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Electron contribution to $(g-2)_\mu$ at four loops

Alexander Kurz
Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), 76128 Karlsruhe, Germany
E-mail: alexander.kurz2@kit.edu

Tao Liu
Department of Physics, University of Alberta, Edmonton AB T6G 2J1, Canada
E-mail: ltao@ualberta.ca

Peter Marquard
Deutsches Elektronen Synchrotron DESY, Platanenallee 6, 15738 Zeuthen, Germany
E-mail: peter.marquard@desy.de

Alexander V. Smirnov
Scientific Research Computing Center, Moscow State University, 119991, Moscow, Russia
E-mail: asmirnov80@gmail.com

Vladimir A. Smirnov
Skobeltsyn Institute of Nuclear Physics of Moscow State University, 119991, Moscow, Russia
E-mail: smirnov@theory.sinp.msu.ru

Matthias Steinhauser*
Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), 76128 Karlsruhe, Germany
E-mail: matthias.steinhauser@kit.edu

In this contribution we summarize the recent calculation of the complete electron contribution to the anomalous magnetic moment of the muon at four-loop order.

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1. Introduction

The anomalous magnetic moment of the muon is among the most precisely known quantities in particle physics. At the same time there is a long-standing deviation between the experimental measurement and the theory prediction which amounts to about three standard deviations. On the experimental side there are upcoming new experiments which either use the same method as in the E821 experiment at BNL [1, 2] but reduce the uncertainties by about a factor four, or even use a completely different technique which would eliminate doubts on possible systematic effects (for details see, e.g., Ref. [3]).

On the theory side it is certainly necessary to improve on the hadronic contributions, both from the vacuum polarization and from light-by-light-type diagrams. Furthermore, it is mandatory to cross check the four-loop QED contribution since the full result has only been obtained by one group [4, 5, 6]. In a series of works [7, 8, 9] the fermionic pieces have been confirmed. The cross check of the purely photonic part is still missing. In this contribution we report on the calculation and results of the diagrams involving closed electron loops [9].

2. The method

It is convenient to sub-divide the contributing four-loop diagrams in twelve classes [6] which are introduced in Fig. 1 with the help of sample Feyman diagrams.

The approach used for the computation of the four-loop diagrams differs from the one applied in Ref. [6] in many ways. In [9] we generate in a first step amplitudes for each individual vertex

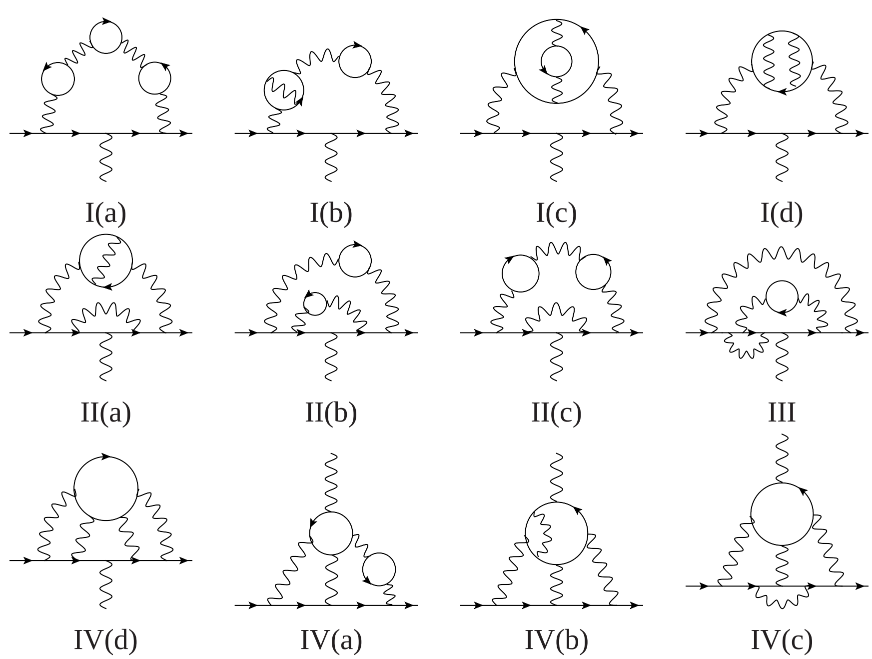

Figure 1: Four-loop diagram classes for a_μ containing at least one closed electron loop. The external solid lines represent muons, the solid loops denote electrons, muons or taus, and the wavy lines represent photons.

diagram which contain both the electron (m_e) and the muon mass (m_μ). At this point we exploit the fact that $m_e \ll m_\mu$ and apply an asymptotic expansion which expresses each amplitude into a sum of so-called sub-diagrams. Each sub-diagram is written as a product of one-scale integrals which are much easier to compute.

The various sub-diagrams involve different types of integrals which have to be treated separately. Most of them are well studied in the literature and analytic results can be obtained. However, there are two types where this is not the case: four-loop on-shell integrals and four-loop integrals involving propagators of the form $1/(2\ell \cdot q)$ where q is the external momentum with $q^2 = m_\mu^2$, in the following called "linear integrals".

At this point we apply an appropriate projector to the magnetic form factor and expand afterwards in the photon momentum to obtain the static limit. Then we perform the traces and decompose each amplitude into a sum of scalar integrals. For simple integral types (like two-loop vacuum integrals) we can directly insert the analytic results for the integrals. The more complicated ones are reduced to so-called master integrals using the program packages `FIRE` [10] and `crusher` [11]. In this way we obtain an analytic result for the muon anomalous magnetic moment in term of a relatively small number [$\mathcal{O}(100)$] of master integrals. This is the case for all coefficients of $(m_e/m_\mu)^n$ (we expanded up to $n = 3$). Note that as far as the four-loop master integrals are concerned the odd powers of m_e/m_μ only involve linear integrals while the even powers get contributions from on-shell and linear integrals.

It is only at this point when we pass on to numerical methods since to date not all master integrals are available in analytic form. This is the origin of the uncertainty in our final results, see below.

The numerical evaluation of the four-loop on-shell master integrals is described in detail in Ref.[12]. A similar approach has also been used for the linear integrals, see also Ref. [9].

3. Results for $(g-2)_\mu$

In this section we present results for the anomalous magnetic moment of the muon. We cast the perturbative expansion in the form

$$\frac{(g-2)_\mu}{2} = a_\mu = \sum_{n=1}^{\infty} a_\mu^{(2n)} \left(\frac{\alpha}{\pi}\right)^n , \tag{3.1}$$

where n counts the number of loops. $a_\mu^{(2n)}$ is conveniently split into several pieces according to the particles present in the loop. In particular, we have for the four-loop term

$$a_\mu^{(8)} = A_1^{(8)} + A_2^{(8)}(m_\mu/m_e) + A_2^{(8)}(m_\mu/m_\tau) + A_3^{(8)}(m_\mu/m_e, m_\mu/m_\tau) , \tag{3.2}$$

where $A_1^{(8)}$ denotes the universal part which includes the pure photonic corrections and closed muon loops. $A_2^{(8)}(m_\mu/m_e)$ ($A_2^{(8)}(m_\mu/m_\tau)$) contains in addition at least one closed electron (tau) loop and $A_3^{(8)}(m_\mu/m_e, m_\mu/m_\tau)$ contains at least one electron and one tau loop.

In Table 1 the results from the individual diagram classes contributing to $A_2^{(8)}(m_\mu/m_e)$ are shown. For practical reasons only the sum is presented for the classes I(b)+I(c) and II(b)+II(c) and

$A_2^{(8)}(m_\mu/m_e)$	[9, 8]	literature	
I(a0)	7.223076	7.223077 ± 0.000029	[4]
		7.223076	[13]
I(a1)	0.494072	0.494075 ± 0.000006	[4]
		0.494072	[13]
I(a2)	0.027988	0.027988 ± 0.000001	[4]
		0.027988	[13]
I(a)	7.745136	7.74547 ± 0.00042	[6]
I(bc0)	8.56876 ± 0.00001	8.56874 ± 0.00005	[4]
I(bc1)	0.1411 ± 0.0060	0.141184 ± 0.000003	[4]
I(bc2)	0.4956 ± 0.0004	0.49565 ± 0.00001	[4]
I(bc)	9.2054 ± 0.0060	9.20632 ± 0.00071	[6]
I(d)	-0.2303 ± 0.0024	-0.22982 ± 0.00037	[6]
		-0.230362 ± 0.000005	[14]
II(a)	-2.77885	-2.77888 ± 0.00038	[6]
		-2.77885	[13]
II(bc0)	-12.212631	-12.21247 ± 0.00045	[4]
II(bc1)	-1.683165 ± 0.000013	-1.68319 ± 0.00014	[4]
II(bc)	-13.895796 ± 0.000013	-13.89457 ± 0.00088	[6]
III	10.800 ± 0.022	10.7934 ± 0.0027	[6]
IV(a0)	116.76 ± 0.02	116.759183 ± 0.000292	[4]
		111.1 ± 8.1	[15]
		117.4 ± 0.5	[16]
IV(a1)	2.69 ± 0.14	2.697443 ± 0.000142	[4]
IV(a2)	4.33 ± 0.17	4.328885 ± 0.000293	[4]
IV(a)	123.78 ± 0.22	123.78551 ± 0.00044	[6]
IV(b)	-0.38 ± 0.08	-0.4170 ± 0.0037	[6]
IV(c)	2.94 ± 0.30	2.9072 ± 0.0044	[6]
IV(d)	-4.32 ± 0.30	-4.43243 ± 0.00058	[6]

Table 1: Final results for the different classes and comparison with the literature.

a further splitting is carried out in case more than one electron loop is present (see Ref. [9] for a detailed discussion.)

It is interesting to note that in some cases our coefficients have smaller uncertainties (e.g. II(bc)) whereas for others we have obtained an uncertainty which is much worse than the one of [6] (e.g. IV(c) or IV(d)). This can be traced back to complicated master integrals which at the moment can only be evaluated with a few-digit precision. Let us stress that, if necessary, the precision of our result can be improved systematically.

Our final result for $A_2^{(8)}(m_\mu/m_e)$ is given by

$$A_2^{(8)} = 126.34(38) + 6.53(30) = 132.86(48)\,, \tag{3.3}$$

where the first number after the first equality sign originates from the light-by-light-type diagrams IV(a), IV(b) and IV(c). Our final numerical uncertainty amounts to approximately $0.5\times(\alpha/\pi)^4 \approx 1.5\times10^{-11}$. It is larger than the uncertainty in Ref. [6]. Nevertheless it is sufficiently accurate as can be seen by the comparison to the difference between the experimental result and theory prediction which is given by [6]

$$a_\mu(\text{exp}) - a_\mu(\text{SM}) \approx 249(87)\times10^{-11}\,. \tag{3.4}$$

Note that the uncertainty in Eq. (3.4) receives approximately the same amount from experiment and theory (i.e. essentially from the hadronic contribution). Even after a projected reduction of the uncertainty by a factor four both in $a_\mu(\text{exp})$ and $a_\mu(\text{SM})$ our numerical precision is a factor ten below the uncertainty of the difference.

4. Conclusions

In this contribution we reported on the calculation of the four-loop QED corrections to a_μ which involve closed electron loops [9, 8] [see $A_2^{(8)}(m_\mu/m_e)$ in Eq. (3.2)]. In Ref. [9] also the contribution $A_3^{(8)}(m_\mu/m_e, m_\mu/m_\tau)$ with at least one electron and one tau loop have been computed and the results for $A_2^{(8)}(m_\mu/m_\tau)$ can be found in Ref. [7]. For all contributions perfect agreement with the results of Ref. [6] have been obtained. The only missing four-loop contribution which still has to be cross-checked is the universal part $A_1^{(8)}$.

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