Five-Loop Running of the QCD coupling constant*

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We analytically compute the five-loop term in the beta function which governs the running of α_s — the quark-gluon coupling constant in QCD. The new term leads to a reduction of the theory uncertainty in α_s taken at the Z-boson scale as extracted from the τ-lepton decays as well as to new, improved by one more order of perturbation theory, predictions for the effective coupling constants of the Standard Model Higgs boson to gluons and for its total decay rate to the quark-antiquark pairs.

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Asymptotic freedom, manifest by a decreasing coupling with increasing energy, can be considered as the basic prediction of nonabelian gauge theories and was crucial for establishing Quantum Chromodynamics (QCD) as the theory of strong interactions [1, 2]. The dominant, leading order prediction was quickly followed by the corresponding two-loop [3, 4] and three-loop [5, 6] results. The next, four-loop calculation was performed almost twenty years later [7] and confirmed in [8]. These results have moved the theory from qualitative agreement with experiment, as observed on the basis of the early results, to precise quantitative predictions, valid over a wide kinematic range, from τ-lepton decays up to LHC results.

Although the agreement between theory predictions and experimental results is impressive already now, it is tempting to push the theory prediction as high as possible. On the one hand one may expect an even better agreement between theory and experiment. On the other hand it is of theoretical interest to push gradually into the region where individual terms of the series might start to increase, thus demonstrating the asymptotic divergence of the perturbative series. At a more modest level we note that predictions for the five-loop term that can be found in the literature are based on a variety of methods and exhibit for some cases quite a dramatic variation of the size of the term (we will give more details later).

There are, of course, a number of phenomenological applications of the five-loop result, which will be discussed in this paper. On the one hand there is the relation between Z-boson and τ-lepton decay rates into hadrons, which involves the strong coupling at two vastly different scales. On the other hand we will discuss the Higgs boson decay rate into bottom quarks and into gluons, which are sensitive to the five-loop running of the QCD coupling.

*In memoriam Dmitry Vasil'evich Shirkov, 1928-2016

Let us start with the definition of the beta function

$$\beta(a_s) = \mu^2 \frac{d}{d\mu^2} a_s(\mu) = -\sum_{i\geq 0} \beta_i a_s^{i+2} \tag{1}$$

which describes the running of the quark-gluon coupling $a_s \equiv \alpha_s/\pi$ as a function of the normalization scale μ within the renormalization group approach [9–11].

Using the same theoretical tools as in the calculations of [12] and [13] we have computed the QCD β-function in five-loop order with the result

$$\beta_0 = \frac{1}{4}\left\{11 - \frac{2}{3}n_f,\right\}, \quad \beta_1 = \frac{1}{4^2}\left\{102 - \frac{38}{3}n_f\right\}, \tag{2}$$

$$\beta_2 = \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right\}, \tag{3}$$

$$\begin{aligned}\beta_3 = \frac{1}{4^4}\Bigg\{ & \frac{149753}{6} + 3564\zeta_3 - \left[\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right] n_f \\ & + \left[\frac{50065}{162} + \frac{6472}{81}\zeta_3\right] n_f^2 + \frac{1093}{729}n_f^3\Bigg\},\end{aligned} \tag{4}$$

$$\begin{aligned}\beta_4 = \frac{1}{4^5}\Bigg\{ & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\ + \; n_f & \left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 \right. \\ & \left. + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right] \\ + \; n_f^2 & \left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right] \\ + \; n_f^3 & \left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\right] \\ + \; n_f^4 & \left[\frac{1205}{2916} - \frac{152}{81}\zeta_3\right]\Bigg\},\end{aligned} \tag{5}$$

where n_f denotes the number of active quark flavors. As expected from the three and four-loop results, the higher transcendentalities ζ_6 and ζ_7 that could be present at five-loop order [42], are actually absent. Note that the contribution in β_4 that is leading in n_f (proportional to n_f^4) was computed long ago with a very different technique [14] for a generic gauge group. For the physical case of $SU(3)$ we find full agreement.

In numerical form the coefficients $\beta_0 - \beta_4$ read

$$\begin{aligned}
\beta_0 &\approx 2.75 - 0.166667\, n_f, \\
\beta_1 &\approx 6.375 - 0.791667\, n_f, \\
\beta_2 &\approx 22.3203 - 4.36892\, n_f + 0.0940394\, n_f^2, \\
\beta_3 &\approx 114.23 - 27.1339\, n_f \\
&\qquad + 1.58238\, n_f^2 + 0.0058567\, n_f^3, \\
\beta_4 &\approx 524.56 - 181.8\, n_f + 17.16\, n_f^2 \\
&\qquad - 0.22586\, n_f^3 - 0.0017993\, n_f^4. \qquad (6)
\end{aligned}$$

Numerically the coefficients are surprisingly small. For example, for the particular cases of $n_f = 3, 4, 5$ and 6 we get:

$$\begin{aligned}
\overline{\beta}(n_f = 3) &= 1 + 1.78\, a_s + 4.47\, a_s^2 + 20.99\, a_s^3 + 56.59\, a_s^4, \\
\overline{\beta}(n_f = 4) &= 1 + 1.54\, a_s + 3.05\, a_s^2 + 15.07\, a_s^3 + 27.33\, a_s^4, \\
\overline{\beta}(n_f = 5) &= 1 + 1.26\, a_s + 1.47\, a_s^2 + 9.83\, a_s^3 + 7.88\, a_s^4, \\
\overline{\beta}(n_f = 6) &= 1 + 0.93\, a_s - 0.29\, a_s^2 + 5.52\, a_s^3 + 0.15\, a_s^4,
\end{aligned}$$

where $\overline{\beta} \equiv \frac{\beta(a_s)}{-\beta_0 a_s^2} = 1 + \sum_{i \geq 1} \bar{\beta}_i a_s^i$. A very modest growth of the coefficients is observed and the (apparent) convergence is better than one would expect from comparison with other examples.

It is instructive to compare β_4 as shown in eq. (6) with a (20 years old!) prediction based on the so-called method of the Asymptotic Pade Approximant (APAP) from [15] (the boxed term was used as input):

$$\beta_4^{APAP} = 740 - 213\, n_f + 20\, n_f^2 - 0.0486\, n_f^3 - \boxed{0.0017993\, n_f^4}.$$

Unfortunately, this strikingly good agreement for all powers of n_f except for n_f^3 term does not always survive for fixed values of n_f due to huge cancellations between contributions proportional to different powers of n_f (see Table I below).

TABLE I: Comparison of the exact results for β_4 with the predictions based on APAP for different values of n_f.

n_f	0	1	2	3	4	5	6
β_4^{exact}	525	360	228	127	57	15	0.27
β_4^{APAP}	741	548	395	281	205	169	170

At this point it may be useful to present the impact of the five-loop term on the running of the strong coupling from low energies, say $\mu = M_\tau$, up to the high energy region $\mu = M_H$, by comparing the predictions based on three and four versus five-loop results [43]. We start from the scale of M_τ with $\alpha_s^{(3)}(M_\tau) = 0.33$ (as given in [16]) and evolve the coupling up to 3 GeV. At this point the four-loop matching from 3 to 4 flavours is performed. The strong coupling now runs up to $\mu = 10$ GeV and, at this point, the number of active quark flavours is switched from the 4 to 5. Subsequently, the strong coupling runs again up to M_Z and, finally, up to the Higgs mass $M_H = 125$ GeV. The relevant values of α_s are listed in Table II. The combined uncertainty in $\alpha_s^{(5)}(M_Z)$ induced by running and matching can be conservatively estimated by the shift in $\alpha_s^{(5)}(M_Z)$ produced by the use of five-loop running (and, consequently) four-loop matching instead of four-loop running (and three-loop matching). It amounts to a minute $8 \cdot 10^{-5}$ which is by a factor of three less than the similar shift made by the use of four-loop running instead of the the three-loop one (see Table II). Note that the final value of $\alpha_s^{(5)}(M_Z)$ which follows from $\alpha_s^{(3)}(M_\tau)$ is in remarkably good agreement with the fit to electroweak precision data (collected in Z boson decays), namely ([16]):

$$\alpha_s^{(5)}(M_Z) = 0.1197 \pm 0.0028. \qquad (7)$$

TABLE II: Running of α_s from $\mu = M_\tau$ to $\mu = M_H$. For the threshold values of c and b heavy quarks we have chosen [17, 18] $m_c(3\ \text{GeV}) = 0.986$ GeV and $m_b(10\ \text{GeV}) = 3.160$ GeV respectively.

# of loops	$\alpha_s^{(3)}(M_\tau)$	$\alpha_s^{(5)}(M_Z)$	$\alpha_s^{(5)}(M_H)$
3	0.33 ± 0.014	0.1195 ± 0.0015	0.1140 ± 0.0015
4	0.33 ± 0.014	0.1197 ± 0.0015	0.1142 ± 0.0015
5	0.33 ± 0.014	0.1198 ± 0.0015	0.1143 ± 0.0015

As anticipated in [13], the running of m_b from low energies, say 10 GeV, is affected by the five-loop term, which in turn, slightly modifies the Higgs boson decay rate into a quark pair. This rate is given by

$$\Gamma(H \to f\bar{f}) = \frac{G_F M_H}{4\sqrt{2}\pi} m_f^2(\mu) R^S(s = m_H^2, \mu), \qquad (8)$$

where μ is the normalization scale and R^S the spectral density of the scalar correlator, known to α_s^4 from [19]

$$\begin{aligned}
&R^S(s = M_H^2, \mu = M_H) \\
&= 1 + 5.667\, a_s + 29.147\, a_s^2 + 41.758\, a_s^3 - 825.7\, a_s^4, \\
&= 1 + 0.2062 + 0.0386 + 0.0020 - 0.00145, \qquad (9)
\end{aligned}$$

where we set $a_s(M_H) = \alpha_s(M_H)/\pi = 0.1143/\pi = 0.0364$ and R^S is evaluated for the Higgs mass value

$M_H = 125\,\mathrm{GeV}$. For the running of the b quark mass the corresponding input is taken from a relatively low scale and has to be evolved up to M_H. The shift from the five-loop term is then given by

$$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1 \cdot 10^{-4} \tag{10}$$

which at present and in the foreseeable future is negligible. We want to stress here that the effect due to the $\mathcal{O}(\alpha_s^4)$ term in (9) are formally of the same order as the one induced by the five-loop running of m_b.

Another application of our result for the β-function is the determination of the effective Higgs-gluon-gluon coupling. In the heavy top limit the Higgs boson couples directly with gluons via the effective Lagrangian of the form [20–22]

$$\mathcal{L}_{eff} = -2^{1/4} G_F^{1/2} H C_1(\mu^2/m_t^2, a_s(\mu))\, G^a_{\nu\rho} G^a_{\nu\rho}. \tag{11}$$

The effective coupling constant $C_1(\mu^2/m_t^2, a_s(\mu))$ appears as a common factor in two quantities important for Higgs physics processes, namely, Higgs decay into gluons (one of the main decay channels for the Standard Model Higgs boson) and Higgs production via the gluon fusion (the main Higgs production mode on LHC). It is expressible through massive tadpoles and was computed at four loops in 1997 [23] (long before the direct calculation of four-loop generic massive tadpoles started to be technically feasible). This happened to be possible due to a low energy theorem (exact in all orders) [23]

$$C_1 = -\frac{1}{2} m_t^2 \frac{\partial}{\partial m_t^2} \ln \zeta_g^2, \quad \alpha_s'(\mu) = \zeta_g^2(\mu^2/m_t^2, \alpha_s(\mu))\, \alpha_s(\mu) \tag{12}$$

which connects C_1 with the corresponding "decoupling" constant ζ_g for α_s. The appearance of the derivative $\frac{\partial}{\partial m_t^2}$ means that the most complicated (that is constant) part of ζ_g^2 does not contribute to C_1, so that one could use the corresponding RG equation to find logs at next loop order (*provided we know the β-function at the same increased loop order!*).

Since the decoupling constant is known at four loops from [24, 25] we can now use (12) and (5) to extend the known four-loop result to one more loop:

$$C_1 = -\frac{1}{12} a_s \Big(1 + 2.750\, a_s + 6.306\, a_s^2 + 9.208\, a_s^3 + 101.485\, a_s^4\Big). \tag{13}$$

In this expression $a_s = \alpha_s^{(6)}(\mu_t)/\pi$, with μ_t being a scale-invariant top quark mass defined as $\mu_t = m_t(\mu_t)$. Note that the contribution due to β_4 to the last coefficient (boxed below) is significant, namely,

$$101.49 = 12.427 + \boxed{89.058}.$$

As another application let us mention the connection with the renormalization group invariant (RGI) mass:

$$m^{RGI} \equiv m(\mu_0)/c(a_s(\mu_0)), \tag{14}$$

with

$$\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}, \quad c(x) = \exp\left\{\int dx' \frac{\gamma_m(x')}{\beta(x')}\right\}, \tag{15}$$

which could be determined in lattice calculations. The function $c(x)$ does depend not only on the quark mass anomalous dimension γ_m (known from [13, 26]) but also on the β-function. In the five-loop approximation we get (for a typical for lattice simulations value of $n_f = 3$)

$$c(x) \underset{n_f=3}{=\!=} x^{4/9}\Big(1 + 0.8950\, x + 1.3714\, x^2 + 1.9517\, x^3 + (15.6982 - 0.11111\, \bar{\beta}_4 = 9.411)\, x^4\Big), \tag{16}$$

with $\bar{\beta}_4 = \beta_4/\beta_0 = 56.59$.

The precise knowledge of the function $c(x)$ (which is a scheme dependent quantity) is required in order to find the mass of the strange quark in a well-defined renormalization scheme (usually the $\overline{\mathrm{MS}}$-one) from m_s^{RGI} measured with lattice simulations at very high energies around 100 GeV [27]. With a typical value of $\alpha_s(2\,\mathrm{GeV})/\pi = 0.1$ we find that the series (16) shows quite good convergence. In contrast, a value of β_4 as large as -2000 as estimated in [28] would lead to a significantly less stable series.

Summary: The exact result for the five-loop term of the QCD β-function allows to relate the strong coupling constant α_s, as determined with NLO^3 accuracy at low energies, say M_τ with the strong coupling as evaluated at high scales, say M_Z or M_H. Including the exact five-loop term has little influence on the central value of the prediction, a consequence of partial cancellations between various contributions from matching and running. However, the five-loop result leads to a considerable further reduction of the theory uncertainty and allows to combine values from low and high energies of appropriate order. It also should be useful in the elimination of the renormalization scheme and scale ambiguities in perturbative QCD within the framework of The Principle of Maximum Conformality and Commensurate Scale Relations [29] or, closely related, the sequential extended BLM approach [30, 31].

We want now to add here some technical details about our calculation. To evaluate the β-function we need to evaluate the following three renormalization constants (RC's) in five-loop order: Z_1^{ccg} for the ghost-ghost-gluon vertex, Z_3^c for the inverted ghost propagator and Z_3 for the inverted gluon propagator. The total number of five-loop diagrams contributing to the RC's (as generated by

QGRAF [32]) amounts to about one and a half million ($1.5 \cdot 10^6$), with the gluon wave-function Z_3 (around $3 \cdot 10^5$ diagrams) being most complicated one. Every power of n_f in (5) was computed separately with the help of the FORM [33, 34] program BAICER, implementing the algorithm of works [35–37].

With a typical set-up of 15-20 workstations (with 8 cores each) running a thread-based version of FORM [38] the calculation of two first subproblems (n_f^4 and n_f^3) took together about a couple of weeks, while the remaining three most complicated pieces (proportional to n_f^2, n_f^1 and n_f^0 correspondingly) required up to 7 months of running time for every particular n_f-slice.

The continued running of our calculations at such computer and time scales would be virtually impossible without the effective support of our computer administration, in particular, Alexander Hasselhuhn, Jens Hoff, David Kunz, Peter Marquard and Matthias Steinhauser, to whom all we express our sincere thanks.

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After our calculations had been finished we have been informed that the subleading in n_f term in the coefficient β_4 (proportional to n_f^3 in eq. (5)) has been confirmed and even extended for the case of a general gauge group in [39]. The authors have used a radically different method which expresses the β-function in terms of completely massive vacuum diagrams.

The work is dedicated to the memory of one of the founders of the renormalization group method—Dmitry Vasil'evich Shirkov, 1928-2016.

[1] D. J. Gross and F. Wilczek, Phys.Rev.Lett. **30**, 1343 (1973).
[2] H. D. Politzer, Phys.Rev.Lett. **30**, 1346 (1973).
[3] W. E. Caswell, Phys.Rev.Lett. **33**, 244 (1974).
[4] D. Jones, Nucl.Phys. **B75**, 531 (1974).
[5] O. Tarasov, A. Vladimirov, and A. Y. Zharkov, Phys.Lett. **B93**, 429 (1980).
[6] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. **B303**, 334 (1993), hep-ph/9302208.
[7] T. van Ritbergen, J. Vermaseren, and S. Larin, Phys.Lett. **B400**, 379 (1997), hep-ph/9701390.
[8] M. Czakon, Nucl.Phys. **B710**, 485 (2005), hep-ph/0411261.
[9] E. Stueckelberg and A. Petermann, Helv. Phys. Acta. **26**, 499 (1953).
[10] M. Gell-Mann and F. Low, Phys.Rev. **95**, 1300 (1954).
[11] N. Bogolyubov and D. Shirkov, Nuovo Cim. **3**, 845 (1956).
[12] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **101**, 012002 (2008), 0801.1821.
[13] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, JHEP **10**, 76 (2014), 1402.6611.
[14] J. Gracey, Phys.Lett. **B373**, 178 (1996), hep-ph/9602214.
[15] J. R. Ellis, I. Jack, D. Jones, M. Karliner, and M. Samuel, Phys.Rev. **D57**, 2665 (1998), hep-ph/9710302.
[16] K. Olive et al. (Particle Data Group), Chin.Phys. **C38**, 090001 (2014).
[17] K. Chetyrkin, J. Kuhn, A. Maier, P. Maierhofer, P. Marquard, et al., Phys.Rev. **D80**, 074010 (2009), 0907.2110.
[18] K. G. Chetyrkin, J. H. Kühn, M. Steinhauser, and C. Sturm, Nucl. Part. Phys. Proc. **261-262**, 19 (2015), 1502.00509.
[19] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **96**, 012003 (2006), hep-ph/0511063.
[20] F. Wilczek, Phys. Rev. Lett. **39**, 1304 (1977).
[21] M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, and V. I. Zakharov, Sov. J. Nucl. Phys. **30**, 711 (1979), [Yad. Fiz.30,1368(1979)].
[22] T. Inami, T. Kubota, and Y. Okada, Z. Phys. **C18**, 69 (1983).
[23] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, Nucl. Phys. **B510**, 61 (1998), hep-ph/9708255.
[24] K. G. Chetyrkin, J. H. Kühn, and C. Sturm, Nucl. Phys. **B744**, 121 (2006), hep-ph/0512060.
[25] Y. Schroder and M. Steinhauser, JHEP **01**, 051 (2006), hep-ph/0512058.
[26] J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, Phys. Lett. **B405**, 327 (1997), hep-ph/9703284.
[27] R. Sommer and U. Wolff, Nucl. Part. Phys. Proc. **261-262**, 155 (2015), 1501.01861.
[28] V. Elias, T. G. Steele, F. Chishtie, R. Migneron, and K. B. Sprague, Phys. Rev. **D58**, 116007 (1998), hep-ph/9806324.
[29] S. J. Brodsky, M. Mojaza, and X.-G. Wu, Phys. Rev. **D89**, 014027 (2014), 1304.4631.
[30] S. V. Mikhailov, JHEP **06**, 009 (2007), hep-ph/0411397.
[31] A. L. Kataev, J. Phys. Conf. Ser. **608**, 012078 (2015), 1411.2257.
[32] P. Nogueira, J. Comput. Phys. **105**, 279 (1993).
[33] J. A. M. Vermaseren (2000), math-ph/0010025.
[34] M. Steinhauser, T. Ueda, and J. A. M. Vermaseren, Nucl. Part. Phys. Proc. **261-262**, 45 (2015), 1501.07119.
[35] P. A. Baikov, Phys. Lett. **B634**, 325 (2006), hep-ph/0507053.
[36] P. A. Baikov, Phys. Lett. **B385**, 404 (1996), hep-ph/9603267.
[37] P. A. Baikov, PoS **RADCOR2007**, 022 (2007).
[38] M. Tentyukov and J. A. M. Vermaseren (2007), hep-ph/0702279.
[39] Th. Luthe, A. Maier, P. Marquard, and Y. Schröder, in preparation (2016).
[40] P. A. Baikov and K. G. Chetyrkin, Nucl. Phys. **B837**, 186 (2010), 1004.1153.
[41] K. G. Chetyrkin, J. H. Kühn, and M. Steinhauser, Comput. Phys. Commun. **133**, 43 (2000), hep-ph/0004189.
[42] For a general analysis of the issue, see [40].
[43] For all practical examples in this paper we have used an extended version of the package RunDec [41].