

# E6Tensors: A Mathematica Package for $E_6$ Tensors

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## Abstract

We present the `Mathematica` package `E6Tensors`, a tool for explicit tensor calculations in  $E_6$  gauge theories. In addition to matrix expressions for the group generators of  $E_6$ , it provides structure constants, various higher rank tensors and expressions for the representations **27**, **78**, **351** and **351'**. This paper comes along with a short manual including physically relevant examples. I further give a complete list of gauge invariant, renormalisable terms for superpotentials and Lagrangians.

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# Program Summary

**Author:** Thomas Deppisch

**Title:** E6Tensors

**Licence:** GNU LGPL

**Programming language / external routines:** Wolfram Mathematica 8-10

**Operating system:** cross-platform

**Computer architecture:** x86, x86\_64

**RAM:** > 1 GB RAM recommended

**Current version:** 1.0.0

**Web page:** <http://e6tensors.hepforge.org>

**Contact:** thomas.deppisch@kit.edu — for bugs, possible improvements and questions

## 1 Introduction

The exceptional Lie group  $E_6$  may be a suitable candidate for describing fundamental symmetries in particle physics [1]. In the discussion of  $E_6$ , most authors rely on abstract group theoretic methods. Besides, there is a paper by Kephart and Vaughn [2] that describes the generators of  $E_6$  in terms of its maximal subgroup  $SU(3) \times SU(3) \times SU(3)$ . In practice, applying these methods may be cumbersome. For writing down Lagrangians and superpotentials of such models it is useful to have explicit expressions for irreducible representations and invariants, preferably in a way suited for computer use. To our knowledge, such a tool is still missing in the literature and the package **E6Tensors** tries to fill this gap.

Our package provides such explicit expressions for the representations **27**, **78**, **351**, **351'** as well as structure constants and higher rank tensors. **E6Tensors** enables the user to study Lagrangians and superpotentials including the aforementioned representations explicitly by components.

The outline of the paper is the following: In Section 2, we state the transformation law for the fundamental representation **27**. From that we construct the matrix expressions for the 78 group generators forming the adjoint representation **78**. Then it is possible to compute the (symmetric) structure constants and other properties of the matrix generators. Since higher-order tensors can be built from the tensor products of (anti-)fundamental representations, we derive expressions for the irreducible representations **351** and **351'** in Section 3. Section 4 then provides a small manual for the **Mathematica** package including some remarks and examples. A summary of possible gauge invariant terms in superpotentials and Lagrangians can be found in Appendix A.

## 2 Matrix Expressions for the Group Generators

In [2], the authors give a prescription how the fundamental representation can be expressed as three  $3 \times 3$  matrices  $L, M, N$  and how the group generators act on them. We briefly revise that

prescription and point out that we performed the calculations described there using symbolic manipulation in `Mathematica`. Throughout the paper we use Greek indices for the fundamental representation of  $E_6$

$$\mu, \nu, \rho, \dots = 1, \dots, 27. \quad (1)$$

For the adjoint representation we use Latin indices  $k, l, m, \dots$

$$k, l, m, \dots = 1, \dots, 78. \quad (2)$$

For describing  $E_6$  by its maximal subgroup  $SU(3)_C \times SU(3)_L \times SU(3)_R$  we also use the  $SU(3)$  indices

$$\alpha, \beta, \gamma, \dots = 1, 2, 3. \quad (3)$$

$$a, b, c, \dots = 1, 2, 3. \quad (4)$$

$$p, q, r, \dots = 1, 2, 3. \quad (5)$$

$$A, B, C, \dots = 1, \dots, 8. \quad (6)$$

## 2.1 Transformation Law of the Fundamental Representation

With respect to its maximal subgroup  $SU(3)_C \times SU(3)_L \times SU(3)_R$ , the fundamental representation of  $E_6$  can be decomposed as

$$\mathbf{27} \rightarrow (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}), \quad (7)$$

and its adjoint representation can be decomposed as

$$\mathbf{78} \rightarrow (\mathbf{8}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{8}) + (\mathbf{3}, \mathbf{3}, \mathbf{3}) + (\bar{\mathbf{3}}, \bar{\mathbf{3}}, \bar{\mathbf{3}}). \quad (8)$$

First, the generators  $T^A$  of the three  $SU(3)$  subgroups corresponding to the first three representations in eq. (8) can be represented by the eight Gell-Mann matrices  $\lambda^A/2$  and their action onto the three matrices  $(L, M, N)$  according to eq. (7) can be expressed by

$$\begin{aligned} T_C^A (L, M, N) &= \left( \frac{1}{2}\lambda^A L, 0, -\frac{1}{2}N\lambda^A \right), \\ T_L^A (L, M, N) &= \left( -\frac{1}{2}L\lambda^A, \frac{1}{2}\lambda^A M, 0 \right), \\ T_R^A (L, M, N) &= \left( 0, -\frac{1}{2}M\lambda^A, \frac{1}{2}\lambda^A N \right). \end{aligned} \quad (9)$$

Second, there are the generators  $T^{\alpha\alpha p}$  and  $\bar{T}^{\alpha\alpha p}$  that mediate shifts between the matrices

$L, M, N$

$$\begin{aligned}
T_{\alpha ap} L_\beta^b &= \varepsilon_{\alpha\beta\gamma} \delta_a^b N_p^\gamma, & \bar{T}^{\alpha ap} L_\beta^b &= -\varepsilon^{abc} \delta_\beta^\alpha M_c^p, \\
T_{\alpha ap} M_b^q &= \varepsilon_{abc} \delta_p^q L_\alpha^c, & \bar{T}^{\alpha ap} M_b^q &= -\varepsilon^{pqr} \delta_b^a N_r^\alpha, \\
T_{\alpha ap} N_q^\beta &= \varepsilon_{pqr} \delta_\alpha^\beta M_\alpha^r, & \bar{T}^{\alpha ap} N_q^\beta &= -\varepsilon^{\alpha\beta\gamma} \delta_q^p L_\gamma^a.
\end{aligned} \tag{10}$$

$\delta_a^b$  is the Kronecker symbol and  $\varepsilon^{abc}$  the Levi-Civita symbol with  $\varepsilon_{123} = \varepsilon^{123} = 1$ . With these set of generators an infinitesimal, unitary  $E_6$  transformation reads

$$U(u, v, w, x, y) = \mathbf{1} + i u_A T_C^A + i v_A T_L^A + i w_A T_R^A + i x_{\alpha ap} T^{\alpha ap} + i y_{\alpha ap} \bar{T}^{\alpha ap} + \dots \tag{11}$$

In total,  $u_A, v_A, w_A, x_{\alpha ap}, y_{\alpha ap}$  are 78 parameters.

## 2.2 Explicit Matrix Expressions for the Group Generators

We now aim at writing the transformation in eq. (11) with one set of 78 matrices  ${}_k T$  and parameters  $\varepsilon^k$  that act on a 27-dimensional vector  $\psi$

$$U(\varepsilon) \psi = (\mathbf{1} + i \varepsilon^k {}_k T + \dots) \psi. \tag{12}$$

For that purpose, we rearrange the transformation parameters and the matrices  $L, M, N$  into column vectors in the following way:

$$\begin{aligned}
\psi &= (L_{11}, L_{12}, \dots, M_{11}, \dots, N_{11}, \dots, N_{33})^T \\
\varepsilon &= (u^1, u^2, \dots, v^1, \dots, w^1, \dots, x^{111}, \dots, y^{111}, \dots, y^{333})^T.
\end{aligned} \tag{13}$$

By comparing coefficients in (11) and (12), the  $27 \times 27$  matrices  ${}_k T$  can be constructed. To adjust the normalisation and obtain Hermiticity, we perform the following change of basis

$$\begin{aligned}
\tilde{T} &= \frac{1}{2} ({}_k T + {}_{k+27} T), & 24 < k < 52 \\
\tilde{T} &= \frac{i}{2} ({}_k T - {}_{k+27} T), & 51 < k < 78.
\end{aligned} \tag{14}$$

The group generators are included in the `Mathematica` package as an  $78 \times 27 \times 27$  dimensional array called `E6gen`. They obey

$$\text{tr}({}_k T {}_l T) = 3 \delta_{kl} \tag{15}$$

$$\sum_{k=1}^{78} {}_k T {}_k T = \frac{26}{3} \mathbf{1}_{27}. \tag{16}$$

This sets the Dynkin index and the quadratic Casimir invariant to

$$C(27) = 3 \quad \text{and} \quad C_2(27) = \frac{26}{3}, \tag{17}$$

satisfying the well-known identity

$$C_2(R) = \frac{\dim(G)}{\dim(R)} C(R), \quad (18)$$

for a representation  $R$  and the adjoint representation  $G$ . In addition to this consistency check,  $C(27)$  and  $C_2(27)$  match the same values Kephart and Vaughn state in their paper [2].

By construction, the generators are ordered in the following way

$${}_1T \dots {}_8T : \quad SU(3)_C, \quad (19)$$

$${}_9T \dots {}_{16}T : \quad SU(3)_L, \quad (20)$$

$${}_{17}T \dots {}_{24}T : \quad SU(3)_R. \quad (21)$$

Therefore, the diagonal generators representing the Cartan subalgebra are

$${}_3T, {}_8T, {}_{11}T, {}_{16}T, {}_{19}T, {}_{24}T. \quad (22)$$

The generators  ${}_{25}T$  to  ${}_{78}T$  are the shifting operators defined in eq. (10). The structure constants  $f_{klm}$  of a Lie Algebra are defined by

$$[{}_kT, {}_lT] = i f_{klm} {}_mT. \quad (23)$$

Applying the normalisation condition gives

$$f_{klm} = -\frac{i}{C(27)} \text{tr}([{}_kT, {}_lT] {}_mT). \quad (24)$$

In the `Mathematica` package they are encoded in the array `E6f`. As a cross-check we calculated the normalisation to be

$$f_{kmn} f_{lmn} = 12 \delta_{kl} \quad (25)$$

which also matches the value in [2]. Since the structure constants are the generators of the adjoint representation, its quadratic Casimir and Dynkin index are

$$C(G) = C_2(G) = 12. \quad (26)$$

The symmetric structure constants  $C_{klm}$  are defined by

$$\{ {}_kT, {}_lT \} = {}_kT {}_lT + {}_lT {}_kT = i C_{klm} {}_mT. \quad (27)$$

This can be rewritten as

$$C_{klm} = -\frac{i}{C(27)} \text{tr}(\{ {}_kT, {}_lT \} {}_mT). \quad (28)$$

Explicit computation then yields

$$C_{klm} = 0 \quad \forall k, l, m = 1, \dots, 78. \quad (29)$$

Hence,  $E_6$  GUT models are in general free of chiral anomalies.

### 3 Higher Rank Tensors

The transformation laws for fundamental **27**, anti-fundamental  $\overline{\mathbf{27}}$  and adjoint representation **78** are as follows

$$\psi_\mu \rightarrow \psi_\mu + i \varepsilon^k ({}_k T)_\mu^\nu \psi_\nu, \quad (30)$$

$$\bar{\psi}^\mu \rightarrow \bar{\psi}^\mu - i \varepsilon^k ({}_k T)_\nu^\mu \bar{\psi}^\nu, \quad (31)$$

$$\phi_l \rightarrow \phi_l + \varepsilon^k f_{klm} \phi_m, \quad (32)$$

with Greek indices running from 1 to 27 and Latin indices running from 1 to 78, cf. eq. (1) and eq. (2).

#### 3.1 Higher Dimensional Representations

The representations **351**, **351'** and **650** are included in the tensor products  $[3]^2$

$$\overline{\mathbf{27}} \otimes \overline{\mathbf{27}} = \mathbf{27} \oplus \mathbf{351} \oplus \mathbf{351}', \quad (33)$$

$$\overline{\mathbf{27}} \otimes \mathbf{27} = \mathbf{1} \oplus \mathbf{78} \oplus \mathbf{650}. \quad (34)$$

Therefore, they can be represented as rank-two tensors. Their transformation properties are implicitly given by (30) and (31). For **351** and **351'**, we labelled the entries for that tensors  $\chi_1, \dots, \chi_{351}$  and choose them in a way that

$$\bar{X}^{\mu\nu} X_{\mu\nu} = \bar{\chi}^1 \chi_1 + \dots + \bar{\chi}^{351} \chi_{351}, \quad \mu, \nu = 1, \dots, 27, \quad (35)$$

ensuring a canonical normalisation of the kinetic term. A comment on this normalisation is given in Appendix B.

**351** can be represented by a second rank tensor  $A_{\mu\nu}$  antisymmetric in its indices. In the `Mathematica` package it is included as an  $27 \times 27$  dimensional array called **E6A**.

**351'** is also a rank two tensor  $S_{\mu\nu}$  but symmetric in its indices and additionally satisfying

$$d^{\mu\nu\lambda} S_{\mu\nu} = 0, \quad \forall \lambda = 1, \dots, 27, \quad (36)$$

with  $d^{\mu\nu\lambda}$  defined below in eq. (39). **351'** is named **E6S** in the package.

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<sup>2</sup>The notation for **351** and **351'** differs in the literature. In our notation, **351** is symmetric, whereas **351'** is anti-symmetric.

**650** has a fundamental and an anti-fundamental index with vanishing trace  $\psi_\mu^\mu = 0$  and

$${}_k T_\nu^\mu \psi_\mu^\nu = 0. \quad (37)$$

It is not (yet) included in the package due to its memory usage.

### 3.2 Invariants for the Fundamental Representation

The Kronecker symbol  $\delta_\nu^\mu$  is the most simple way to define a quadratic invariant of a fundamental and an anti-fundamental representation

$$\delta_\nu^\mu \bar{\psi}^\nu \psi_\mu. \quad (38)$$

A cubic invariant can be defined in the following way [2]

$$d^{\mu\nu\lambda} \psi_\mu \psi_\nu \psi_\lambda = \det(L + M + N) - \text{tr}(LMN). \quad (39)$$

The entries of the tensor  $d^{\mu\nu\lambda}$  can be obtained by comparing the coefficients of the field components on each side of the equation. It is provided as **E6d** in the **Mathematica** package. Together with  $d_{\mu\nu\lambda}$ , defined by

$$d_{\mu\nu\lambda} \bar{\psi}^\mu \bar{\psi}^\nu \bar{\psi}^\lambda = \det(L^\dagger + M^\dagger + N^\dagger) - \text{tr}(N^\dagger M^\dagger L^\dagger), \quad (40)$$

it is normalised to

$$d_{\mu\nu\lambda} d^{\mu\nu\sigma} = 10 \delta_\lambda^\sigma. \quad (41)$$

Additionally there are some compound tensors that are used to construct the invariants in Appendix A. A tensor with four indices is

$$D_{\mu\nu}^{\sigma\tau} = d_{\mu\nu\lambda} d^{\lambda\sigma\tau}. \quad (42)$$

In the package it is called **E6D**.

There is also a compound tensor carrying five indices:

$$D_\lambda^{\mu\nu,\sigma\tau} = (d^{\mu\sigma\xi} d^{\nu\tau\eta} - d^{\mu\tau\xi} d^{\nu\sigma\eta}) d_{\xi\eta\lambda}, \quad (43)$$

$$D_{\mu\nu,\sigma\tau}^\lambda = (d_{\mu\sigma\xi} d_{\nu\tau\eta} - d_{\mu\tau\xi} d_{\nu\sigma\eta}) d^{\xi\eta\lambda}. \quad (44)$$

They are not included in the package to avoid excessive memory usage.

### 3.3 Invariants for the Adjoint Representation

The normalisation condition for the generators can be used to define a quadratic invariant for the adjoint representation  $\phi$ .

$$\delta_{kl} \phi_k \phi_l = \frac{1}{3} \text{tr}({}_k T_l T) \phi_k \phi_l \quad (45)$$

With the structure constants one can form an invariant of three different adjoint representations  $\phi, \phi', \phi''$ :

$$f_{klm} \phi_k \phi'_l \phi''_m. \quad (46)$$

The completely symmetric tensor

$$\chi_{klmn}^5 = \delta_{kl}\delta_{mn} + \delta_{kl}\delta_{mn} + \delta_{kl}\delta_{mn} \quad (47)$$

in the package is called **E6chi** and can be used to form an quartic invariant  $\chi_{klmn}^5 \phi'_k \phi''_l \phi'''_m \phi''''_n$ .

### 3.4 Mixed Invariants

The generators  $({}_k T)_\mu^\nu$  form a tensor  ${}_k T_\mu^\nu$  with an adjoint, a fundamental and an anti-fundamental index.

Further, there is a tensor that can be constructed from the anti-commutator and Kronecker symbols

$${}_{kl} H_\mu^\nu = \{T_k, T_l\}_\mu^\nu - \frac{2}{9} \delta_{kl} \delta_\mu^\nu, \quad (48)$$

called **E6H** in the package.

Contracting  ${}_k T_\mu^\nu$  with  $d_{\nu\rho\lambda}$  gives the tensor

$${}_k A_{\mu\nu,\lambda} = {}_k T_\mu^\sigma d_{\sigma\nu\lambda} - {}_k T_\nu^\sigma d_{\mu\sigma\lambda} \quad (49)$$

which is antisymmetric w.r.t.  $\mu \leftrightarrow \nu$  and the tensor

$${}_k S_{\mu\nu,\lambda} = -{}_k T_\lambda^\sigma d_{\mu\nu\sigma} \quad (50)$$

which is symmetric w.r.t.  $\mu \leftrightarrow \nu$ . They are called **E6kA** and **E6kS**, respectively.

## 4 E6Tensors.m - A short Manual

### 4.1 Download and Installation

**E6Tensors** can be downloaded from <http://e6tensors.hepforge.org>. On that page, there are also some instructions on how to install it. Currently, there are two versions at the download section: **e6tensors\_full-1.0.0.tar.gz** and **e6tensors\_small-1.0.0.tar.gz**.

**e6tensors\_small-1.0.0.tar.gz** contains the following files: **install.sh** calls the command line version of **Mathematica**<sup>3</sup> and runs **create\_E6Tensors.m**. This script uses **E6gen.m** and **E6d.m** to create the higher dimensional tensors and saves them as arrays to **E6Tensors.m**. **examples.nb** shows some well-documented examples how **E6Tensors.m** can be used. Note that **E6Tensors.m** has a size of roughly 150 MB. Therefore make sure to provide enough RAM for

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<sup>3</sup>We assume it to be called **math**. Change that if it has another name on your system.

loading it. Running `create_E6Tensors.m` also may need some time. On a quad-core i5 machine this took about half an hour working on four subkernels. To use parallelisation, change `LaunchKernels[1]` in `create_E6Tensors.m` to the appropriate value.

For most users, we recommend to download `e6tensors_full-1.0.0.tar.gz`. After extracting the tarball, it is ready to use and contains all files of `e6tensors_small-1.0.0.tar.gz` including `E6Tensors.m`.

You probably will not need all tensors in a single project. In that case you can comment out the unnecessary tensors in `create_E6Tensors.m` and run it to get your own customised file `E6Tensors.m`.

## 4.2 Structure of the Package

`E6Tensors.m` is a simple text file. It contains the definitions of all tensors as nested lists. In this way, it is very flexible to use: You can write your own functions and procedures that fit the problem you want to solve. As an example, the Pauli matrices would look like

```
{{{0,1},{1,0}},{{0,-I},{I,0}},{{1,0},{0,-1}}}
```

The generators `E6gen` have exactly the same structure. Hence,

```
E6gen[[k,mu,nu]]
```

is the element in the  $\mu^{\text{th}}$  row and the  $\nu^{\text{th}}$  column of the  $k^{\text{th}}$  generator. All tensors are listed in Table 1. There, you can also find their symbolic name, the indices they carry and a short explanation.

For instance, `E6gen` has indices  $78, \overline{27}, 27$  referring to the gauge index  $k = 1, \dots, 78$ , the row index  $\mu = 1, \dots, 27$  and the column index  $\nu = 1, \dots, 27$ . The order follows the convention of the `Part[]` function in `Mathematica`. Keep in mind that `Mathematica` does not make any difference between row and column vectors.

## 4.3 Known Issues

It is not recommended to open `E6Tensors.m` via the graphical frontend of `Mathematica`. To load it use

```
Get["##/E6Tensors.m"]
```

instead, where `##` refers to the correct path.

## 4.4 Examples

In the download version, there is a notebook `examples.nb` that contains some possible applications of the package. It starts with loading `E6gen.m` and `E6d.m` which are sufficient for the first examples.

Name in [2]	Name in E6Tensors	Indices	Comment
${}_k T_\mu^\nu$	E6gen	78, $\overline{27}$ , 27	adjoint representation <b>78</b> $\text{tr}({}_k T {}_l T) = 3 \delta_{kl}$
$A_{\mu\nu}$	E6A	27, 27	antisymmetric <b>351</b> 27 × 27 matrix with entries labelled $\chi_1 \dots \chi_{351}$
$S_{\mu\nu}$	E6S	27, 27	symmetric <b>351'</b> $d^{\mu\nu\sigma} S_{\nu\sigma} = 0$ 27 × 27 matrix with entries labelled $\chi_1 \dots \chi_{351}$
$d_{\mu\nu\lambda}$	E6d	27, 27, 27	fully symmetric invariant $d_{\mu\nu\lambda} d^{\mu\nu\sigma} = 10 \delta_\lambda^\sigma$
$f_{klm}$	E6f	78, 78, 78	structure constants: $[{}_k T, {}_l T] = i f_{klm} {}_m T$ $f_{kmn} f_{lmn} = 12 \delta_{kl}$
$D_{\mu\nu}^{\sigma\tau}$	E6D	$\overline{27}$ , $\overline{27}$ , 27, 27	$D_{\mu\nu}^{\sigma\tau} = \delta_\mu^\sigma \delta_\nu^\tau + \delta_\nu^\tau \delta_\mu^\sigma$
$\chi_{klmn}^5$	E6chi	78, 78, 78, 78	$\chi_{klmn}^5 = \delta_{kl} \delta_{mn} + \delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm}$
${}_{kl} H_\mu^\nu$	E6H	27, $\overline{27}$ , 78, 78	${}_{kl} H_\mu^\nu = \{T_k, T_l\}_\mu^\nu - \frac{2}{9} \delta_{kl} \delta_\mu^\nu$
${}_k A_{\mu\nu, \lambda}$	E6kA	78, 27, 27, 27	antisymmetric in $\mu, \nu$ ${}_k A_{\mu\nu, \lambda} = {}_k T_\mu^\sigma d_{\sigma\nu\lambda} - {}_k T_\nu^\sigma d_{\mu\sigma\lambda}$
${}_k S_{\mu\nu, \lambda}$	E6kS	78, 27, 27, 27	symmetric in $\mu, \nu$ ${}_k S_{\mu\nu, \lambda} = -{}_k T_\lambda^\sigma d_{\mu\nu\sigma}$

Table 1: Overview of Tensors in E6Tensors.m.

We identify the Standard Model generators among the  $E_6$  generators: By construction, we can choose the gluons to be the first eight generators. The generators of  $SU(2)_L$  can then be defined as

$$T^9, T^{10}, T^{11}, \quad (51)$$

and hypercharge as

$$Y = \sqrt{\frac{3}{5}} \left( -\sqrt{\frac{1}{3}} T^{16} - T^{19} - \sqrt{\frac{1}{3}} T^{24} \right). \quad (52)$$

There are two additional U(1) charges, which we can define by

$$Y' = \sqrt{\frac{1}{40}} \left( -2\sqrt{3} T^{16} - T^{19} + 3\sqrt{3} T^{24} \right), \quad (53)$$

$$Y'' = \sqrt{\frac{1}{40}} \left( -2\sqrt{3} T^{16} + 4T^{19} - 2\sqrt{3} T^{24} \right). \quad (54)$$

In this basis, there is a singlet in the fundamental representation for  $Y'$  and  $Y''$  each.

The generators for  $SU(2)_R$  can be defined as

$$T^{17}, T^{18}, T^{19}, \quad (55)$$

and  $B - L$  as

$$B - L = -\sqrt{\frac{1}{2}} (T^{16} + T^{24}) = \sqrt{\frac{5}{2}} Y + \sqrt{\frac{3}{2}} T^{19}. \quad (56)$$

As a first check, we can write the fundamental representation as a list of field names and show their quantum numbers in a table. We also check for the GUT normalisation of the  $U(1)$  charges and the correct commutation relation for  $SU(2)_L$ ,  $SU(2)_R$  and  $Y$ .

In a next step, we use  $d^{\mu\nu\lambda}$  to write down the trilinear coupling in the superpotential

$$\mathcal{W} = \frac{\lambda}{6} d^{\mu\nu\rho} \psi_\mu \psi_\nu \psi_\rho. \quad (57)$$

For instruction, we once write out the explicit sum over all indices and once use the `Dot[]` operator. In many cases, the latter one will be the faster way to do it. The same holds for functions like `TensorContract[]`.

For the next examples `E6Tenors.m` must be located in the same directory. We first test the tensors for the higher dimensional representations **351** and **351'**, i.e. their defining properties

$$d^{\mu\nu\lambda} S_{\mu\nu} = 0, \quad S_{\mu\nu} = S_{\nu\mu}, \quad (58)$$

and

$$A_{\mu\nu} = -A_{\nu\mu}. \quad (59)$$

The normalisation of the kinetic terms gives the wanted result.

The couplings to matter fields can be described by  $\mathcal{W} = \bar{S}^{\mu\nu} \psi_\mu \psi_\nu$  and  $\mathcal{W} = \bar{A}^{\mu\nu} \psi_\mu \psi_\nu$ . Now, we can read off the fields that couple to down quarks. Since  $A^{\mu\nu}$  is anti-symmetric, it does not

couple fields of the same representation (e.g. the same flavour) to each other.

For a more advanced example, we discuss possible vacuum expectation values (VEVs) for **351'** ( $S^{\mu\nu}$ ) and **351** ( $A^{\mu\nu}$ ). Since they are contained in the tensor product  $\mathbf{27} \otimes \mathbf{27}$ , we can write an infinitesimal  $E_6$  transformation as

$$S^{\mu\nu} \rightarrow (\delta_\rho^\mu + i\alpha_k {}_k T_\rho^\mu) (\delta_\sigma^\nu + i\alpha_k {}_k T_\sigma^\nu) S^{\rho\sigma} = S^{\mu\nu} + i\alpha_k ({}_k T_\nu^\sigma S_{\mu\sigma} + {}_k T_\mu^\rho S_{\rho\nu}). \quad (60)$$

$\alpha_k$  is a set of parameters. For a VEV  $s^{\mu\nu} = \langle S^{\mu\nu} \rangle$  that transforms trivially under a set of generators  $\{{}_k T\}$ , the last term in (60) must vanish. Using the permutation symmetry of  $S^{\mu\nu}$ , this can be written as a matrix equation

$$({}_k T \cdot s)^T + {}_k T \cdot s = 0. \quad (61)$$

$A^{\mu\nu}$  is antisymmetric, therefore the condition reads

$$({}_k T \cdot a)^T - {}_k T \cdot a = 0. \quad (62)$$

The conditions are implemented in `example.nb`. As set of generators we use the gluons and

$$Q = I_L^3 + \sqrt{\frac{5}{3}} Y. \quad (63)$$

This ensures that the vacuum does not carry electric charge or colour. We then calculate the resulting mass terms that are generated by this VEV.

## 5 Acknowledgements

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## A Renormalisable Potentials

For completeness, we write down all possible renormalisable and gauge invariant terms that may occur in superpotentials and Lagrangians. They can also be found in [2].

## Mass Terms

Possible mass terms for the various representations are:

$$\begin{aligned}
\overline{27} \cdot 27 & : \quad \bar{\psi}^\mu \psi_\mu \\
78^2 & : \quad \text{tr}({}_k T_l T) \phi_k \phi'_l = 3 \phi_k \phi'_k \\
\overline{351} \cdot 351 & : \quad \bar{A}^{\mu\nu} A_{\mu\nu} \\
\overline{351'} \cdot 351' & : \quad \bar{S}^{\mu\nu} S_{\mu\nu}
\end{aligned}$$

## Cubic Terms

$$\begin{aligned}
27^3 & : \quad d^{\mu\nu\lambda} \psi_\mu \psi_\nu \psi_\lambda \\
351'^3 & : \quad d^{\mu\nu\lambda} d^{\sigma\tau\rho} S_{\mu\sigma} S_{\nu\tau} S_{\lambda\rho} \\
27^2 \cdot \overline{351'} & : \quad \psi_\mu \psi_\nu \bar{S}^{\mu\nu} \\
351'^2 \overline{27} & : \quad D_\lambda^{\mu\nu, \sigma\tau} A_{\mu\nu} A_{\sigma\tau} \bar{\psi}^\lambda \\
27 \cdot 351 \cdot 78 & : \quad {}_k T_\mu^\tau d^{\mu\sigma\lambda} \psi_\lambda A_{\sigma\tau} \phi_k \\
78^3 & : \quad f_{klm} \phi_k \phi'_l \phi''_m \\
27 \cdot \overline{27} \cdot 78 & : \quad {}_k T_\mu^\nu \bar{\psi}^\mu \psi_\nu \phi_k \\
351 \cdot \overline{351} \cdot 78 & : \quad {}_k T_\mu^\nu \bar{A}^{\mu\lambda} A_{\nu\lambda} \phi_k \\
351' \cdot 351' \cdot 78 & : \quad {}_k T_\mu^\nu \bar{S}^{\mu\lambda} S_{\nu\lambda} \phi_k
\end{aligned}$$

## Quartic Terms

For the fundamental and adjoint representations, there are

$$\begin{aligned}
27^2 \cdot \overline{27}^2 & : \quad \bar{\psi}^\mu \psi_\mu \bar{\psi}^\nu \psi_\nu \\
& \quad D_{\sigma\tau}^{\mu\nu} \bar{\psi}^\sigma \bar{\psi}^\tau \psi_\mu \psi_\nu \\
78^4 & : \quad (\phi_k \phi_k)^2 \\
& \quad (\phi_l \phi_l)^2 (\phi'_k \phi'_k)^2 \\
& \quad (\phi_l \phi'_l)^2 (\phi_k \phi'_k)^2 \\
& \quad \phi_k \phi_l \phi_m \phi_n \text{tr}(\{{}_k T_{,l} T\} \{{}_m T_{,n} T\}) \\
& \quad \chi_{klmn}^5 \phi_k \phi_l \phi_m \phi_n \\
27 \cdot \overline{27} \cdot 78^2 & : \quad \bar{\psi}^\mu \psi_\mu \phi_k \phi_k \\
& \quad ({}_{kl} H_\mu^\nu) \bar{\psi}^\mu \psi_\nu \phi_k \phi_l
\end{aligned}$$

Including **351** and **351'** gives

$$\begin{aligned}
\mathbf{351}^2 \overline{\mathbf{351}}^2 : & \quad (A_{\mu\nu} \bar{A}^{\mu\nu})^2 \\
& \quad A_{\mu\nu} \bar{A}^{\nu\sigma} A_{\sigma\tau} \bar{A}^{\tau\mu} \\
& \quad d^{\mu\nu\lambda} d_{\xi\eta\lambda} A_{\mu\sigma} A_{\nu\tau} \bar{A}^{\xi\sigma} \bar{A}^{\eta\tau} \\
& \quad d^{\mu\nu\alpha} d^{\sigma\tau\beta} d_{\xi\eta\alpha} d_{\lambda\rho\beta} A_{\mu\sigma} \bar{A}_{\nu\tau} A^{\xi\lambda} \bar{A}^{\eta\rho} \\
& \quad d_{\mu\nu\alpha} d^{\sigma\beta\gamma} d_{\xi\eta\beta} d_{\lambda\alpha\gamma} A_{\mu\sigma} A_{\nu\tau} \bar{A}_\xi^\lambda \bar{A}^{\eta\tau} \\
\mathbf{351}'^2 \overline{\mathbf{351}'}^2 : & \quad (A_{\mu\nu} \bar{A}^{\mu\nu})^2 \\
& \quad S_{\mu\nu} \bar{S}^{\nu\sigma} S_{\sigma\tau} \bar{S}^{\tau\mu} \\
& \quad d^{\mu\nu\lambda} d_{\xi\eta\lambda} S_{\mu\sigma} S_{\nu\tau} \bar{S}^{\xi\sigma} \bar{S}^{\eta\tau} \\
& \quad d^{\mu\nu\alpha} d^{\sigma\tau\beta} d_{\xi\eta\alpha} d_{\lambda\rho\beta} S_{\mu\sigma} \bar{S}_{\nu\tau} S^{\xi\lambda} \bar{S}^{\eta\rho} \\
\mathbf{351} \cdot \overline{\mathbf{351}} \cdot \mathbf{78}^2 : & \quad \bar{A}^{\mu\nu} A_{\mu\nu} \phi_k \phi_k \\
& \quad ({}_{kl}H_\mu^\nu) \bar{A}^{\mu\lambda} A_{\nu\lambda} \phi_k \phi_l \\
& \quad ({}_{kl}H_\mu^\nu) d^{\mu\sigma\alpha} d_{\nu\tau\alpha} \bar{A}^{\tau\lambda} A_{\sigma\lambda} \phi_k \phi_l \\
& \quad ({}_kT_\mu^\sigma) d^{\mu\lambda\alpha} ({}_lT_\tau^\nu) d_{\nu\rho\alpha} \bar{A}^{\rho\tau} A_{\alpha\lambda} \phi_k \phi_l \\
\mathbf{351}' \cdot \overline{\mathbf{351}'} \cdot \mathbf{78}^2 : & \quad \bar{S}^{\mu\nu} S_{\mu\nu} \phi_k \phi_k \\
& \quad ({}_{kl}H_\mu^\nu) \bar{S}^{\mu\lambda} S_{\nu\lambda} \phi_k \phi_l \\
& \quad ({}_{kl}H_\mu^\nu) d^{\mu\sigma\alpha} d_{\nu\tau\alpha} \bar{S}^{\tau\lambda} S_{\sigma\lambda} \phi_k \phi_l \\
\mathbf{27} \cdot \overline{\mathbf{27}} \cdot \mathbf{351} \cdot \overline{\mathbf{351}} : & \quad \bar{\psi}^\mu \psi_\mu \bar{A}^{\sigma\tau} A_{\sigma\tau} \\
& \quad \bar{\psi}^\mu \psi_\nu \bar{A}^{\nu\tau} A_{\mu\tau} \\
& \quad d_{\mu\nu\lambda} d^{\xi\eta\lambda} \bar{\psi}^\mu \psi_\xi \bar{A}^{\nu\tau} A_{\eta\tau} \\
\mathbf{27} \cdot \overline{\mathbf{27}} \cdot \mathbf{351}' \cdot \overline{\mathbf{351}'} : & \quad \bar{\psi}^\mu \psi_\mu \bar{S}^{\sigma\tau} S_{\sigma\tau} \\
& \quad \bar{\psi}^\mu \psi_\nu \bar{S}^{\nu\tau} S_{\mu\tau} \\
\mathbf{27}^2 \mathbf{351}^2 : & \quad d^{\mu\sigma\xi} d^{\nu\tau\eta} \psi_\mu \psi_\nu A_{\sigma\tau} A_{\xi\eta} \\
& \quad d^{\mu\sigma\alpha} d^{\nu\xi\beta} d_{\alpha\beta\gamma} d^{\gamma\tau\eta} \psi_\mu \psi_\nu A_{\sigma\tau} A_{\xi\eta} \\
\mathbf{27}^2 \mathbf{351}'^2 : & \quad d^{\mu\sigma\xi} d^{\nu\tau\eta} \psi_\mu \psi_\nu S_{\sigma\tau} S_{\xi\eta} \\
\mathbf{351}^3 \mathbf{78} : & \quad ({}_kT_\rho^\eta) d^{\mu\sigma\alpha} d^{\nu\tau\beta} d_{\alpha\beta\gamma} d^{\gamma\xi\rho} A_{\mu\nu} A_{\sigma\tau} A_{\xi\eta} \phi_k
\end{aligned}$$

## B On the Normalisation of **351'**

The symmetric tensor **351'** ( $S_{\mu\nu}$ ) is defined by

$$S_{\nu\mu} = S_{\mu\nu} \quad \text{and} \quad d^{\mu\nu\lambda} S_{\mu\nu} = 0 \quad \forall \lambda = 1, \dots, 27. \quad (64)$$

The first condition is easy to construct: We label the off-diagonal entries  $\phi_1, \dots, \phi_{351}$  and the diagonal ones  $\phi_{352}, \dots, \phi_{378}$ . The second condition then eliminates 27 entries. For  $\lambda = 1$ , it

reads

$$\phi_{122} + \phi_{207} + \phi_{226} + \phi_{244} - \phi_{102} = 0. \quad (65)$$

It is now tempting to solve e.g. for  $\phi_{102}$  and substitute that in  $S_{\mu\nu}$ . But then the kinetic term  $\partial^\alpha S^{\mu\nu} \partial_\alpha S_{\mu\nu}$  is not canonically normalised.<sup>4</sup> Another solution is to introduce new field names  $\psi_1, \psi_2, \psi_3, \psi_4$  with

$$\phi_{102} = a(\psi_1 + \psi_2 + \psi_3 + \psi_4) \quad (66)$$

$$\phi_{122} = \psi_1 - b(\psi_1 + \psi_2 + \psi_3 + \psi_4) \quad (67)$$

$$\phi_{207} = \psi_2 - b(\psi_1 + \psi_2 + \psi_3 + \psi_4) \quad (68)$$

$$\phi_{226} = \psi_3 - b(\psi_1 + \psi_2 + \psi_3 + \psi_4) \quad (69)$$

$$\phi_{244} = \psi_4 - b(\psi_1 + \psi_2 + \psi_3 + \psi_4) \quad (70)$$

For  $a = 1 - 4b$  and  $b = (5 + \sqrt{5})/20$ , the defining condition is fulfilled and the kinetic term for  $S^{\mu\nu}$  takes the form

$$\partial_\alpha \bar{S}^{\mu\nu} \partial^\alpha S_{\mu\nu} = \dots + \partial_\alpha \bar{\psi}_1 \partial^\alpha \psi_1 + \partial_\alpha \bar{\psi}_2 \partial^\alpha \psi_2 + \partial_\alpha \bar{\psi}_3 \partial^\alpha \psi_3 + \partial_\alpha \bar{\psi}_4 \partial^\alpha \psi_4 + \dots \quad (71)$$

The same procedure also works for all other values of  $\lambda$ . It is important, that the component with the relative minus sign ( $\phi_{102}$  in eq. (65)) is replaced by the expression with  $a$  in it. This procedure is implemented in `create_E6Tensors.m` and used to construct **E6S**.

## References

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<sup>4</sup> $\alpha$  is a space-time index in this case.