Five-Loop Running of the QCD coupling constant

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We analytically compute the five-loop term in the beta function which governs the running of \( \alpha_s \) — the quark-gluon coupling constant in QCD. The new term leads to a reduction of the theory uncertainty in \( \alpha_s \) taken at the Z-boson scale as extracted from the \( \tau \)-lepton decays as well as to new, improved by one more order of perturbation theory, predictions for the effective coupling constants of the Standard Model Higgs boson to gluons and for its total decay rate to the quark-antiquark pairs.

Asymptotic freedom, manifest by a decreasing coupling with increasing energy, can be considered as the basic prediction of nonabelian gauge theories and was crucial for establishing Quantum Chromodynamics (QCD) as the theory of strong interactions \([1, 2]\). The dominant, leading order prediction was quickly followed by the corresponding two-loop \([3, 4]\) and three-loop \([5, 6]\) results. The next, four-loop calculation was performed almost twenty years later \([7]\) and confirmed in \([8]\). These results have moved the theory from qualitative agreement with experiment, as observed on the basis of the early results, to precise quantitative predictions, valid over a wide kinematic range, from \( \tau \)-lepton decays up to LHC results.

Although the agreement between theory predictions and experimental results is impressive already now, it is tempting to push the theory prediction as high as possible. On the other hand one may expect an even better agreement between theory and experiment. On the other hand it is of theoretical interest to push gradually into the region where individual terms of the series might start to increase, thus demonstrating the asymptotic divergence of the perturbative series. At a more modest level we note that predictions for the five-loop term that can be found in the literature are based on a variety of methods and exhibit for some cases quite a dramatic variation of the size of the term (we will give more details later).

Let us start with the definition of the beta function

\[
\beta(\alpha_s) = \mu^2 \frac{d}{d\mu^2} \alpha_s(\mu) = -\sum_{i \geq 0} \beta_i \alpha_s^{i+2}
\]

which describes the running of the quark-gluon coupling \( \alpha_s = \alpha_s(\mu) / \pi \) as a function of the normalization scale \( \mu \) within the renormalization group approach \([9, 11]\).

Using the same theoretical tools as in the calculations of \([12]\) and \([13]\) we have computed the QCD \( \beta \)-function in five-loop order with the result

\[
\beta_0 = \frac{1}{4} \left\{ 11 - 2 \frac{2}{3} n_f \right\}, \quad \beta_1 = \frac{1}{4^2} \left\{ 102 - \frac{38}{3} n_f \right\},
\]

\[
\beta_2 = \frac{1}{4^3} \left\{ \frac{2857}{2} - \frac{5033}{18} \mu_f + \frac{325}{54} n_f^2 \right\},
\]

\[
\beta_3 = \frac{1}{4^4} \left\{ \frac{149753}{6} + \frac{564}{3} \zeta_1 - \left[ \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right] n_f \right. + \left[ \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right] n_f^2 + \frac{1093}{729} n_f^3 \right\},
\]

\[
\beta_4 = \frac{1}{4^5} \left\{ \frac{8157455}{16} + \frac{621885}{2} \zeta_3 - \frac{88209}{2} \zeta_4 - 288090 \zeta_5 \right. + \left. \frac{336460813}{1944} - \frac{4811164}{81} \zeta_3 \right. + \left. 33935 \frac{1358995}{6} \zeta_4 + \frac{1358995}{27} \zeta_5 \right\},
\]

\[
+ \left[ \frac{25960913}{1944} + \frac{698531}{81} \zeta_3 - \frac{10526}{9} \zeta_4 - \frac{381760}{81} \zeta_5 \right] n_f^2 + \frac{60359}{5832} \zeta_3 + \frac{1618}{27} \zeta_4 + \frac{460}{9} \zeta_5 \right\},
\]

where \( n_f \) denotes the number of active quark flavors. As expected from the three and four-loop results, the higher

*In memoriam Dmitry Vasil’evich Shirkov, 1928-2016*
transcendentalities $\zeta_6$ and $\zeta_7$ that could be present at five-loop order \cite{42} are actually absent. Note that the contribution in $\beta_4$ that is leading in $n_f$ (proportional to $n_f^4$) was computed long ago with a very different technique \cite{14} for a generic gauge group. For the physical case of $SU(3)$ we find full agreement.

In numerical form the coefficients $\beta_0 - \beta_4$ read

$\beta_0 \approx 2.75 - 0.16667 n_f$, \\
$\beta_1 \approx 6.375 - 0.79167 n_f$, \\
$\beta_2 \approx 22.3203 - 4.36892 n_f + 0.0940394 n_f^2$, \\
$\beta_3 \approx 114.23 - 27.1339 n_f + 1.58238 n_f^2 + 0.0058567 n_f^3$, \\
$\beta_4 \approx 524.56 - 181.8 n_f + 17.16 n_f^2 - 0.22586 n_f^3 - 0.0017993 n_f^4$. (6)

Numerically the coefficients are surprisingly small. For example, for the particular cases of $n_f = 3, 4, 5$ and 6 we get:

$\overline{\beta}(n_f = 3) = 1 + 1.78 a_s + 4.47 a_s^2 + 20.99 a_s^3 + 56.59 a_s^4$, \\
$\overline{\beta}(n_f = 4) = 1 + 1.54 a_s + 3.05 a_s^2 + 15.07 a_s^3 + 27.33 a_s^4$, \\
$\overline{\beta}(n_f = 5) = 1 + 1.26 a_s + 1.47 a_s^2 + 9.83 a_s^3 + 7.88 a_s^4$, \\
$\overline{\beta}(n_f = 6) = 1 + 0.93 a_s - 0.29 a_s^2 + 5.52 a_s^3 + 0.15 a_s^4$,

where $\overline{\beta} \equiv \beta(a_s) = 1 + \sum_{i \geq 1} \beta_i a_s^i$. A very modest growth of the coefficients is observed and the (apparent) convergence is better than one would expect from comparison with other examples.

It is instructive to compare $\beta_4$ as shown in eq. (5) with a (20 years old!) prediction based on the so-called method of the Asymptotic Padé Approximant (APAP) from \cite{13} (the boxed term was used as input):

$\beta_4^{APAP} = 740 - 213 n_f + 20 n_f^2 - 0.486 n_f^3 - 0.0017993 n_f^4$

Unfortunately, this strikingly good agreement for all powers of $n_f$ except for $n_f^4$ term does not always survive for fixed values of $n_f$ due to huge cancellations between contributions proportional to different powers of $n_f$ (see Table I below).

<table>
<thead>
<tr>
<th>$n_f$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_4^{exact}$</td>
<td>525</td>
<td>360</td>
<td>228</td>
<td>127</td>
<td>57</td>
<td>15</td>
<td>0.27</td>
</tr>
<tr>
<td>$\beta_4^{APAP}$</td>
<td>740</td>
<td>548</td>
<td>395</td>
<td>281</td>
<td>205</td>
<td>169</td>
<td>170</td>
</tr>
</tbody>
</table>

At this point it may be useful to present the impact of the five-loop term on the running of the strong coupling from low energies, say $\mu = M_F$, up to the high energy region $\mu = M_H$, by comparing the predictions based on three and four versus five-loop results \cite{43}. We start from the scale of $M_F$ with $\alpha_s^{(5)}(M_F) = 0.33$ (as given in \cite{14}) and evolve the coupling up to 3 GeV. At this point the four-loop matching from 3 to 4 flavours is performed. The strong coupling now runs up to $\mu = 10$ GeV and, at this point, the number of active quark flavours is switched from the 4 to 5. Subsequently, the strong coupling runs again up to $M_Z$ and, finally, up to the Higgs mass $M_H = 125$ GeV. The relevant values of $\alpha_s$ are listed in Table II. The combined uncertainty in $\alpha_s^{(5)}(M_Z)$ induced by running and matching can be conservatively estimated by the shift in $\alpha_s^{(5)}(M_Z)$ produced by the use of five-loop running (and, consequently) four-loop matching instead of four-loop running (and three-loop matching). It amounts to a minute $8 \times 10^{-5}$ which is by a factor of three less than the similar shift made by the use of four-loop running instead of the the three-loop one (see Table II).

Note that the final value of $\alpha_s^{(5)}(M_Z)$ which follows from $\alpha_s^{(3)}(M_F)$ is in remarkably good agreement with the fit to electroweak precision data (collected in Z boson decays), namely (16):

$\alpha_s^{(5)}(M_Z) = 0.1197 \pm 0.0028$. (7)

<table>
<thead>
<tr>
<th># of loops</th>
<th>$\alpha_s^{(3)}(M_F)$</th>
<th>$\alpha_s^{(3)}(M_Z)$</th>
<th>$\alpha_s^{(5)}(M_H)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.33 ± 0.014</td>
<td>0.1195 ± 0.0015</td>
<td>0.1140 ± 0.0015</td>
</tr>
<tr>
<td>4</td>
<td>0.33 ± 0.014</td>
<td>0.1197 ± 0.0015</td>
<td>0.1142 ± 0.0015</td>
</tr>
<tr>
<td>5</td>
<td>0.33 ± 0.014</td>
<td>0.1198 ± 0.0015</td>
<td>0.1143 ± 0.0015</td>
</tr>
</tbody>
</table>

As anticipated in \cite{13}, the running of $m_b$ from low energies, say 10 GeV, is affected by the five-loop term, which in turn, slightly modifies the Higgs boson decay rate into a quark pair. This rate is given by

$\Gamma(H \to f \bar{f}) = \frac{G_F M_H^2 m_f^2(\mu)}{4\sqrt{2}\pi} R_S(s = m^2_H, \mu), \quad (8)$

where $\mu$ is the normalization scale and $R_S$ the spectral density of the scalar correlator, known to $\alpha_s^4$ from \cite{13}

$R_S(s = M^2_H, \mu = M_H) = 1 + 5.667 a_s + 29.147 a_s^2 + 41.758 a_s^3 - 825.7 a_s^4$, \\
$= 1 + 0.2062 + 0.0386 + 0.0020 - 0.0015$. (9)

where we set $a_s(M_H) = \alpha_s(M_H)/\pi = 0.1143/\pi = 0.0364$ and $R_S$ is evaluated for the Higgs mass value.
$M_H = 125 \text{ GeV}$. For the running of the $b$ quark mass the corresponding input is taken from a relatively low scale and has to be evolved up to $M_H$. The shift from the five-loop term is then given by

$$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1 \cdot 10^{-4} \quad (10)$$

which at present and in the foreseeable future is negligible. We want to stress here that the effect due to the $O(\alpha_s^5)$ term in \[9\] are formally of the same order as the one induced by the five-loop running of $m_b$.

Another application of our result for the $\beta$-function is the determination of the effective Higgs-gluon-gluon coupling. In the heavy top limit the Higgs boson couples directly with gluons via the effective Lagrangian of the form \[20\]–\[22\]

$$L_{eff} = -2^{1/4} G_F^{1/2} H C_1 (\mu^2/m_t^2, \alpha_s(\mu)) G_{\nu\rho} G^{\nu\rho}. \quad (11)$$

The effective coupling constant $C_1 (\mu^2/m_t^2, \alpha_s(\mu))$ appears as a common factor in two quantities important for Higgs physics processes, namely, Higgs decay into gluons (one of the main decay channels for the Standard Model Higgs boson) and Higgs production via the gluon fusion (the main Higgs production mode on LHC). It is expressible through massive tadpoles and was computed at four loops in 1997 \[23\] (long before the direct calculation of four-loop generic massive tadpoles started to be technically feasible). This happened to be possible due to a low energy theorem (exact in all orders) \[23\]

$$C_1 = -\frac{1}{2} m_t^2 \frac{\partial}{\partial m_t^2} \ln \frac{\alpha_s'}{\alpha_s} = \frac{3}{2} \alpha_s^2 (\mu^2/m_t^2, \alpha_s(\mu)) \alpha_s(\mu) \quad (12)$$

which connects $C_1$ with the corresponding “decoupling” constant $\zeta_2$ for $\alpha_s$. The appearance of the derivative $\frac{\partial}{\partial m_t^2}$ means that the most complicated (that is constant) part of $c_g^2$ does not contribute to $C_1$, so that one could use the corresponding RG equation to find logs at next loop order (provided we know the $\beta$-function at the same increased loop order!)

Since the decoupling constant is known at four loops from \[24\]–\[25\] we can now use (12) and (5) to extend the known four-loop result to one more loop:

$$C_1 = -\frac{1}{12} \alpha_s \left( 1 + 2.750 \alpha_s + 6.306 \alpha_s^2 + 9.208 \alpha_s^3 + 101.485 \alpha_s^4 \right) \quad (13)$$

In this expression $\alpha_s = \alpha_s^{(6)}(\mu_t)/\pi$, with $\mu_t$ being a scale-invariant top quark mass defined as $\mu_t = m_t(\mu_t)$. Note that the contribution due to $\beta_4$ to the last coefficient (boxed below) is significant, namely,

$$101.49 = 12.427 + 89.058$$

As another application let us mention the connection with the renormalization group invariant (RGI) mass:

$$m^{RGI} \equiv m(\mu_0)/c(\alpha_s(\mu_0)), \quad (14)$$

with

$$\frac{m(\mu)}{m(\mu_0)} = \frac{c(\alpha_s(\mu))}{c(\alpha_s(\mu_0))}, \quad c(x) = \exp \left\{ \int^x dx \frac{\gamma_m(x')}{\beta(x')} \right\}, \quad (15)$$

which could be determined in lattice calculations. The function $c(x)$ does depend not only on the quark mass anomalous dimension $\gamma_m$ (known from \[13\]–\[26\]) but also on the $\beta$-function. In the five-loop approximation we get (for a typical for lattice simulations value of $n_f = 3$)

$$c(x) \equiv x^{4/9} \left( 1 + 0.8950 x + 1.3714 x^2 + 1.9517 x^3 \right) + (15.6982 - 0.11111 \beta_4) x^4, \quad (16)$$

with $\beta_4 = \beta_4/\beta_0 = 56.59$.

The precise knowledge of the function $c(x)$ (which is a scheme dependent quantity) is required in order to find the mass of the strange quark in a well-defined renormalization scheme (usually the $\overline{\text{MS}}$-one) from $m_s^{RGI}$ measured with lattice simulations at very high energies around 100 GeV \[27\]. With a typical value of $\alpha_s(2 \text{ GeV})/\pi = 0.1$ we find that the series \[16\] shows quite good convergence. In contrast, a value of $\beta_4$ as large as $-2000$ as estimated in \[28\] would lead to a significantly less stable series.

Summary: The exact result for the five-loop term of the QCD $\beta$-function allows to relate the strong coupling constant $\alpha_s$, as determined with NLO\(^3\) accuracy at low energies, say $M_t$, with the strong coupling as evaluated at high scales, say $M_Z$ or $M_H$. Including the exact five-loop term has little influence on the central value of the prediction, a consequence of partial cancellations between various contributions from matching and running. However, the five-loop result leads to a considerable further reduction of the theory uncertainty and allows to combine values from low and high energies of appropriate order. It also should be useful in the elimination of the renormalization scheme and scale ambiguities in perturbative QCD within the framework of The Principle of Maximum Conformality and Commensurate Scale Relations \[29\] or, closely related, the sequential extended BLM approach \[30\]–\[31\].

We want now to add some technical details about our calculation. To evaluate the $\beta$-function we need to evaluate the following three renormalization constants (RC’s) in five-loop order: $Z_1^{ccg}$ for the ghost-ghost-gluon vertex, $Z_3^c$ for the inverted ghost propagator and $Z_3$ for the inverted gluon propagator. The total number of five-loop diagrams contributing to the RC’s (as generated by
QGRAF \cite{32} amounts to about one and a half million (1.5 \cdot 10^6), with the gluon wave-function $Z_3$ (around 3 \cdot 10^5 diagrams) being most complicated one. Every power of $n_f$ in \cite{42} was computed separately with the help of the FORM \cite{33, 34} program BAICER, implementing the algorithm of works \cite{35, 37}.

With a typical set-up of 15-20 workstations (with 8 cores each) running a thread-based version of FORM \cite{38} the calculation of two first subproblems ($n_f^3$ and $n_f^2$) took together about a couple of weeks, while the remaining three most complicated pieces (proportional to $n_f^2$, $n_f^1$ and $n_f^0$ correspondingly) required up to 7 months of running time for every particular $n_f$-slice.

The continued running of our calculations at such computer and time scales would be virtually impossible without the effective support of our computer administration, in particular, Alexander Hasselluhn, Jens Hoff, David Kunz, Peter Marquard and Matthias Steinhauser, to whom all we express our sincere thanks.

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After our calculations had been finished we have been informed that the subleading in $n_f$ term in the coefficient $\beta_1$ (proportional to $n_f^3$ in eq. \cite{42}) has been confirmed and even extended for the case of a general gauge group in \cite{39}. The authors have used a radically different method which expresses the $\beta$-function in terms of completely massive vacuum diagrams.

The work is dedicated to the memory of one of the founders of the renormalization group method—Dmitry Vasil’evich Shirkov, 1928-2016.

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[42] For a general analysis of the issue, see \cite{40}.
[43] For all practical examples in this paper we have used an extended version of the package RunDec \cite{41}.