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Analytic three-loop static potential

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We present analytic results for the three-loop static potential of two heavy quarks. The analytic calculation of the missing ingredients is outlined and results for the singlet and octet potential are provided.

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I. INTRODUCTION

The static potential between two heavy quarks belongs to the fundamental quantities of QCD. In lowest order it is described by the Coulomb potential adapted to QCD. Such an approach has already been used more than 40 years ago [1] to describe the bound state of heavy quarks. Shortly afterwards the one-loop corrections were computed [2, 3] and the two-loop terms were added towards the end of the nineties [4–6]. Light quark mass effects at two loops can be found in Ref. [7]. About eight years ago the three-loop corrections have been computed by two groups in Refs. [8–10]. However, in contrast to the lower-order expressions, the three-loop results could only be presented in numerical form. In fact, in Refs. [8, 9] three coefficients in the expansion of the master integrals around $d = 4$, where d is the space-time dimension, could only be evaluated numerically (see also below). The evaluation of one of them is described in detail in Ref. [11] (in a broader context) and the remaining two coefficients are considered in Section II of this paper. We are thus in the position to present analytic results at three loops. The corresponding expressions can be found in Section III.

A generalization of the three-loop singlet potential has been considered in Ref. [12]. It is still assumed that the heavy colour sources form a singlet state, however, the colour representation is kept general.

The new results can also be used to present analytic expressions for the so-called octet potential which describes the situation where the quark and anti-quark do not form a colour-singlet but a colour-octet state. Two- and (numerical) three-loop results have been obtained in Refs. [13, 14] and [15], respectively. Analytic results for the octet potential are presented in Section IV.

In order to fix the notation we write the momentum space potential in the form

$$V^{[c]}(|\vec{q}\,|) \;=\; -4\pi C^{[c]}\frac{\alpha_s(|\vec{q}\,|)}{\vec{q}^{\,2}}\left[1+\frac{\alpha_s(|\vec{q}\,|)}{4\pi}a_1^{[c]}+\left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^2 a_2^{[c]}+\left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^3\left(a_3^{[c]}+8\pi^2C_A^3\ln\frac{\mu^2}{\vec{q}^{\,2}}\right)+\cdots\right], \quad (1)$$

with $C^{[1]} = C_F$ for the colour-singlet and $C^{[8]} = C_F - C_A/2$ for the colour-octet case. Here, $C_A = N_c$ and $C_F = (N_c^2-1)/(2N_c)$ are the eigenvalues of the quadratic Casimir operators of the adjoint and fundamental representations of the $SU(N_c)$ colour gauge group, respectively. The strong coupling α_s is defined in the $\overline{\rm MS}$ scheme and for the renormalization scale we choose $\mu = |\vec{q}\,|$ in order to suppress the corresponding logarithms. The general results, both in momentum and coordinate space, can, e.g., be found in Appendix A of Ref. [15].

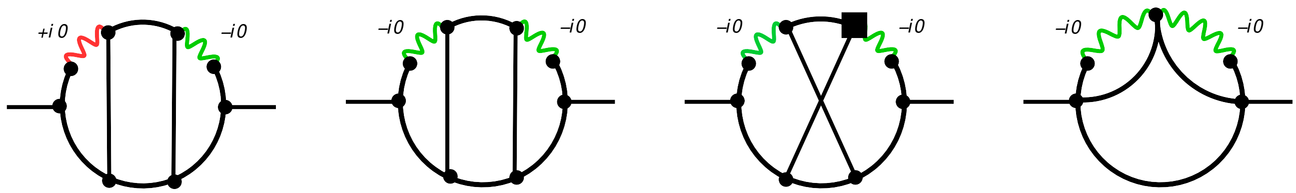
The logarithmic term in Eq. (1) has its origin in an infra-red divergence which is present for the first time at three loops as has been pointed out on Ref. [16]. The corresponding pole has been subtracted minimally. Its presence can be understood in the context of methods of regions and potential non-relativistic QCD [17–21] where $V^{[c]}$ appears as a matching coefficients. Thus, the infrared divergence cancels against ultraviolet divergences of the ultrasoft contributions. The latter have been studied in Refs. [20, 22, 23]. For the resummation of leading and next-to-leading ultrasoft logarithms we refer to [24–26].

For later convenience we decompose the three-loop corrections according to the number of closed fermion loops

$$a_3^{[c]} \;=\; a_3^{[c],(3)}n_l^3 + a_3^{[c],(2)}n_l^2 + a_3^{[c],(1)}n_l + a_3^{[c],(0)}\,, \quad (2)$$

where n_l is the number of light (massless) quarks. We furthermore consider the difference between the singlet and octet contributions and write ($i = 0, 1, 2, 3$)

$$a_3^{[8],(i)} \;=\; a_3^{[1],(i)} + \delta a_3^{[8],(i)}\,. \quad (3)$$



(a) I_{11} (b) I_{16} (c) I_{18} (d) I_{14}

FIG. 1: (a)-(c): Master integrals entering $a_3^{[c]}$ which were only known numerically. Solid lines denote relativistic scalar propagators and wavy lines refer to static propagators. For the latter the causality prescription is given explicitly where $\pm i0$ indicates a propagator of the form $1/(-k_0 \pm i0)$ with k_0 being the zeroth component of the momentum flowing through the corresponding line. The square in I_{18} indicates a convenient choice for the numerator which is specified in Ref. [11]. I_{18} is finite and only the $\mathcal{O}(\epsilon^0)$ term is needed. For I_{11} and I_{16} also the $\mathcal{O}(\epsilon^1)$ terms enter $a_3^{[c]}$. (d): Master integral which is needed for the computation of the integrals in (b) and (c). The integral I_{15} belongs to the same integral family as I_{14}, however, has an additional dot on the lower line.

In Section IV we provide analytical results for $\delta a_3^{[8],(i)}$.

The three-loop coefficient of the colour singlet potential, $a_3^{[1]}$, has entered a number of physical applications as building block (see also Ref. [27] for a recent review on applications of non-relativistic QCD to high-energy processes). To name a few of them we want to mention the next-to-next-to-next-to-leading order corrections to the leptonic decay width of the $\Upsilon(1S)$ meson [28] and the top quark threshold production in electron positron colliders [29]. Furthermore, a_3 has entered analyses to determine precise values for the charm and bottom quark masses [30–33] and the strong coupling constant [34].

II. CALCULATION OF I_{11} AND I_{16}

The calculation of $a_3^{[1]}$ as performed in [9] requires the evaluation of 41 master integrals which can be sub-divided into three different classes: There are ten integrals which do not have any static line and are thus known since long. Furthermore, we have 14 integrals with a massless one-loop insertion. They can easily be integrated in terms of Γ functions using standard techniques. The corresponding results have been presented in Ref. [35]. Results for 16 more complicated integrals can be found in Ref. [36] as expansions in $\epsilon = (4-d)/2$ to the necessary order except for two integrals (I_{11} and I_{16} of Ref. [36], see also Fig. 1(a) and (b)). Their $\mathcal{O}(\epsilon)$ terms enter $a_3^{[1]}$, however, they were only known numerically. The evaluation of these coefficients will be described in the remainder of this section. For completeness we want to mention that the third numerical ingredient required in [9] comes from the finite diagram in Fig. 1(c) (the 41$^{\text{th}}$ master integral) which has been computed in a parallel article [11].

Let us also mention that techniques which have been used to compute master integrals in [10] can be found in Ref. [37], see also [38] for a more recent summary of the status.

The method which is used to compute I_{11} and I_{16} is based on the dimensional recurrence relation and analyticity with respect to space-time dimensionality d (the so-called "DRA method") and has been developed in Ref. [39]. In Ref. [40] this method has been applied for the first time to the case with more than one master integral in a sector. Some integrals taken from families of integrals for the three-loop static quark potential and denoted in [40] by I_{14} and I_{15} (see Fig. 1(d)) have been calculated. Note that I_{14} and I_{15} are the only nontrivial integrals entering the right-hand side of the dimensional recurrence relation for I_{16}. Therefore, in principle, the results of Ref. [40] make the calculation of I_{16} straightforward.

However, the numerical issues related to the calculation of contributions to the inhomogeneous terms proportional to I_{14} and I_{15} in the right-hand side of dimensional recurrence relations for I_{16} are quite involved. The most complicated part of this contribution has the form

$$T(\nu) = \sum_{k=0}^{\infty} v^T(\nu+k) \sum_{n=k}^{\infty} \left(\prod_{l=k}^{n} \mathrm{M}(\nu+l) \right) u(\nu+n)\,, \tag{4}$$

where $\nu = d/2$, $v^T(x)$, $\mathrm{M}(x)$, and $u(x)$ are a row-vector, a 2×2 matrix and a column-vector, respectively. Their components are rational functions of the variable x. In order to calculate the sums in Eq. (4) without nested loops, we apply the standard trick of the DRA method, see Ref. [41]. Namely, let us denote

$$F(k)=\sum_{n=k}^{\infty}\mathrm{P}(k,n)u(\nu+n)\,, \tag{5}$$

where $\mathrm{P}(k,n)=\prod_{l=k}^{n}\mathrm{M}(\nu+l)$. Then

$$T(\nu)=\sum_{k=0}^{\infty}v^T(\nu+k)F(k)\,. \tag{6}$$

Using Eq. (5), the function $F(k)$ can be calculated for given k in one loop if one takes into account the recurrence relation $\mathrm{P}(k,n+1)=\mathrm{P}(k,n)\mathrm{M}(\nu+n+1)$. Now we note that $F(k)$ satisfies the recurrence relation

$$F(k+1)=\mathrm{M}^{-1}(\nu+k)F(k)-u(\nu+k)\,. \tag{7}$$

Therefore, in order to calculate consecutive terms of the sum in Eq. (6) we need to use Eq. (5) only once, and then use the recurrence relation (7). However, the price we have to pay is much higher than for scalar sums. This is connected with the multiplication by the inverse matrix $\mathrm{M}^{-1}(\nu+k)$. For $x\to\infty$ the elements of $\mathrm{M}(x)$ are of order unity, while its determinant tends to $1/1024$. Due to this fact, the multiplication by M^{-1} involves large cancellations which result in rapid precision loss. For example, using a precision of 7000 digits in the initial expression we obtain only about 370 digits in the final result.

Besides, it appears that the sum over n in the definition of $F(k)$ converges very slowly, with the summand behaving as $n^{-\alpha}$ $(\alpha>1)$ at large n. So, in order to obtain the high-precision numerical result suitable for using PSLQ [42], one has to apply the matrix analog of the convergence acceleration algorithm described in Ref. [43]. In particular, one needs to know the exponent α of the power-like decay. This appears to be possible thanks to Ref. [44], where a method for finding the asymptotic behaviour of the solutions of recurrence relations was developed. Once we dealt with these numerical issues, we have obtained the result

$$\begin{aligned}
I_{16} \;=\; & -\frac{56\pi^4}{135\epsilon}-\left(\frac{112\pi^4}{135}+\frac{16\pi^2\zeta(3)}{9}+\frac{8\zeta(5)}{3}\right)+\left(\frac{968\zeta(5)}{3}-16\pi^4 l_2+\frac{136\zeta(3)^2}{3}+\frac{400\pi^2\zeta(3)}{9}-\frac{838\pi^6}{2835}\right.\\
&\left.+\frac{1792\pi^4}{135}\right)\epsilon+\left(\frac{6144 s_6 l_2}{7}-\frac{6144 s_{7\mathrm{a}}}{7}+\frac{15360 s_{7\mathrm{b}}}{7}+1536\alpha_4\zeta(3)+1024\pi^2\alpha_5-256\pi^2\alpha_4-\frac{64}{9}\pi^4 l_2^3\right.\\
&-2976\zeta(5)l_2^2-64\pi^2\zeta(3)l_2^2-\frac{112}{3}\pi^4 l_2^2-\frac{7680\zeta(3)^2 l_2}{7}-\frac{544\pi^6 l_2}{315}+128\pi^4 l_2+\frac{306202\zeta(7)}{21}-\frac{12182\pi^2\zeta(5)}{7}\\
&\left.+\frac{64\zeta(5)}{3}-\frac{1168\zeta(3)^2}{3}-\frac{11828\pi^4\zeta(3)}{945}+\frac{1664\pi^2\zeta(3)}{9}+\frac{1376\pi^6}{135}-\frac{12544\pi^4}{135}+768 s_6\right)\epsilon^2+O\left(\epsilon^3\right)\,,
\end{aligned} \tag{8}$$

where $\zeta(n)$ is Riemann's zeta function evaluated at n and

$$\begin{aligned}
l_2 &= \log(2)\,,\\
\alpha_n &= \mathrm{Li}_n(1/2)+\frac{(-\log 2)^n}{n!}\,,\\
s_6 &= \zeta(-5,-1)+\zeta(6)\,,\\
s_{7a} &= \zeta(-5,1,1)+\zeta(-6,1)+\zeta(-5,2)+\zeta(-7)\,,\\
s_{7b} &= \zeta(7)+\zeta(5,2)+\zeta(-6,-1)+\zeta(5,-1,-1)\,.
\end{aligned} \tag{9}$$

$\zeta(m_1,\ldots,m_k)$ are multiple zeta values given by

$$\zeta(m_1,\ldots,m_k)\;=\;\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_k=1}^{i_{k-1}-1}\prod_{j=1}^{k}\frac{\mathrm{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}\,. \tag{10}$$

In order to apply the DRA method to I_{11}, one has to take into account that the dimensional recurrence relation for I_{11} contains now two non-trivial integrals denoted in [40] by I_9 and I_{10}. So, in a first step one has to apply the DRA method to these two integrals. Fortunately, they can be calculated along the same lines as I_{14} and I_{15} from which they differ only by the $\pm i0$ prescription in one of the linear denominators. In particular, the summing factor has the same form as in Ref. [40] (see Eq. (4.14) of that paper). Plugging the results for I_9 and I_{10} in the dimensional recurrence relation for I_{11} and applying the DRA method, we obtain

$$\begin{aligned}
I_{11} \;=\; & \frac{64\pi^4}{135\epsilon}+\left(\frac{128\pi^4}{135}+\frac{32\pi^2\zeta(3)}{9}-\frac{8\zeta(5)}{3}\right)+\left(16\pi^4 l_2+\frac{968\zeta(5)}{3}+\frac{136\zeta(3)^2}{3}-\frac{800\pi^2\zeta(3)}{9}+\frac{548\pi^6}{2835}-\frac{2048\pi^4}{135}\right)\epsilon \\
&+\left(\frac{6144 s_6 l_2}{7}-\frac{6144 s_{7\mathrm{a}}}{7}+\frac{15360 s_{7\mathrm{b}}}{7}+1536\alpha_4\zeta(3)-2048\pi^2\alpha_5+512\pi^2\alpha_4-\frac{64}{9}\pi^4 l_2^3-2976\zeta(5)l_2^2\right. \\
&-64\pi^2\zeta(3)l_2^2+\frac{80}{3}\pi^4 l_2^2-\frac{7680\zeta(3)^2 l_2}{7}-\frac{208\pi^6 l_2}{315}-128\pi^4 l_2+\frac{306202\zeta(7)}{21}+\frac{1482\pi^2\zeta(5)}{7}+\frac{64\zeta(5)}{3} \\
&\left.-\frac{1168\zeta(3)^2}{3}-\frac{70208\pi^4\zeta(3)}{945}-\frac{3328\pi^2\zeta(3)}{9}-\frac{1504\pi^6}{135}+\frac{14336\pi^4}{135}+768 s_6\right)\epsilon^2+O\left(\epsilon^3\right)\,. \qquad (11)
\end{aligned}$$

Note that the $\mathcal{O}(\epsilon^2)$ terms of I_{16} and I_{11} in Eqs. (8) and (11) are not needed for $a_3^{[c]}$. We nevertheless provide these results to demonstrate the powerfulness of the DRA method.

In principle, the DRA method is also applicable to the calculation of I_{18}. However, the difficulties related to the slow convergence of certain matrix sums and the corresponding precision loss appear to be overwhelming. For this reason, the method of differential equations has been applied to I_{18}, see Ref. [11].

III. SINGLET POTENTIAL

In this Section we present analytic expressions for $a_3^{[1]}$. One- and two-loop results using the same notation can be found in Ref. [15]. Analytic results for the coefficients of n_l^3 and n_l^2 have already been presented in Ref. [8]. Here, they are repeated for completeness

$$\begin{aligned}
a_3^{[1],(3)} &= -\left(\frac{20}{9}\right)^3 T_F^3\,, \\
a_3^{[1],(2)} &= \left(\frac{12541}{243}+\frac{368\zeta(3)}{3}+\frac{64\pi^4}{135}\right)C_A T_F^2+\left(\frac{14002}{81}-\frac{416\zeta(3)}{3}\right)C_F T_F^2\,. \qquad (12)
\end{aligned}$$

Let us now turn to the n_l^1 and n_l^0 term. Expressed in terms of the eigenvalues of the Casimir operators we obtain for the linear-n_l term the analytic result

$$\begin{aligned}
a_3^{[1],(1)} \;=\; & \frac{d_F^{abcd}d_F^{abcd}}{N_A}\left\{\pi^2\left(\frac{1264}{9}-\frac{976\zeta(3)}{3}+l_2\left(64+672\zeta(3)\right)\right)+\pi^4\left(-\frac{184}{3}+\frac{32 l_2}{3}-32 l_2^2\right)+\frac{10\pi^6}{3}\right\} \\
&+T_F\left\{C_F^2\left(\frac{286}{9}+\frac{296\zeta(3)}{3}-160\zeta(5)\right)+C_A C_F\left(-\frac{71281}{162}+264\zeta(3)+80\zeta(5)\right)\right. \\
&+C_A^2\left[-\frac{58747}{486}+\pi^2\left(\frac{17}{27}-32\alpha_4+l_2\left(-\frac{4}{3}-14\zeta(3)\right)-\frac{19\zeta(3)}{3}\right)-356\zeta(3)\right. \\
&\left.\left.+\pi^4\left(-\frac{157}{54}-\frac{5 l_2}{9}+l_2^2\right)+\frac{1091\zeta(5)}{6}+\frac{57(\zeta(3))^2}{2}+\frac{761\pi^6}{2520}-48 s_6\right]\right\}, \qquad (13)
\end{aligned}$$

and the gluonic part is given by

$$\begin{aligned}
a_3^{[1],(0)} \;=\; & \frac{d_F^{abcd}d_A^{abcd}}{N_A}\left\{\pi^2\left[\frac{7432}{9}-4736\alpha_4+l_2\left(\frac{14752}{3}-3472\zeta(3)\right)-\frac{6616\zeta(3)}{3}\right]\right. \\
&\left.+\pi^4\left(-156+\frac{560 l_2}{3}+\frac{496 l_2^2}{3}\right)+\frac{1511\pi^6}{45}\right\}+C_A^3\left\{\frac{385645}{2916}+\pi^2\left[-\frac{953}{54}+\frac{584\alpha_4}{3}+\frac{175\zeta(3)}{2}\right.\right. \\
&\left.+l_2\left(-\frac{922}{9}+\frac{217\zeta(3)}{3}\right)\right]+\frac{584\zeta(3)}{3}+\pi^4\left(\frac{1349}{270}-\frac{20 l_2}{9}-\frac{40 l_2^2}{9}\right)-\frac{1927\zeta(5)}{6}-\frac{143(\zeta(3))^2}{2} \\
&\left.-\frac{4621\pi^6}{3024}+144 s_6\right\}. \qquad (14)
\end{aligned}$$

The numerical evaluation of the analytic results is in full agreement (including all digits) with [8, 9].

It is interesting to note that the contributions proportional to $d_F^{abcd}d_F^{abcd}$ and $d_F^{abcd}d_A^{abcd}$ only involve π^2, π^4 and π^6 terms. Note that these colour structures appear for the first time at three-loop order. On the other hand, the other colour structures basically involve all constants one expects up to transcendentality weight six. Note, however, that the constant s_6 is only present in the most non-abelian parts, i.e., $T_F C_A^2$ and C_A^3. Let us also mention that log(2) terms are present to first, second and fourth power but there are no cubic terms.

In a next step we specify to $\mathrm{SU}(N_c)$ and replace the colour factors by

$$\begin{aligned} &C_A = N_c\,, \quad C_F = \frac{N_c^2-1}{2N_c}\,, \quad T_F = \frac{1}{2}\,, \quad N_A = N_c^2 - 1\,, \\ &\frac{d_F^{abcd}d_F^{abcd}}{N_A} = \frac{18-6N_c^2+N_c^4}{96N_c^2}\,, \quad \frac{d_F^{abcd}d_A^{abcd}}{N_A} = \frac{N_c(N_c^2+6)}{48}\,. \end{aligned} \tag{15}$$

This leads to

$$\begin{aligned} a_3^{[1],(1)} &= \frac{66133}{648} + \pi^2\left(-\frac{79}{9} + l_2\left(-4-42\zeta(3)\right) + \frac{61\zeta(3)}{3}\right) - \frac{272\zeta(3)}{3} + \pi^4\left(\frac{23}{6} - \frac{2l_2}{3} + 2l_2^2\right) + 20\zeta(5) - \frac{5\pi^6}{24} \\ &+ \frac{1}{N_c^2}\left\{\frac{143}{36} + \pi^2\left[\frac{79}{3} - 61\zeta(3) + l_2\left(12+126\zeta(3)\right)\right] + \frac{37\zeta(3)}{3} + \pi^4\left(-\frac{23}{2} + 2l_2 - 6l_2^2\right) - 20\zeta(5)\right. \\ &\left. + \frac{5\pi^6}{8}\right\} + N_c^2\left\{-\frac{323615}{1944} + \pi^2\left(\frac{16}{9} - 16\alpha_4 - \frac{59\zeta(3)}{9}\right) - \frac{299\zeta(3)}{3} + \pi^4\left(-\frac{113}{54} - \frac{l_2}{6} + \frac{l_2^2}{6}\right)\right. \\ &\left. + \frac{1091\zeta(5)}{12} + \frac{57(\zeta(3))^2}{4} + \frac{13\pi^6}{70} - 24s_6\right\}, \\ a_3^{[1],(0)} &= N_c\left\{\pi^2\left[\frac{929}{9} - 592\alpha_4 + l_2\left(\frac{1844}{3} - 434\zeta(3)\right) - \frac{827\zeta(3)}{3}\right] + \pi^4\left(-\frac{39}{2} + \frac{70l_2}{3} + \frac{62l_2^2}{3}\right) + \frac{1511\pi^6}{360}\right\} \\ &+ N_c^3\left\{\frac{385645}{2916} + \pi^2\left(-\frac{4}{9} + 96\alpha_4 + \frac{374\zeta(3)}{9}\right) + \frac{584\zeta(3)}{3} + \pi^4\left(\frac{943}{540} + \frac{5l_2}{3} - l_2^2\right)\right. \\ &\left. - \frac{1927\zeta(5)}{6} - \frac{143(\zeta(3))^2}{2} - \frac{29\pi^6}{35} + 144s_6\right\}. \end{aligned} \tag{16}$$

Finally, for $N_c = 3$ we have

$$\begin{aligned} a_3^{[1],(1)} &= -\frac{452213}{324} + \pi^2\left[\frac{274}{27} - \frac{409\zeta(3)}{9} - 144\alpha_4 + l_2\left(-\frac{8}{3} - 28\zeta(3)\right)\right] - \frac{26630\zeta(3)}{27} \\ &+ \pi^4\left(-\frac{293}{18} - \frac{35l_2}{18} + \frac{17l_2^2}{6}\right) + \frac{30097\zeta(5)}{36} + \frac{1931\pi^6}{1260} + \frac{513(\zeta(3))^2}{4} - 216s_6\,, \end{aligned} \tag{17}$$

$$\begin{aligned} a_3^{[1],(0)} &= \frac{385645}{108} + \pi^2\left[\frac{893}{3} + 816\alpha_4 + l_2\left(1844-1302\zeta(3)\right) + 295\zeta(3)\right] + 5256\zeta(3) \\ &+ \pi^4\left(-\frac{227}{20} + 115l_2 + 35l_2^2\right) - \frac{17343\zeta(5)}{2} - \frac{1643\pi^6}{168} - \frac{3861(\zeta(3))^2}{2} + 3888s_6\,, \end{aligned} \tag{18}$$

which in numerical form is given by

$$a_3^{[1]} = 13432.5648565 - 3289.9052968\,n_l + 185.9900266\,n_l^2 - 1.3717421\,n_l^3\,. \tag{19}$$

IV. OCTET POTENTIAL

In this Section we proceed similar to the previous one and present results for $\delta a_3^{[8],(i)}$ defined in Eq. (3). We refrain from presenting results in terms of C_A, C_F, etc. and immediately use Eq. (15) to obtain expressions in terms of N_c.

We have $\delta a_3^{[8],(i)} = 0$ for $i=2$ and $i=3$ and for the linear-n_l and n_l-independent terms we get

$$\begin{aligned}
\delta a_3^{[8],(1)} &= \pi^2\left[-\frac{11}{3} - 31\zeta(3) + l_2\left(4+42\zeta(3)\right)\right] + \pi^4\left(-\frac{7}{6}+\frac{2l_2}{3}-2l_2^2\right)+\frac{5\pi^6}{24} \\
&\quad + N_c^2\left[\pi^2\left(\frac{8}{9}+48\alpha_4+25\zeta(3)\right)+\pi^4\left(\frac{2}{3}+\frac{2l_2}{3}\right)-\frac{13\pi^6}{20}\right], \\
\delta a_3^{[8],(0)} &= N_c^3\left\{\pi^2\left[\frac{139}{9}+304\alpha_4+15\zeta(3)+l_2\left(-\frac{1844}{3}+434\zeta(3)\right)\right]\right. \\
&\quad \left. + \pi^4\left(\frac{295}{6}-30l_2-\frac{62l_2^2}{3}\right)-\frac{1187\pi^6}{360}\right\},
\end{aligned} \tag{20}$$

which for $N_c = 3$ leads to

$$\begin{aligned}
\delta a_3^{[8],(1)} &= -\frac{677\pi^6}{120}+\pi^4\left(\frac{29}{6}+\frac{20l_2}{3}-2l_2^2\right)+\pi^2\left[\frac{13}{3}+432\alpha_4+194\zeta(3)+l_2\left(4+42\zeta(3)\right)\right], \\
\delta a_3^{[8],(0)} &= \pi^2\left[417+8208\alpha_4+405\zeta(3)+l_2\left(-16596+11718\zeta(3)\right)\right]+\pi^4\left(\frac{2655}{2}-810l_2-558l_2^2\right) \\
&\quad -\frac{3561\pi^6}{40}.
\end{aligned} \tag{21}$$

It is interesting to note that $\delta a_3^{[8],(0)}$ and $\delta a_3^{[8],(1)}$ have an overall factor π^2 which was predicted in Ref. [15] on the basis of the involved master integrals. Although they could not be computed analytically it was possible to show that there is an overall factor π^2, a feature which is also observed at two-loop order in QCD [13, 14] and in $\mathcal{N}=4$ supersymmetric Yang Mills theories [45].

In numerical form we obtain for the complete three-loop coefficient

$$\delta a_3^{[8]} = -2634.7351731 + 367.9626044\, n_l\,. \tag{22}$$

V. CONCLUSIONS

The interaction of a slowly moving heavy quark-anti-quark pair can be described with the help of a static potential, a concept which is familiar from ordinary quantum mechanics. Its perturbative part is obtained from the exchange of soft gluons which are conveniently considered in the framework of non-relativistic QCD. Numerical results for the three-loop potential, which have entered a number of physical observables, have been obtained eight years ago by two independent groups [8–10]. The obtained precision has been sufficient for all physical applications where a_3 entered as a building block. However, from the aesthetic point of view it is important to obtain analytic results for higher order quantum corrections. This has been achieved in this paper. We have obtained analytic results for the three-loop corrections to the singlet and octet potential which are presented in Sections III and IV, respectively.

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