) 089 doi:10.1007/JHEP10(2016)089 [arXiv:1608.02605 [hep-ph]].
- [39] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725 doi:10.1142/S0217751X00000367 [hep-ph/9905237].
- [40] P. A. Baikov and K. G. Chetyrkin, Nucl. Phys. B **837** (2010) 186 doi:10.1016/j.nuclphysb.2010.05.004 [arXiv:1004.1153 [hep-ph]].
- [41] R. N. Lee, A. V. Smirnov and V. A. Smirnov, Nucl. Phys. B **856** (2012) 95 doi:10.1016/j.nuclphysb.2011.11.005 [arXiv:1108.0732 [hep-th]].
- [42] A. V. Smirnov and M. Tentyukov, Nucl. Phys. B **837** (2010) 40 doi:10.1016/j.nuclphysb.2010.04.020 [arXiv:1004.1149 [hep-ph]].
- [43] D. Maitre, Comput. Phys. Commun. **174** (2006) 222 [arXiv:hep-ph/0507152].
- [44] W. Wasow, “Asymptotic expansions for ordinary differential equations”, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, Pure and Applied Mathematics, Vol. XIV, 1965.
- [45] T. Becher and M. Neubert, JHEP **0906** (2009) 081 Erratum: [JHEP **1311** (2013) 024] doi:10.1088/1126-6708/2009/06/081, 10.1007/JHEP11(2013)024 [arXiv:0903.1126 [hep-ph]].