

Two-loop Bhabha Scattering at High Energy beyond Leading Power Approximation

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ABSTRACT: We evaluate the two-loop $\mathcal{O}(m_e^2/s)$ contribution to the wide-angle high-energy electron-positron scattering in the double-logarithmic approximation. The origin and the general structure of the power-suppressed double logarithmic corrections are discussed in detail.

KEYWORDS: Precision QED, Effective field theories

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1 Introduction

High-energy electron-positron or *Bhabha* scattering [1] is among the classical applications of the perturbative quantum electrodynamics (QED). Beside its phenomenological importance as a standard candle for luminosity calibration at the electron-positron colliders, Bhabha scattering has become a testing ground for new techniques of multiloop calculations. The analysis of high-order corrections to this process often sheds new light on perturbative structure of gauge theories. In general the radiative corrections for the scattering of two massive particles are known only in the one-loop approximation. Despite significant progress over the last decade [2–7], the two-loop corrections have been computed only in the high energy limit neglecting the terms suppressed by the ratio of the electron mass m_e to the center-of-mass energy \sqrt{s} [8–15].¹ The logarithmically enhanced two-loop electroweak corrections are available in this approximation as well [17–21]. At the same time the power-suppressed terms in two loops are still beyond the reach of existing computational techniques. In general the power-suppressed contributions are of great interest. At intermediate energies the power corrections in many cases are phenomenologically important. Moreover, in contrast to the leading-power contribution very little is known about the infrared structure of the power-suppressed terms. This problem has been studied already in early days of QED [22] and currently attracts much attention in various context [23–27]. However, a systematic renormalization group analysis

¹For a review see Ref. [16]

of the high-energy behavior of on-shell amplitudes beyond the leading-power approximation is still elusive for the existing effective field theory methods.

In this paper we consider the $\mathcal{O}(m_e^2/s)$ two-loop QED corrections to the differential cross section of the high-energy large-angle Bhabha scattering. The corrections are evaluated in the double-logarithmic approximation *i.e.* retaining the terms enhanced by two powers of the large logarithm $\ln(s/m_e^2)$ per each power of the coupling constant. These terms dominate the power-suppressed contribution and in a wide energy interval are numerically comparable to the nonlogarithmic leading-power terms. The leading power-suppressed double-logarithmic corrections have been obtained in Ref. [26] to all orders in fine structure constant α for the electromagnetic form factor of electron. In this paper we elaborate the approach [26] and apply it to the electron-positron scattering amplitude in two-loop approximation. Our main result is given by Eq. (4.10).

The paper is organized as follows. In the next section we describe the perturbative expansion of the cross section at high energy. In Sect. 3 we discuss the origin and general structure of the double-logarithmic corrections. In Sect. 4 we describe the evaluation of the one and two-loop double-logarithmic power-suppressed corrections to Bhabha scattering. Sect. 5 is our summary and conclusion.

2 Perturbative expansion of the cross section at high energy

We consider the electron-positron scattering $e^-(p_1)e^+(p_2) \rightarrow e^-(p_3)e^+(p_4)$ at high energy and large angle when all the kinematic invariants $s_{ij} = (p_i + p_j)^2$ for $i \neq j$ are of the same scale much larger than m_e^2 .² In this limit the cross section can be written as a series in a small ratio $\rho = m_e^2/s$

$$\sigma = \frac{\alpha^2}{s} \sum_{n=0}^{\infty} \rho^n \sigma_n, \quad (2.1)$$

where σ_n are the functions of $x = -t/s \sim 1$.³ These functions in turn can be computed as series in α . Up to $\mathcal{O}(\alpha)$ the result for the cross section is known in a closed analytical form (see *e.g.* [5]) and the coefficients in Eq. (2.1) can be found for any n . The second order result is available only for the leading-power contribution σ_0 . The series (2.1) is asymptotic and after the expansion in α its coefficients in general have logarithmic dependence on ρ . In the high-energy limit the double-logarithmic contributions enhanced by two powers of the large logarithm $\ln \rho$ per each power of the coupling constant dominate the expansion of σ_n in α . In the double-logarithmic approximation perturbative expansion for these coefficients can be written as series in $\tau = \frac{\alpha}{4\pi} \ln^2 \rho$

$$\sigma_n = \exp \left[-\frac{2\alpha}{\pi} B(\rho) \ln(\lambda^2/m_e^2) \right] \sum_{m=0}^{\infty} \tau^m \sigma_n^{(m)}. \quad (2.2)$$

² All the external momenta are defined to be incoming and on-shell so that $p_i^2 = m_e^2$ and the Mandelstam variables are $s = s_{12}$, $t = s_{13}$, and $u = s_{14}$.

³The variable x is related to the scattering angle θ in the center of mass frame, $x = (1 - 4\rho)(1 - \cos \theta)/2$.

In Eq. (2.2) the exponential prefactor with $B(\rho) = \ln \rho + \mathcal{O}(1)$ accounts for the universal singular dependence of the amplitude on the auxiliary photon mass λ introduced to regulate the infrared divergences [28]. For the leading-power term the double-logarithmic corrections are known to factorize and exponentiate [29–37]. In this approximation the all-order dependence of the differential cross section on τ is given by the expression

$$\frac{d\sigma_0}{d\Omega} = e^{-4\tau} \frac{d\sigma_0^{(0)}}{d\Omega}, \quad (2.3)$$

where the Born term reads

$$\frac{d\sigma_0^{(0)}}{d\Omega} = \left(\frac{1 - x + x^2}{x} \right)^2. \quad (2.4)$$

The goal of this paper is to compute the coefficient $\sigma_1^{(2)}$.

3 General structure of double-logarithmic corrections

The double-logarithmic terms are in general associated with the soft and collinear divergences of the amplitudes due to radiation of the soft virtual particles by highly energetic on-shell charged particles. At the same time the structure of the double-logarithmic corrections crucially depends on their origin. Below we consider two types of the double-logarithmic corrections, which play the central role in our analysis.

3.1 Sudakov logarithms

Sudakov double-logarithmic corrections are induced by the soft photon exchange. In the leading order of the high energy/small mass expansion the Sudakov double logarithms exponentiate and result in a strong universal suppression of any electron scattering amplitude with a fixed number of emitted photons in the limit when all the kinematic invariants of the process are large, Eq. (2.3). A crucial observation of Ref. [26] is that “Sudakov” photons do not generate $\mathcal{O}(\rho)$ double-logarithmic corrections to the scattering amplitudes. Below we present a detailed derivation of this result.

Let us outline our approach to the analysis of the power-suppressed double-logarithmic contributions. We use the expansion by regions method [38, 39] to get a systematic expansion of the Feynman integrals in ρ . Within this method every Feynman integral is given by the sum over contributions of different virtual momentum regions. Each contribution is represented by a homogeneous Feynman integral, which is in general divergent even if the original integral before the expansion is finite. These spurious divergences result from the process of scale separation and have to be dimensionally regulated. The singular terms cancel out in the sum of all regions but can be used to find the logarithmic terms. The double-logarithmic contributions are determined by the leading singular behavior of the integrals and can be found by the method developed in Ref. [29] (see also [22, 31]). Though the method is blind to the power corrections, it can be applied in this case since the expansion by regions provides the integrals, which are *homogeneous* in the expansion parameter. Let us first consider an

exchange of a virtual photon with the momentum l between on-shell fermion lines with the momenta p_i and p_j . The Sudakov double logarithmic contribution originates from the region where the photon momentum is small. Thus we can neglect it in the numerator of the fermion propagators since the integral with the additional power of the photon momentum is not sufficiently singular to develop the double-logarithmic behavior. Then by using the equations of motion $(\not{p}_i - m_e)\psi(p_i) = 0$ the Sudakov photon contribution can be reduced to the integral

$$I = \int d^d l \frac{(p_i p_j)}{l^2((p_i - l)^2 - m_e^2)((p_j + l)^2 - m_e^2)}. \quad (3.1)$$

In the above equation we neglected the photon mass and use the dimensional regularization with $d = 4 - 2\varepsilon$. The soft divergence in this case appear as a pole in ε . This modifies the form of the exponent in Eq. (2.2) but does not affect the structure of the expansion in ρ . The integral gets contributions from the hard and two (symmetric) collinear regions $I = I_h + I_{c-i} + I_{c-j}$. Since the singularities of the hard and collinear regions are not independent, it is sufficient to consider only the contribution of a single region, *e.g.* the i -collinear one I_{c-i} . We set the parameter of dimensional regularization to be $\mu^2 \sim s_{ij}$, so that the expansion of the hard region contribution with $l \sim \sqrt{s_{ij}}$ in ε does not produce large logarithms. For the large-angle scattering we can choose the light-cone coordinates where $p_1 \approx p_{i-}$ and $p_j \approx p_{j+}$. Then the i -collinear region is defined by the following scaling of the virtual momentum components $l_+ \sim m_e^2/\sqrt{s_{ij}}$, $l_- \sim \sqrt{s_{ij}}$, $l_\perp \sim m_e$, so that $l^2 \sim m_e^2$. It is convenient to introduce the light-like vectors \tilde{p}_i , \tilde{p}_j such that $p_i = \tilde{p}_i + \frac{m_e^2}{s_{ij}}\tilde{p}_j$ and $p_j = \tilde{p}_j + \frac{m_e^2}{s_{ij}}\tilde{p}_i$, where $(\tilde{p}_i \tilde{p}_j) = \tilde{s}_{ij}$. In the i -collinear region the electron propagator is substituted by the series

$$\frac{1}{(p_j + l)^2 - m_e^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(2(m_e^2/\tilde{s}_{ij})(l\tilde{p}_i) + l^2)^n}{(l\tilde{p}_j)^{(n+1)}}, \quad (3.2)$$

which results in a series

$$I_{c-i} = \int d^d l \frac{(p_i p_j)}{l^2(-2(p_i l) + l^2)(l\tilde{p}_j)} \left[1 - \frac{2(m_e^2/\tilde{s}_{ij})(l\tilde{p}_i) + l^2}{(l\tilde{p}_j)} + \mathcal{O}(m_e^4/\tilde{s}_{ij}^2) \right]. \quad (3.3)$$

Let us consider the second term in Eq. (3.3). In the limit when the virtual momentum is soft and collinear to p_i either l^2 or $p_i l$ factor in the denominator is cancelled and the integrand is therefore not singular enough to develop the double-logarithmic contribution. At the same time by integrating the first term one gets

$$(i\pi^2) \frac{2(p_i p_j)}{\tilde{s}_{ij}} \left[-\frac{1}{\varepsilon} \ln \left(\frac{m_e^2}{\tilde{s}_{ij}} \right) + \frac{1}{2} \ln^2 \left(\frac{m_e^2}{\tilde{s}_{ij}} \right) \right], \quad (3.4)$$

where only the double-logarithmic contribution is retained and the pole corresponds to the soft divergence not regulated by the electron mass. Since $2(p_i p_j)/\tilde{s}_{ij} = 1 + \mathcal{O}(m_e^4/\tilde{s}_{ij}^2)$, Eq. (3.4) can be written as follows

$$(i\pi^2) \left[-\frac{1}{\varepsilon} \ln \rho + \frac{1}{2} \ln^2 \rho + \mathcal{O}(\rho^2) \right], \quad (3.5)$$

i.e. the first term of the expansion (3.3) does not generate $\mathcal{O}(\rho)$ double-logarithmic corrections as well. The above analysis can be generalized to an arbitrary number of Sudakov photons. After neglecting all the photon momenta in the numerators the Lorentz/spinor reduction becomes straightforward. By using the equations of motion and the on-shell conditions one gets the factor $(p_i p_j)$ per each photon connecting the lines with the momenta p_i and p_j for any i and j . At the same time the structure of the expansion by regions becomes more involved. For the multiloop diagrams it also includes ultra-collinear regions, which are obtained by multiplying the collinear scaling rules with a power of (m_e^2/\tilde{s}_{ij}) . All these regions should be taken into account to find the total double-logarithmic contribution. As an example let us consider all the virtual momenta l_k to be i -collinear. It represents the most complicated case since the integrations over different l_k do not factorize. After the expansion one gets eikonal propagators of the form (3.2), which depend on a sum of several virtual momenta l_k with identical scaling. Since the expansion by regions generates homogeneous integrals, the leading term of the expansion is proportional to a product of $(p_i p_j)/\tilde{s}_{ij}$ factors for different i and j and therefore does not produce any $\mathcal{O}(\rho)$ terms. Then for the analysis of the next-to-leading term we use the method [29] to extract the double-logarithmic asymptotic behavior of a given integral. According to [22, 29, 31] the double-logarithmic contribution originates from the region of strongly ordered virtual momenta determined by a set of conditions $(l_{k_1} p_m) \ll (l_{k_2} p_m) \ll \dots \ll (l_{k_n} p_m)$ for any m and some permutations of the indices k_i . Thus in the double-logarithmic region one can neglect all the virtual momenta but one in each eikonal propagator and the problem effectively reduces to the one-loop case considered above, where the other virtual momenta only play a role of an infrared or ultraviolet cutoff for the double-logarithmic integration. Due to a natural ordering of the momenta with different collinearity the analysis of the double-logarithmic contribution of the corresponding mixed regions does not differ from the case considered above.

Thus we have shown that Sudakov photons do not produce double logarithms in the first order in ρ . We have verified the absence of the $\mathcal{O}(\rho)$ double-logarithmic terms by explicit evaluation of the collinear regions contribution to the two-loop scalar integrals, which appear in the analysis of the Bhabha scattering. This observation agrees with the analysis [40] of the cusp anomalous dimension, which determines the double-logarithmic corrections to the light-like Wilson line with a cusp. For the large cusp angle corresponding to the limit $\rho \rightarrow 0$ from the result of Ref. [40] one gets

$$\Gamma_{cusp} = -\frac{\alpha}{\pi} \ln \rho (1 + \mathcal{O}(\rho^2)), \quad (3.6)$$

with vanishing first-order term in ρ . Our result, however, is more general since it also implies the absence of “kinematic” $\mathcal{O}(\rho)$ corrections, which multiply the leading-order cusp anomalous dimension when the scattering amplitude is related to the Wilson line.

Note that the double-logarithmic $\mathcal{O}(\rho)$ corrections do vanish only for the *amplitudes*. When the amplitudes are squared one gets $\mathcal{O}(\rho)$ terms, which multiply the Sudakov expo-

mental factor and produce the $\mathcal{O}(\rho)$ double-logarithmic corrections of the form

$$e^{-4\tau} \frac{d\sigma_1^{(0)}}{d\Omega}. \quad (3.7)$$

3.2 Non-Sudakov logarithms

The $\mathcal{O}(\rho)$ double-logarithmic contributions to the amplitudes originate from a completely different virtual momentum configuration. Let us consider an electron propagator $S(p_i - l)$, where l is the momentum of a virtual photon with the propagator $D_{\mu\nu}(l)$. In the soft-photon limit $l \rightarrow 0$ the electron propagator becomes eikonal

$$S(p_i - l) \approx -\frac{\not{p}_i + m_e}{2(p_i l)} \quad (3.8)$$

and develops a collinear singularity when l is parallel to p_i . Alternatively, we may consider the soft-electron limit $l' \rightarrow 0$, where $l' = p_i - l$. Then the electron propagator becomes scalar

$$S(l') \approx \frac{m_e}{l'^2 - m_e^2} \quad (3.9)$$

while the photon propagator becomes eikonal

$$D_{\mu\nu}(l') \approx \frac{g_{\mu\nu}}{-2(p_i l') + m_e^2 - \lambda^2}. \quad (3.10)$$

Thus the roles of the electron and photon propagators are exchanged. Due to the explicit factor m_e in the scalar electron propagator this region can only generate a mass-suppressed double-logarithmic contribution. The existence of non-Sudakov double-logarithmic contributions due to soft electron exchange has actually been known for a long time [22]. They are typical for the amplitudes that are mass suppressed at high energy. In contrast to the Sudakov case such logarithms do not factorize and exponentiate. As a result very little is known about the all-order structure of the power-suppressed non-Sudakov logarithms. Only a few examples of the non-Sudakov resummation are known so far [22, 23, 26, 27]. At the same time due to explicit power suppression factor the soft-electron double-logarithmic contribution in a given order of perturbation theory can be determined within the original method of Ref. [29].

For the calculation we in general follow the procedure formulated in [26] for the analysis of the form factor (see also Ref. [27]). The structure of the two-loop non-Sudakov corrections to the electron-positron scattering amplitude has an important difference though. For the one-loop vertex corrections the virtual momentum configuration discussed above does not produce a double-logarithmic contribution because the momentum shift distorts the eikonal structure of the second electron propagator and removes the soft singularity at small l' necessary to get the second power of the large logarithm. As a consequence the $\mathcal{O}(\rho)$ double-logarithmic corrections to the electron form factor appear first in two loops in a diagram with soft electron pair exchange [26]. On the other hand in the one-loop box diagrams both photon propagators after the momentum shift become eikonal and provide the necessary infrared structure. Thus

the $\mathcal{O}(\rho)$ double-logarithmic corrections to the scattering amplitude appear already in one loop due to a single soft electron exchange. Therefore for the calculation of the two-loop $\mathcal{O}(\rho)$ corrections to the Bhabha scattering one has to take into account the diagrams with both soft electron and soft photon in addition to the soft electron pair contribution. We discuss the details of the calculation in the next section.

4 Double-logarithmic $\mathcal{O}(\rho)$ corrections to Bhabha scattering

In the analysis of the Feynman diagrams we always choose the momentum routing in such a way that a soft electron or photon line carries a single virtual momentum only. For the determination of the double-logarithmic contribution we can use the effective Feynman rules, which retain the leading infrared behavior of the full theory. For a soft electron line we make the following approximation

$$\frac{\hat{l} + m_e}{l^2 - m_e^2} \rightarrow \frac{m_e}{l^2 - m_e^2}, \quad (4.1)$$

so that it effectively becomes scalar. For an electron carrying a single external momentum we use the eikonal approximation

$$\frac{\hat{p}_i + \hat{l} + m_e}{(p_i + l)^2 - m_e^2} \rightarrow \frac{\hat{p}_i + m_e}{2(p_i l)}. \quad (4.2)$$

An electron line with two different external momenta corresponds to a far off-shell or “hard” electron propagator

$$\frac{\hat{p}_i + \hat{p}_j + \hat{l} + m_e}{(p_i + p_j + l)^2 - m_e^2} \rightarrow \frac{\hat{p}_i + \hat{p}_j + m_e}{s_{ij}}, \quad (4.3)$$

which effectively reduces to a local interaction vertex. Similar approximations are used for the eikonal and hard photons

$$\frac{g_{\mu\nu}}{(p_i + l)^2 - \lambda^2} \rightarrow \frac{g_{\mu\nu}}{2(p_i l) + m_e^2 - \lambda^2}, \quad \frac{g_{\mu\nu}}{(p_i + p_j + l)^2 - \lambda^2} \rightarrow \frac{g_{\mu\nu}}{s_{ij}}. \quad (4.4)$$

In principle the Feynman rules can be further simplified by using the light-cone coordinates. In this case the soft photon and eikonal electron propagators have only the light-cone components, the eikonal photon propagator has only transverse components, and so on. In two loops, however, the use of the standard computational tools for performing Lorentz and spinor algebra in a covariant form is more convenient.

The next big simplification is related to the treatment of the dependence of the corrections on the photon mass. In the diagrams without soft photon lines both soft and collinear divergences of the virtual momentum integration are regulated by the electron mass. These diagrams are not sensitive to the photon mass and in the double-logarithmic approximation can be computed either with $\lambda = 0$ or $\lambda = m_e$ with the identical result. The diagrams with both soft photon and soft electron exchanges do depend on λ . This dependence, however, can be determined from the general properties of soft photon contribution. Indeed, the

virtual momentum space of the soft photons with $|\mathbf{l}| \ll m_e$ is known to factorize [28]. For high-energy scattering the integration over such momenta results in the exponent of the one-loop contribution in Eq. (2.2). When we perform the calculation with $\lambda \sim m_e$, this part of the virtual momentum space is eliminated so that the exponent in Eq. (2.2) reduces to a nonlogarithmic factor and we directly obtain the coefficients $\sigma_n^{(m)}$. In this way we reduce the number of different scales in the problem, which significantly simplifies the analysis. It is important to note that the above factorization works only for the sum of all the diagrams in a given order of perturbation theory. Individual diagrams, even infrared finite, may have different double-logarithmic behavior for $\lambda = 0$ and $\lambda = m_e$.

4.1 One-loop contributions

According to the discussion of Sect. 3 the one-loop leading-power corrections have two distinct sources. The soft photon part is determined by the product of the standard Sudakov double-logarithmic corrections to the scattering amplitudes and the $\mathcal{O}(\rho)$ Born cross section. It is given by the first term of the expansion of Eq. (3.7) in τ . The non-Sudakov contribution is generated by the box diagrams with one soft and one hard electron line and two eikonal photon propagators. We compute it by using the effective Feynman rules introduced in the previous section. The total result for the one-loop double-logarithmic power-suppressed contribution is

$$\frac{d\sigma_1^{(1)}}{d\Omega} = -4 \frac{d\sigma_1^{(0)}}{d\Omega} + \frac{6 - 20x + 24x^2 - 20x^3 + 6x^4}{(1-x)x^2}, \quad (4.5)$$

where the first and the second terms correspond to the soft photon and soft electron contributions, respectively. It agrees with the known analytic one-loop result [7] expanded to $\mathcal{O}(\rho)$.

4.2 Two-loop contributions

In two loops the double-logarithmic power-suppressed contribution can be decomposed as follows

$$\frac{d\sigma_1^{(2)}}{d\Omega} = \left. \frac{d\sigma_1^{(2)}}{d\Omega} \right|_{1l \times 1l} + \left. \frac{d\sigma_1^{(2)}}{d\Omega} \right|_{1PR} + \left. \frac{d\sigma_1^{(2)}}{d\Omega} \right|_{1PI}, \quad (4.6)$$

where three terms correspond to the one-loop by one-loop amplitude interference, the two-loop one-particle reducible and one-particle irreducible corrections to the amplitude, respectively. The calculation of the interference term is straightforward and gives

$$\left. \frac{d\sigma_1^{(2)}}{d\Omega} \right|_{1l \times 1l} = -4 \frac{d\sigma_1^{(0)}}{d\Omega} - 2 \frac{d\sigma_1^{(1)}}{d\Omega}. \quad (4.7)$$

The two-loop one-particle reducible contribution is determined by the corrections to the electron form factor. Its soft photon part is given by the interference of the two-loop Sudakov form factor and square of the one-loop Sudakov form factor with the $\mathcal{O}(\rho)$ part of the Born cross section. The non-Sudakov corrections are generated by the two-loop soft electron pair

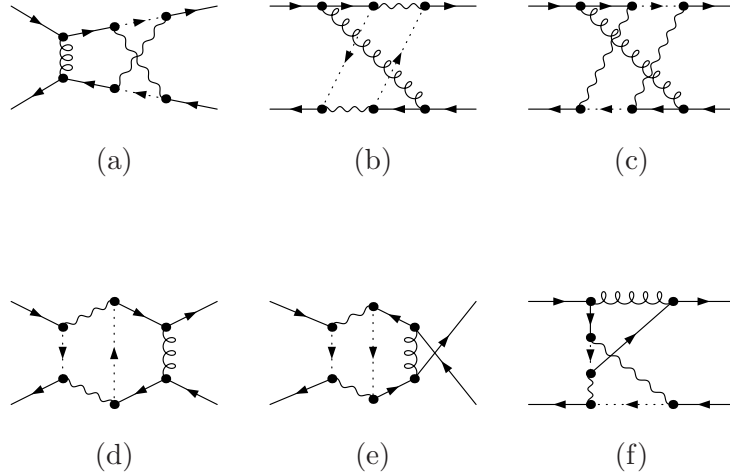


Figure 1. Two-loop one-particle irreducible diagrams with soft electron pair exchange. Dashed arrows correspond to the scalar soft quarks. The loopy (wavy) lines correspond to the hard (eikonal) photons. Symmetric diagrams are not shown.

exchange and can be found in [26] (see also [41, 42] for the full theory calculation). The total reducible contribution reads

$$\left. \frac{d\sigma_1^{(2)}}{d\Omega} \right|_{1PR} = 4 \frac{d\sigma_1^{(0)}}{d\Omega} - \frac{4 - 6x + 8x^2 - 8x^3 + 6x^4 - 4x^5}{3x^3}, \quad (4.8)$$

where the first and the second terms correspond to the soft photon and soft electron pair contributions, respectively.

The irreducible part gets contributions from the Feynman diagrams with soft electron pair exchange given in Fig. 1 and the Feynman diagrams with both soft electron and soft photon exchanges, Fig. 2. Note that the double-logarithmic corrections due to two soft photon exchanges cancel out in the irreducible part according to the general factorization property of the Sudakov logarithms. To compute the irreducible part we use the effective Feynman rules described in the beginning of Sect. 4. The full set of contributing diagrams is generated with QGRAF [43]. Its output is processed by a MATHEMATICA program, which automatically chooses the routing of internal and external momenta through the diagram in such a way that the soft particle propagators carry only a single loop momentum and no external momenta. The program generates FORM-readable expressions. By a custom code written in FORM [44, 45] the spin chains appearing in the diagrams are projected onto an irreducible basis, which allows to easily square the amplitude. The output is then mapped onto a set of five two-loop “master” integrals I_i , which are evaluated in the double-logarithmic approximation in the Appendix. The soft electron pair contribution is similar to the form factor corrections discussed in [26] and can be reduced to nonplanar and planar

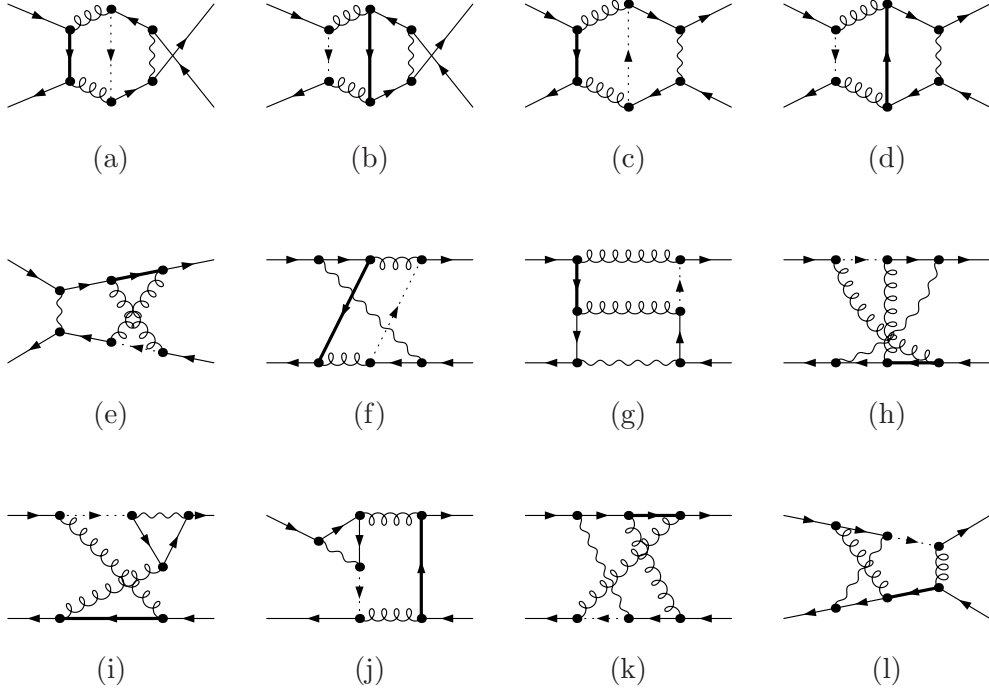


Figure 2. Two-loop diagrams with soft electron and soft photon exchange. Dashed (thick) arrows correspond to the scalar soft (hard) electrons. The loopy (wavy) lines correspond to the eikonal (soft) photons. Symmetric diagrams are not shown.

scalar vertex integrals $I_{1,2}$. The irreducible diagrams with soft photon exchange between eikonal lines, Figs. 2(a-d), are expressed through I_2 and the product of the one-loop integrals. The reduction of the diagrams Figs. 2(e-h) includes the integral I_3 , which depends on three external momenta. The diagrams with soft photon emission off the soft electron line, Figs. 2(i-l), are reduced to the vector integrals I_4 and I_5 , respectively. The total one-particle reducible contribution reads

$$\left. \frac{d\sigma_1^{(2)}}{d\Omega} \right|_{1PI} = \frac{1}{3} \frac{1+x^4}{(1-x)x^2} - \frac{21-70x+84x^2-70x^3+21x^4}{3(1-x)x^2} - \frac{8-9x+9x^2-8x^3}{x^2}, \quad (4.9)$$

where the three terms correspond to the soft electron pair exchange, the soft photon and soft electron exchange between the eikonal lines, and the soft photon emission off the soft electron line, respectively.

The total result for the two-loop double-logarithmic power-suppressed term is given by

the sum of Eqs. (4.7-4.9) and reads

$$\begin{aligned} \frac{d\sigma_1^{(2)}}{d\Omega} &= 8 \frac{d\sigma_1^{(0)}}{d\Omega} - \frac{4 + 70x - 227x^2 + 266x^3 - 227x^4 + 70x^5 + 4x^6}{3(1-x)x^3} \\ &= -\frac{4 + 166x - 323x^2 + 266x^3 - 323x^4 + 166x^5 + 4x^6}{3(1-x)x^3}. \end{aligned} \quad (4.10)$$

To estimate the numerical impact of the power-suppressed terms let us consider the scattering at $\theta \sim 30^\circ$. For this scattering angle the correction to the cross section is maximal at the energy $\sqrt{s} \approx 8 m_e \approx 4$ MeV, where

$$\rho \tau^2 \frac{d\sigma_1^{(2)}}{d\Omega} \approx -24.4 \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_0^{(0)}}{d\Omega}. \quad (4.11)$$

The effect decreases with the increasing energy but even for $\sqrt{s} \sim 300 m_e \approx 150$ MeV corresponding to $\rho \approx 10^{-5}$ the numerical coefficient in Eq. (4.11) is approximately equal to -1 , *i.e.* the double logarithmic power-suppressed term is comparable to the nonlogarithmic two-loop leading-power corrections.

5 Summary

In this paper we have developed a systematic approach for the calculation of the leading power correction to the high-energy scattering processes in the double logarithmic approximation. We focus on the two-loop electron-positron scattering in QED but the analysis can be extended to more complicated processes and to nonabelian gauge theories. The higher order double-logarithmic corrections in QED can in principle be resummed by using the method described in Refs. [26, 27]. The general feature of the high-energy expansion is the absence of the leading power-suppressed double-logarithmic pure Sudakov corrections to the amplitudes due to the soft virtual photon exchange. At the same time the structure of the corrections to the two-particle scattering amplitudes turns out to be more diverse than for the form factors describing single particle scattering in an external field. In particular the non-Sudakov double logarithms appear already in one-loop scattering amplitude due to a single soft electron exchange. For the energies ranging from a few to a few hundred MeV where $|\ln \rho| \gg 1$ and $\rho \ln^4 \rho \sim 1$, the calculated two-loop double-logarithmic terms saturate the power-suppressed contribution and are comparable in magnitude with the two-loop nonlogarithmic leading-power corrections. This effectively sets up the low boundary of the energy region where the leading power approximation for the $\mathcal{O}(\alpha^2)$ cross section can be used.

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A Evaluation of two-loop integrals

The one-particle irreducible diagrams in Fig. 1 and Fig. 2(a-d) can be reduced to two scalar integrals

$$I_1(p_i, p_j) = \int d^4 l_1 d^4 l_2 D(l_1) D(l_2) D(p_i + l_1 + l_2) D(p_i + l_1) D(p_j - l_1 - l_2) D(p_j - l_2), \quad (\text{A.1})$$

$$I_2(p_i, p_j) = \int d^4 l_1 d^4 l_2 D(l_1) D(l_2) D(p_i + l_1 + l_2) D(p_i + l_1) D(p_j - l_1 - l_2) D(p_j - l_1), \quad (\text{A.2})$$

where $D(k) = 1/(k^2 - m_e^2)$. Let us consider the nonplanar case (A.1). To compute the integral in the double-logarithmic approximation we follow Ref. [29] and introduce the Sudakov parametrization of each virtual momentum $l_k = u_k p_i + v_k p_j + l_{k\perp}$. Integration over the transverse momentum components $l_{k\perp}$ is performed by taking the residue of a soft propagator pole

$$D(l_k) \rightarrow -i\pi\delta(l_k^2 - m_e^2) = -i\pi\delta(su_k v_k - l_{k\perp}^2 - m_e^2). \quad (\text{A.3})$$

For $\rho < u_k, v_k < 1$ the eikonal propagators become

$$\begin{aligned} D(p_i + l_1) &\approx \frac{1}{s_{ij}v_1}, & D(p_i + l_1 + l_2) &\approx \frac{1}{s_{ij}(v_1 + v_2)}, \\ D(p_j - l_2) &\approx -\frac{1}{s_{ij}u_2}, & D(p_j - l_1 - l_2) &\approx -\frac{1}{s_{ij}(u_1 + u_2)}. \end{aligned} \quad (\text{A.4})$$

Then the double-logarithmic region is given by the interval $\rho < v_1 < v_2 < 1$, $\rho < u_2 < u_1 < 1$ with an additional constraint $\rho < u_k v_k$, which ensures that the soft propagators can go on-shell. Thus in the double-logarithmic approximation the two-loop nonplanar integral reads

$$I_1(p_i, p_j) \approx \left(\frac{i\pi^2}{s_{ij}}\right)^2 \int_{\rho}^1 \frac{dv_1}{v_1} \int_{v_1}^1 \frac{dv_2}{v_2} \int_{\rho/v_1}^1 \frac{du_1}{u_1} \int_{\rho/v_2}^{u_1} \frac{du_2}{u_2}. \quad (\text{A.5})$$

By introducing the normalized logarithmic variables $\eta_k = \ln v_k / \ln \rho$ and $\xi_k = \ln u_k / \ln \rho$ Eq. (A.5) can be transformed to

$$I_1(p_i, p_j) \approx N_{ij}^2 \int \theta(1 - \eta_1 - \xi_1) \theta(1 - \eta_2 - \xi_2) \theta(\eta_2 - \eta_1) \theta(\xi_1 - \xi_2) d\eta_1 d\eta_2 d\xi_1 d\xi_2 = \frac{N_{ij}^2}{12}, \quad (\text{A.6})$$

where $N_{ij} = i\pi^2 \ln^2 \rho / s_{ij}$ and the integration goes over the four-dimensional cube $0 < \eta_k, \xi_k < 1$. The only difference in the calculation of the planar two-loop integral (A.2) is the ordering of the variables $\eta_2 < \eta_1$, which provides the double-logarithmic scaling of the integrand. Thus one gets

$$I_2(p_i, p_j) \approx N_{ij}^2 \int \theta(1 - \eta_1 - \xi_1) \theta(1 - \eta_2 - \xi_2) \theta(\eta_1 - \eta_2) \theta(\xi_1 - \xi_2) d\eta_1 d\eta_2 d\xi_1 d\xi_2 = \frac{N_{ij}^2}{24}. \quad (\text{A.7})$$

The diagrams where the soft photon and soft electron emitted by the same eikonal line end on different eikonal lines, Fig. 2(e-h), include the scalar integral depending on three external momenta, which can be evaluated in the same way

$$I_3(p_i, p_j, p_k) = \int d^4l_1 d^4l_2 D(l_1) D(l_2) D(p_i + l_1) D(p_j - l_1) D(p_j - l_1 - l_2) D(p_k + l_2) \approx \frac{N_{ij} N_{jk}}{8}. \quad (\text{A.8})$$

The diagrams with the soft photon emission off the soft electron line, Fig. 2(i-l), include the electron propagator, which depend on both soft momenta. When the soft momenta are close to the mass shell the propagator becomes eikonal

$$\frac{\hat{l}_1 + \hat{l}_2 + m_e}{(l_1 + l_2)^2 - m_e^2} \approx \frac{\hat{l}_1 + \hat{l}_2}{2(l_1 l_2)} \quad (\text{A.9})$$

and is sufficiently singular to produce the double-logarithmic contribution despite the presence of soft momenta in the numerator [27]. In total we have to take into account two vector master integrals, which depend on two and three external momenta. It is convenient to project them on the external momenta and consider the following quantities

$$I_4(p_i, p_j) = \int d^4l_1 d^4l_2 D(l_1) D(l_2) D(l_1 - l_2) D(p_i + l_1) D(p_i + l_2) D(p_j - l_1) \times (l_1 p_i), \quad (\text{A.10})$$

$$I_5(p_i, p_j, p_k) = \int d^4l_1 d^4l_2 D(l_1) D(l_2) D(l_1 + l_2) D(p_i - l_1) D(p_j - l_2) D(p_k + l_1 + l_2) \times \{(l_1 p_i), (l_1 p_j), (l_1 p_k)\}. \quad (\text{A.11})$$

Let us consider the calculation of the integral (A.10). We introduce the Sudakov parameters in a slightly different way $l_1 = u_1 p_i + v_1 p_j + l_{1\perp}$, $l_2 = u_2 p_i + v_2 l_1 + l_{2\perp}$. Then

$$D(l_1 - l_2) \approx -\frac{1}{s_{ij} v_1 u_2} \quad (\text{A.12})$$

and the extra factor v_1 in the denominator cancels the one from the scalar product $(l_1 p_i) \approx s_{ij} v_1 / 2$ in the numerator providing the double logarithmic scaling of the integrand. The double-logarithmic integration region is now defined by the intervals $\rho < v_1, u_1 < 1, \rho/v_1 < v_2, u_2 < 1, \rho < u_1 v_1, \rho/v_1 < u_2 v_2$ and one gets

$$I_4(p_i, p_j) \approx s_{ij} N_{ij}^2 \int \theta(1 - \eta_1 - \xi_1) \theta(1 - \eta_1 - \eta_2 - \xi_2) \theta(1 - \eta_1 - \eta_2) d\eta_1 d\eta_2 d\xi_1 d\xi_2 = \frac{N_{ij}^2 s_{ij}}{8}. \quad (\text{A.13})$$

In the same way we obtain

$$I_5(p_i, p_j, p_k) \approx \frac{1}{24} \{N_{ik}^2 s_{ik}, 2N_{jk}^2 s_{jk}, 2N_{ij}^2 s_{ij}\}. \quad (\text{A.14})$$

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