

# The $\beta$ -function of Quantum Chromodynamics and the effective Higgs-gluon-gluon coupling in five-loop order

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We present the result of the analytical computation of the five-loop term in the beta function which governs the running of  $\alpha_s$  — the quark-gluon coupling constant in QCD. Using the result, a low-energy theorem and the four-loop QCD decoupling formula, we derive the effective Higgs-gluon-gluon coupling constant in five-loop approximation.

*Loops and Legs in Quantum Field Theory  
24-29 April 2016  
Leipzig, Germany*

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Asymptotic freedom, manifest by a coupling constant decreasing with increasing energy, is one of the basic predictions of nonabelian gauge theories. It is quantitatively encoded in the behaviour of the  $\beta$ -function of Quantum Chromodynamics (QCD):

$$\beta(a_s) = \mu^2 \frac{d}{d\mu^2} a_s(\mu) = - \sum_{i \geq 0} \beta_i a_s^{i+2} \quad (1)$$

which describes the running of the quark-gluon coupling constant  $a_s \equiv \alpha_s/\pi$  as a function of the renormalization scale  $\mu$ . The four lowest terms were calculated in Refs. [1–8] with the result (we use the  $\overline{\text{MS}}$  renormalization scheme throughout the paper):

$$\begin{aligned} \beta_0 &= \frac{1}{4} \left\{ 11 - \frac{2}{3} n_f \right\}, \\ \beta_1 &= \frac{1}{4^2} \left\{ 102 - \frac{38}{3} n_f \right\}, \\ \beta_2 &= \frac{1}{4^3} \left\{ \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right\}, \\ \beta_3 &= \frac{1}{4^4} \left\{ \frac{149753}{6} + 3564 \zeta_3 - \left[ \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right] n_f + \left[ \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right] n_f^2 + \frac{1093}{729} n_f^3 \right\}, \end{aligned}$$

where  $n_f$  denotes the number of active quark flavors. These results have moved the theory from qualitative agreement with experiment to precise quantitative predictions, covering a wide kinematical range, from  $\tau$ -lepton decays up to LHC results.

Recently, the five-loop term has been evaluated with the result:

$$\begin{aligned} \beta_4 &= \frac{1}{4^5} \left\{ \frac{8157455}{16} + \frac{621885}{2} \zeta_3 - \frac{88209}{2} \zeta_4 - 288090 \zeta_5 \right. \\ &\quad + n_f \left[ -\frac{336460813}{1944} - \frac{4811164}{81} \zeta_3 + \frac{33935}{6} \zeta_4 + \frac{1358995}{27} \zeta_5 \right] \\ &\quad + n_f^2 \left[ \frac{25960913}{1944} + \frac{698531}{81} \zeta_3 - \frac{10526}{9} \zeta_4 - \frac{381760}{81} \zeta_5 \right] \\ &\quad + n_f^3 \left[ -\frac{630559}{5832} - \frac{48722}{243} \zeta_3 + \frac{1618}{27} \zeta_4 + \frac{460}{9} \zeta_5 \right] \\ &\quad \left. + n_f^4 \left[ \frac{1205}{2916} - \frac{152}{81} \zeta_3 \right] \right\}. \quad (2) \end{aligned}$$

As expected from the three and four-loop results,  $\beta_4$  does not contain the higher transcendentality  $\zeta_3^2$ ,  $\zeta_6$  and  $\zeta_7$  which do appear in individual diagrams. Note that the  $n_f^4$  term is in full agreement with the result of [9] derived for a generic gauge group long time ago with a very different technique. Furthermore, quite recently, the  $n_f^3$  term has been confirmed and even extended for a generic gauge group in [10].

In numerical form the coefficients  $\beta_0 - \beta_4$  read

$$\begin{aligned}
\beta_0 &\approx 2.75 - 0.166667 n_f, \\
\beta_1 &\approx 6.375 - 0.791667 n_f, \\
\beta_2 &\approx 22.3203 - 4.36892 n_f + 0.0940394 n_f^2, \\
\beta_3 &\approx 114.23 - 27.1339 n_f + 1.58238 n_f^2 + 0.0058567 n_f^3, \\
\beta_4 &\approx 524.558 - 181.799 n_f + 17.156 n_f^2 - 0.22586 n_f^3 - 0.0017993 n_f^4.
\end{aligned} \tag{3}$$

For the particular cases of  $n_f = 3, 4, 5$  and  $6$  we get:

$$\begin{aligned}
\bar{\beta}(n_f = 3) &= 1 + 1.78 a_s + 4.47 a_s^2 + 20.99 a_s^3 + 56.59 a_s^4, \\
\bar{\beta}(n_f = 4) &= 1 + 1.54 a_s + 3.05 a_s^2 + 15.07 a_s^3 + 27.33 a_s^4, \\
\bar{\beta}(n_f = 5) &= 1 + 1.26 a_s + 1.47 a_s^2 + 9.83 a_s^3 + 7.88 a_s^4, \\
\bar{\beta}(n_f = 6) &= 1 + 0.93 a_s - 0.29 a_s^2 + 5.52 a_s^3 + 0.15 a_s^4,
\end{aligned} \tag{4}$$

where  $\bar{\beta} \equiv \frac{\beta(a_s)}{-\beta_0 a_s^2} = 1 + \sum_{i \geq 1} \bar{\beta}_i a_s^i$ . A very modest growth of the coefficients is observed and the (apparent) convergence is better than one would expect from comparison with other examples. The smallness of the five-loop contributions in eqs. (4) implies that the final result is hardly effected when running  $\alpha_s$  from the scale of the  $\tau$ -lepton mass  $m_\tau$  to the mass of the Z-boson.

The combined uncertainty in  $\alpha_s^{(5)}(M_Z)$  induced by running and matching from  $m_\tau$  to  $M_Z$  can be conservatively estimated by the shift in  $\alpha_s^{(5)}(M_Z)$  produced by the use of five-loop running (and, consequently) four-loop matching instead of four-loop running (and three-loop matching). It amounts to a  $8 \cdot 10^{-5}$  which is by a factor of three less than the similar shift made by the use of four-loop running instead of the the three-loop one. The final value of  $\alpha_s^{(5)}(M_Z) = 0.1143 \pm 0.0015$  which follows from  $\alpha_s^{(3)}(M_\tau)$  is in remarkably good agreement with the fit to electroweak precision data ([11])  $\alpha_s^{(5)}(M_Z) = 0.1197 \pm 0.0028$ .

Our result for the  $\beta$ -function leads to a determination of the effective Higgs-gluon-gluon coupling, improved by one more loop. In the heavy top limit the Higgs boson couples directly to gluons via the effective Lagrangian of the form [12–15]

$$\mathcal{L}_{eff} = -2^{1/4} G_F^{1/2} H C_1(\mu^2/m_t^2, a_s(\mu)) G_{\nu\rho}^a G_{\nu\rho}^a. \tag{5}$$

The effective coupling constant  $C_1(\mu^2/m_t^2, a_s(\mu))$  appears as a common factor in two quantities important for Higgs physics processes, namely, Higgs decay into gluons (one of the main decay channels for the Standard Model Higgs boson) and Higgs production via gluon fusion (the main Higgs production mode on LHC). It is expressible through massive tadpoles and was computed at four loops in 1997 [16] (long before the direct calculation of four-loop generic massive tadpoles started to be technically feasible). This happened to be possible due to a low energy theorem<sup>1</sup> (exact in all orders) [16]

$$C_1 = -\frac{1}{2} m_h^2 \frac{\partial}{\partial m_h^2} \ln \zeta_g^2, \tag{6}$$

<sup>1</sup>Very recently the theorem has been redirived with a different technique [17] and even extended on the case of multi-Higgs couplings to gluons in Ref. [18].

which connects  $C_1$  with the corresponding ‘‘decoupling’’ constant  $\zeta_g$  defined as a conversion factor in the relation

$$\alpha'_s = \alpha'_s(\mu, \alpha_s, m_h) = \zeta_g^2(\mu, \alpha_s, m_h) \alpha_s(\mu), \quad \zeta_g^2 = 1 + \sum_{i \geq 1} a_s^i(\mu) d_{\text{MS},i},$$

where  $\alpha'_s$  denotes the coupling constant in the effective QCD with the heavy quark, integrated out and  $n_l$  is the number of remaining active quark flavours.

Eq. (6) assumes that we are dealing with QCD with one heavy quark  $h$  and  $n_l$  massless ones and that Higgs field is coupled with the scalar heavy quark current via

$$-\frac{H}{v} m_h \bar{h} h.$$

The total number of active flavours is  $n_f = n_l + 1$ . For the case of the SM Higgs one should set  $m_h = m_t$  and  $n_f = 6$ ,  $n_l = 5$ . A more convenient form of the rhs of eq. (6) was found in Ref. [16]

$$C_1 = \frac{\pi}{2\alpha'_s [1 - 2\gamma_m(\alpha_s)]} \left[ \beta'(\alpha'_s) - \beta(\alpha_s) \frac{\partial \alpha'_s}{\partial \alpha_s} \right]. \quad (7)$$

By counting the powers of  $\alpha_s$  in the rhs of eq. (7) one can easily see that, say, the  $(L+1)$ -loop contribution to  $C_1$  is constructed from the  $(L+1)$  loop  $\beta$ -function as well as from the quark mass anomalous dimension  $\gamma_m$  and the decoupling constant  $\zeta_g^2$ , with the latter two quantities being sufficient to be known at  $L$  loops *only*.

The decoupling constant  $\zeta_g^2$  is currently known at four loops; the corresponding coefficients  $d_i$  read<sup>2</sup>

$$d_{\text{MS},1} = -\frac{1}{6} \ell_{\mu m}, \quad (8)$$

$$d_{\text{MS},2} = \frac{11}{72} - \frac{11}{24} \ell_{\mu m} + \frac{1}{36} \ell_{\mu m}^2, \quad (9)$$

$$\begin{aligned} d_{\text{MS},3} = & \frac{564731}{124416} - \frac{82043}{27648} \zeta_3 - \frac{955}{576} \ell_{\mu m} + \frac{53}{576} \ell_{\mu m}^2 - \frac{1}{216} \ell_{\mu m}^3 \\ & + n_l \left[ -\frac{2633}{31104} + \frac{67}{576} \ell_{\mu m} - \frac{1}{36} \ell_{\mu m}^2 \right], \end{aligned} \quad (10)$$

$$\begin{aligned} d_{\text{MS},4} = & \Delta_{\text{MS},4} + \frac{7391699}{746496} \ell_{\mu m} - \frac{2529743}{165888} \zeta_3 \ell_{\mu m} + \frac{2177}{3456} \ell_{\mu m}^2 - \frac{1883}{10368} \ell_{\mu m}^3 + \frac{1}{1296} \ell_{\mu m}^4 \\ & + n_l \left[ -\frac{110341}{373248} \ell_{\mu m} + \frac{110779}{82944} \zeta_3 \ell_{\mu m} - \frac{1483}{10368} \ell_{\mu m}^2 - \frac{127}{5184} \ell_{\mu m}^3 \right] \\ & + n_l^2 \left[ \frac{6865}{186624} \ell_{\mu m} - \frac{77}{20736} \ell_{\mu m}^2 + \frac{1}{324} \ell_{\mu m}^3 \right], \end{aligned} \quad (11)$$

<sup>2</sup>The original result for  $\zeta_g^2$  as obtained in Refs. [19, 20] was not completely analytical as a pair of relevant master integrals was then known only numerically. In the formula below we have used finding of Refs. [21, 22] to display the  $\Delta_{\text{MS},4}$  in a completely analytical form (it was also done in Ref. [23]).

$$\begin{aligned}
\Delta_{\text{MS},4} = & \frac{291716893}{6123600} - \frac{76940219}{195955200} \pi^4 - \frac{2362581983}{87091200} \zeta_3 - \frac{12057583}{483840} \zeta_5 \\
& + \frac{9318467}{32659200} \pi^4 \ln 2 - \frac{3031309}{1306368} \pi^2 \ln^2 2 + \frac{340853}{816480} \pi^2 \ln^3 2 + \frac{3031309}{1306368} \ln^4 2 \\
& - \frac{340853}{1360800} \ln^5 2 + \frac{3031309}{54432} a_4 + \frac{340853}{11340} a_5 \\
& + n_l \left[ -\frac{4770941}{2239488} - \frac{541549}{14929920} \pi^4 + \frac{3645913}{995328} \zeta_3 + \frac{115}{576} \zeta_5 \right. \\
& \quad \left. - \frac{685}{124416} \pi^2 \ln^2 2 + \frac{685}{124416} \ln^4 2 + \frac{685}{5184} a_4 \right] \\
& + n_l^2 \left[ -\frac{271883}{4478976} + \frac{167}{5184} \zeta_3 \right]. \tag{12}
\end{aligned}$$

Here  $\ell_{\mu m} = \ln \frac{\mu^2}{m^2(\mu)}$ ,  $m(\mu)$  is the (running) heavy quark mass and  $a_n = \text{Li}_n(1/2) = \sum_{i=1}^{\infty} 1/(2^i)^n$ . The quark mass anomalous dimension  $\gamma_m$  is known at four loops since long [24, 25]. Finally, using our result for the  $\beta$ -function,  $\zeta_g$  and  $\gamma_m$  we arrive at ( $\mu_h$  is defined as  $\mu_h = m_h(\mu_h)$ ):

$$\begin{aligned}
C_1 = & -\frac{1}{12} a_s(\mu_h) \left\{ 1 + \frac{11}{4} a_s(\mu_h) + a_s^2(\mu_h) \left[ \frac{2821}{288} + n_l \left( -\frac{67}{96} \right) \right] \right. \\
& + a_s^3(\mu_h) \left[ -\frac{4004351}{62208} + \frac{1305893}{13824} \zeta(3) + n_l \left( \frac{115607}{62208} - \frac{110779}{13824} \zeta(3) \right) + n_l^2 \left( -\frac{6865}{31104} \right) \right] \\
& + a_s^4(\mu_h) \left[ \left( -\frac{348837110927}{223948800} + \frac{1959268669}{111974400} \pi^4 + \frac{35074684379}{24883200} \zeta_3 + \frac{127050929}{138240} \zeta_5 \right. \right. \\
& \quad - \frac{121140071}{9331200} \pi^4 \ln 2 + \frac{39407017}{373248} \pi^2 \ln^2 2 - \frac{4431089}{233280} \pi^2 \ln^3 2 - \frac{39407017}{373248} \ln^4 2 \\
& \quad \left. \left. + \frac{4431089}{388800} \ln^5 2 - \frac{39407017}{15552} a_4 - \frac{4431089}{3240} a_5 \right) \right. \\
& + n_l \left( \frac{223712155249}{1567641600} + \frac{190184767}{348364800} \pi^4 - \frac{108268867471}{348364800} \zeta_3 - \frac{22007549}{483840} \zeta_5 \right. \\
& \quad + \frac{9318467}{10886400} \pi^4 \ln 2 - \frac{3896297}{580608} \pi^2 \ln^2 2 + \frac{340853}{272160} \pi^2 \ln^3 2 + \frac{3896297}{580608} \ln^4 2 \\
& \quad \left. \left. - \frac{340853}{453600} \ln^5 2 + \frac{3896297}{24192} a_4 + \frac{340853}{3780} a_5 \right) \right. \\
& + n_l^2 \left( -\frac{8059709}{4478976} - \frac{576757}{4976640} \pi^4 + \frac{455777}{36864} \zeta_3 - \frac{115}{384} \zeta_5 - \frac{685}{41472} \pi^2 \ln^2 2 + \frac{685}{41472} \ln^4 2 + \frac{685}{1728} a_4 \right) \\
& \left. + n_l^3 \left( -\frac{270407}{1492992} + \frac{211}{1728} \zeta_3 \right) \right\},
\end{aligned}$$

or, numerically,

$$\begin{aligned}
C_1 \approx & -\frac{1}{12} a_s(\mu_h) \left[ 1 + 2.75 a_s(\mu_h) + (9.795 - 0.698 n_l) a_s^2(\mu_h) \right. \\
& \quad + (49.183 - 7.774 n_l - 0.221 n_l^2) a_s^3(\mu_h) \\
& \quad \left. + (326.716 - 64.163 n_l + 1.594 n_l^2 - 0.0343 n_l^3) a_s^4(\mu_h) \right]. \tag{13}
\end{aligned}$$

The last expression should be modified if one want to use the pole quark mass  $M_h$  instead of the running  $\overline{\text{MS}}$  one. Using the corresponding conversion formula from [26–28] we arrive at

$$C_1 \approx -\frac{1}{12} a_s(M_h) \left[ 1 + 2.75 a_s(M_h) + (9.351 - 0.698 n_l) a_s^2(M_h) + (43.609 - 6.538 n_l - 0.221 n_l^2) a_s^3(M_h) + (240.383 - 30.231 n_l + 0.8001 n_l^2 - 0.0343 n_l^3) a_s^4(M_h) \right]. \quad (14)$$

For the case of the SM Higgs we obtain:

$$C_1 = -\frac{1}{12} a_s(\mu_t) \left[ 1 + 2.750 a_s(\mu_t) + 6.306 a_s^2(\mu_t) + 4.794 a_s^3(\mu_t) + 41.447 a_s^4(\mu_t) \right]. \quad (15)$$

Note that the contribution due to  $\beta_4$  to the last coefficient (boxed below) is significant, namely,

$$41.447 = -47.611 + \boxed{89.058}. \quad (16)$$

After transformation to the pole top quark mass eq. (15) reads

$$C_1 = -\frac{1}{12} a_s(M_t) \left[ (1 + 2.750 a_s(M_t) + 5.861 a_s^2(M_t) + 5.3996 a_s^3(M_t) + 54.951 a_s^4(M_t)) \right], \quad (17)$$

with the last term decomposition assuming the form:

$$54.951 = -34.107 + \boxed{89.058}. \quad (18)$$

Let us close this paper with a brief discussion of its technical details<sup>3</sup>. The total number of five-loop diagrams contributing to the  $\beta$ -function (as generated by QGRAF [30]) amounts to about one and a half million. Every power of  $n_f$  in (2) was computed separately with the help of the FORM [31, 32] program BAICER, implementing the algorithm of works [33–35].

Our analytical results are also available in computer readable form under the URL <http://www.ttp.kit.edu/Progdata/ttp16/ttp16-032>

We thank Johannes Blümlein for his interest to our work and encouragement.

The work by K. G. Chetyrkin and J. H. Kühn was supported by the Deutsche Forschungsgemeinschaft through CH1479/1-1. The work of P. A. Baikov is supported in part by grant NSh-7989.2016.2 of the President of Russian Federation.

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<sup>3</sup>More information about our calculation and its implications can be found in Ref. [29].

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