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Leading QCD-induced four-loop contributions to the β-function of the Higgs self-coupling in the SM and vacuum stability

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Abstract: We present analytical results for the leading top-Yukawa and QCD contribution to the β-function for the Higgs self-coupling λ of the Standard Model at four-loop level, namely the part $\propto y_t^4 g_s^6$. We also give the contribution $\propto y_t^2 g_s^6$ of the anomalous dimension of the Higgs field as well as the terms $\propto y_t g_s^8$ to the top-Yukawa β-function which can also be derived from the anomalous dimension of the top quark mass. Together with the recently computed top-Yukawa and QCD contributions to β_{g_s} [1, 2] these constitute the leading four-loop contributions to the evolution of the Higgs self-coupling. A numerical estimate of these terms at the scale of the top-quark mass is presented as well as an analysis of the impact on the evolution of λ up to the Planck scale and the vacuum stability problem.

Keywords: Renormalization Group, Standard Model, QCD

Contents

1 Introduction

The evolution of the Higgs self-coupling and of the Higgs field are important ingredients for the Renormalization Group (RG) improved Higgs potential and the study of vacuum stability. A precise determination of the Higgs self-coupling in the Standard Model extended up to the Planck scale is important because this parameter is close to zero at the Planck scale and the question whether the SM vacuum state is stable or not can only be answered definitively by reducing the uncertainties.[1] The largest source of uncertainty is the experimentally measured top mass M_t. At a future linear e^+e^- collider it could, however, be measured with a precision which matches that of the theory input to the vacuum stability analysis (see Fig. 5 in [12]).

On the theory side there are three sources of uncertainty. The first is the difference between the effective Higgs potential $V_{\text{eff}}(\Phi_{\text{cl}})$ [18] and the approximation of the RG-improved potential

$$V_{\text{RG}}(\Phi_{\text{cl}}) := \lambda(t)\left[\Phi_{\text{cl}}\cdot\exp\left(-\frac{1}{2}\int_0^t \mathrm{d}t'\gamma_\Phi(t')\right)\right]^4,\quad t := \ln\left(\frac{\Phi_{\text{cl}}^2}{\mu_0^2}\right), \tag{1.1}$$

where $\Phi_{\text{cl}} = \langle 0|\Phi|0\rangle$ is the classical field strength of the scalar SU(2) doublet Φ, γ_Φ the anomalous dimension of Φ and μ_0 the scale where we start the evolution of fields and

[1]During the last years many detailed studies of the vacuum stability issue in the SM have been performed [3–16]. A recent extension to the MSSM can be found in [17].

couplings, e. g. $\mu_0 = M_t$. This uncertainty is negligible at large values of $\Phi_{\rm cl}$, e. g. close to the Planck scale [19–23]. In this approximation the SM vacuum is stable up to the scale $\Lambda \sim M_{\rm Planck}$ if $\lambda(\mu) > 0$ for $\mu \leq \Lambda$.

Another source of theoretical uncertainties is the matching of experimental parameters, e. g. $M_t, \alpha_s(M_Z), M_H$, to the parameters of the SM Lagrangian, $y_t(\mu_0), g_s(\mu_0), \lambda(\mu_0), \ldots$ at some initial scale μ_0 renormalized in the $\overline{\rm MS}$-scheme. State of the art is the full numerical two-loop matching [15, 24]. In order to improve precision here three-loop calculations might be attempted and different mass definitions than the pole mass could be used for the top mass.

In this paper we improve on the third source of uncertainty, namely we increase the precision in the β-functions, calculated in the $\overline{\rm MS}$-scheme. The β-function for a coupling X is defined as

$$\beta_X(X, X_1, X_2, \ldots) = \mu^2 \frac{dX}{d\mu^2} = \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n} \beta_X^{(n)} \tag{1.2}$$

and the anomalous dimension of a field f as

$$\gamma_2^f(X, X_1, X_2, \ldots) = -\mu^2 \frac{d\ln Z_f^{-1}}{d\mu^2} = \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n} \gamma_2^{f\,(n)}, \tag{1.3}$$

where Z_f is the field strength renormalization constant, where $X, X_1, X_2, \ldots$ are the couplings of the theory which we want to include in the analysis.

The RG functions of the SM were computed at three-loop accuracy during the last years [9, 25–31]. The four-loop β-function for the strong coupling g_s was first computed in pure QCD [32, 33] and recently extended to the gaugeless limit of the SM, namely to include the dependence on the top-Yukawa coupling y_t and the Higgs self-coupling λ [1, 2].

The paper is structured as follows: In the next section we briefly describe the technical details of the calculation. Then the leading four-loop terms for β_λ, β_{y_t} and γ_2^Φ are given and the relevance of the four-loop terms numerically determined at the scale of the top quark mass. Finally we investigate the impact of the new contributions on the evolution of λ in order to estimate the uncertainty reduction due to four-loop β-functions.

2 Technicalities

2.1 The Model: QCD plus minimal top-Yukawa contributions

For this calculation we start with the SM Lagrangian in the broken phase where

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \to \begin{pmatrix} \Phi^+ \\ \frac{1}{\sqrt{2}}(v + H - i\chi) \end{pmatrix}. \tag{2.1}$$

The UV renormalization constants in the $\overline{\rm MS}$-scheme do not depend on masses and are the same as in the unbroken phase. Hence we can use all renormalization constants determined up to three-loop level in previous calculations [9] and set all masses to zero. There is no γ_5 in the $t\bar{t}H$-vertex $\propto y_t$ as opposed to the $t\bar{t}\Phi_2$-vertex of the unbroken phase. γ_5 now only appears in the Yukawa vertices with χ and $\Phi^{\pm}$. As at low scales (where we start the evolution of couplings and fields) the strong coupling is the largest we take as the leading contribution to the vertex and self-energy corrections those where H only appears as an external field. This means that no Higgs or Goldstone propagators appear. The electroweak gauge-couplings as well as λ and all Yukawa couplings except y_t are neglected. For the top-Yukawa vertex and the top self-energy these are the pure QCD corrections (see Fig. 1). For the Higgs self-energy and the quartic Higgs-vertex these are gluon insertions into the one-loop diagram $\propto y_t^2$ (see Fig. 3 (a)) and $\propto y_t^4$ (see Fig. 2 (a)) as well as diagrams with two fermion loops (see Fig. 3 (b) and Fig. 2 (b)). Thus we get the four-loop contributions which are numerically most significant to the evolution of λ avoiding γ_5 and its treatment in $D \neq 4$ dimensions completely.

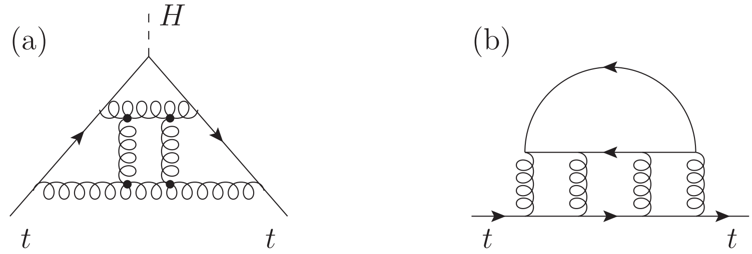

Figure 1: Some diagrams contributing to the Yukawa correction (a) and the top quark self-energy (b)

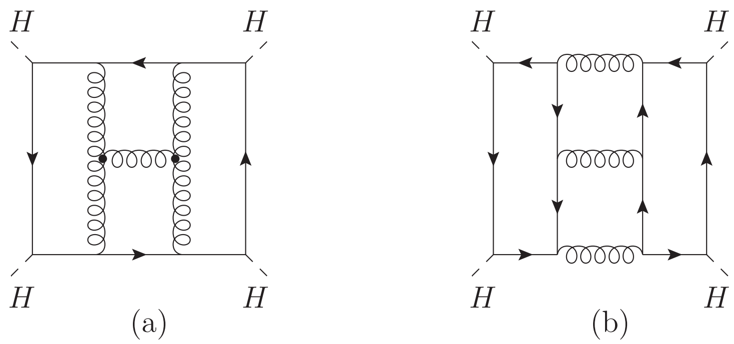

Figure 2: Some diagrams contributing to the quartic Higgs self-interaction

We compute the field strength renormalization constants $Z_2^{(HH)}$ from the Higgs and $Z_2^{(\bar{t}t)}$ from the top self-energies. The quartic Higgs-vertex is renormalized with $Z_1^{(4H)}$ and the top-Yukawa-vertex with $Z_1^{(\bar{t}tH)}$.[2]

[2]In the notation of [9] these renormalization constants are $Z_2^{(HH)} = Z_2^{(2\Phi)}$, $Z_2^{(\bar{t}t)} = Z_2^{(2t)}$, $Z_1^{(4H)} = Z_1^{(4\Phi)}$ and $Z_1^{(\bar{t}tH)} = Z_1^{(tb\Phi)}$.

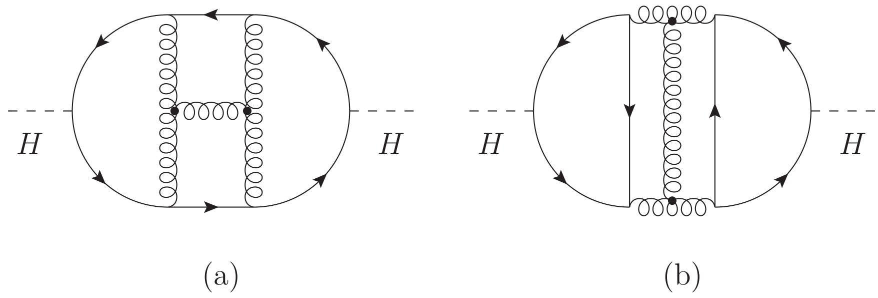

Figure 3: Some diagrams contributing to the Higgs self-energy

>From these we compute

$$\delta Z_\lambda = \frac{\lambda - \delta Z_1^{(4H)}}{\left(Z_2^{(HH)}\right)^2} - \lambda \tag{2.2}$$

and

$$Z_{y_t} = \frac{Z_1^{(\bar{t}tH)}}{Z_{2,}^{(\bar{t}t)}\sqrt{Z_2^{(HH)}}}. \tag{2.3}$$

All divergent integrals are regularized in $D = 4 - 2\varepsilon$ space time dimensions and the renormalization constants are defined as $Z = 1 + \delta Z$ in the $\overline{\text{MS}}$-scheme.

2.2 Calculation with massive tadpole integrals

For the computation of the four-loop terms we use the setup described in detail in [2]. The generation of all necessary Feynman diagrams was done with QGRAF [34]. The C++ programs Q2E and EXP [35, 36] are then used to identify the topology of the diagram. The Taylor expansion in external momenta, the fermion traces and the insertion of counterterms in lower loop diagrams was performed with FORM [37, 38]. All colour factors were computed with the FORM package COLOR [39].

In the momentum space part of the diagrams we introduce the same auxiliary mass parameter M^2 in every propagator denominator. The self-energy diagrams are then expanded to second order in the external momentum q after applying a projector $\propto \not{q}$ to the top self-energy diagrams and taking the trace over the external fermion line. Then we divide by q^2 before q is set to zero. In all vertex correction diagrams we can set $q \to 0$ from the beginning. This is allowed as $\overline{\text{MS}}$ renormalization constants do not depend on external momenta. After this we are left with tadpole integrals. Subdivergences $\propto M^2$ are canceled by counterterms

$$\frac{M^2}{2} \delta Z_{M^2}^{(2g)} \, A_\mu^a A^{a\,\mu} \tag{2.4}$$

computed from and inserted in lower loop diagrams. This is the same method for computing UV renormailzation constants as in our previous calculations [2, 9, 28]. It was first introduced in [40] and then further developed in [41]. A detailed explanation of the calculation of Z-factors with an auxiliary mass can be found in [13].

Up to three-loop order the tadpole integrals were computed with the FORM-based package MATAD[42]. The four-loop tadpoles are reduced to Master integrals using FIRE [43, 44]. The needed four-loop Master integrals can be found in [33].

3 Analytical Results

In this section we give our results which can be found in machine readable format on
http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp16/ttp16-008/
For a gerneric SU(N_c) gauge group the colour factors are expressed through the quadratic Casimir operators C_F and C_A of the fundamental and the adjoint representation of the corresponding Lie algebra. The dimension of the fundamental representation is called d_R. The adjoint representation has dimension n_g and the trace T_F is defined by $T_F\delta^{ab} = \mathbf{Tr}\left(T^aT^b\right)$ with the group generators T^a of the fundamental representation. Higher order invariants are constructed from the symmetric tensors

$$\begin{aligned} d_F^{abcd} = \frac{1}{6}\mathrm{Tr}\Big(&T^aT^bT^cT^d + T^aT^bT^dT^c + T^aT^cT^bT^d \\ +\ &T^aT^cT^dT^b + T^aT^dT^bT^c + T^aT^dT^cT^b\Big)\,. \end{aligned} \tag{3.1}$$

of the generators of the fundamental representation and analogously d_A^{abcd} from the generators of the adjoint representation. The combinations needed and their SU(N_c) values are

$$d_F^{abcd}d_F^{abcd} = n_g\left(\frac{N_c^4 - 6N_c^2 + 18}{96N_c^2}\right), \quad d_F^{abcd}d_A^{abcd} = n_g\left(\frac{N_c(N_c^2+6)}{48}\right). \tag{3.2}$$

Furthermore for SU(N_c) we have

$$d_R = N_c, \quad T_F = \frac{1}{2}, \quad C_F = \frac{N_c^2-1}{2N_c}, \quad C_A = N_c, \quad n_g = N_c^2 - 1. \tag{3.3}$$

The number of active fermion flavours is denoted by n_f. The leading four-loop contributions to the β-functions for the Higgs self-coupling and the top-Yukawa coupling are found to be

$$
\begin{aligned}
\beta_\lambda^{(4)} =& y_t^4 g_s^6\, d_R \,\Big(-\frac{2942}{3} C_F^3 - 64 T_F C_F^2 + \frac{3584}{3} C_A C_F^2 + \frac{5888}{9} C_A T_F C_F \\
& - \frac{121547}{243} C_A^2 C_F + \frac{562}{3} n_f T_F C_F^2 - \frac{256}{9} n_f T_F^2 C_F - \frac{2644}{243} n_f C_A T_F C_F \\
& + \frac{10912}{243} n_f^2 T_F^2 C_F + 160\zeta_5 C_F^3 + 720\zeta_5 C_A C_F^2 - 160\zeta_5 C_A T_F C_F \\
& - 520\zeta_5 C_A^2 C_F + 288\zeta_4 C_F^3 + 32\zeta_4 C_A C_F^2 - 88\zeta_4 C_A^2 C_F \\
& - 160\zeta_4 n_f T_F C_F^2 + 128\zeta_4 n_f C_A T_F C_F + 48\zeta_3 C_F^3 - \frac{3304}{3}\zeta_3 C_A C_F^2 \\
& + 352\zeta_3 C_A T_F C_F + \frac{1880}{3}\zeta_3 C_A^2 C_F + \frac{32}{3}\zeta_3 n_f T_F C_F^2 + 16\zeta_3 n_f C_A T_F C_F \\
& - \frac{128}{3}\zeta_3 n_f^2 T_F^2 C_F \Big) \\
& + \mathcal{O}(y_t^6) + \mathcal{O}(\lambda) + \mathcal{O}(g_2) + \mathcal{O}(g_1)
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
\beta_{y_t}^{(4)} =& y_t g_s^8 \Bigg(32 \frac{d_F^{abcd} d_A^{abcd}}{d_R} + \frac{1261}{8} C_F^4 - \frac{15349}{12} C_A C_F^3 + \frac{34045}{36} C_A^2 C_F^2 - \frac{70055}{72} C_A^3 C_F \\
& - 64 n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R} + \frac{280}{3} n_f T_F C_F^3 + \frac{8819}{27} n_f C_A T_F C_F^2 + \frac{65459}{162} n_f C_A^2 T_F C_F \\
& - \frac{304}{27} n_f^2 T_F^2 C_F^2 - \frac{1342}{81} n_f^2 C_A T_F^2 C_F + \frac{664}{81} n_f^3 T_F^3 C_F - 440\zeta_5 C_A^2 C_F^2 \\
& + 440\zeta_5 C_A^3 C_F + 480\zeta_5 n_f T_F C_F^3 - 80\zeta_5 n_f C_A T_F C_F^2 - 400\zeta_5 n_f C_A^2 T_F C_F \\
& + 264\zeta_4 n_f C_A T_F C_F^2 - 264\zeta_4 n_f C_A^2 T_F C_F - 96\zeta_4 n_f^2 T_F^2 C_F^2 + 96\zeta_4 n_f^2 C_A T_F^2 C_F \\
& - 240\zeta_3 \frac{d_F^{abcd} d_A^{abcd}}{d_R} + 336\zeta_3 C_F^4 - 316\zeta_3 C_A C_F^3 + 152\zeta_3 C_A^2 C_F^2 \\
& - \frac{1418}{9}\zeta_3 C_A^3 C_F + 480\zeta_3 n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R} - 552\zeta_3 n_f T_F C_F^3 - 368\zeta_3 n_f C_A T_F C_F^2 \\
& + \frac{2684}{3}\zeta_3 n_f C_A^2 T_F C_F + 160\zeta_3 n_f^2 T_F^2 C_F^2 - 160\zeta_3 n_f^2 C_A T_F^2 C_F - \frac{128}{9}\zeta_3 n_f^3 T_F^3 C_F \Bigg) \\
& + \mathcal{O}(y_t^3) + \mathcal{O}(\lambda) + \mathcal{O}(g_2) + \mathcal{O}(g_1).
\end{aligned}
\tag{3.5}
$$

The leading contribution to the anomalous dimension of the Higgs field (or equivalently the scalar SU(2) doublet Φ) at four-loop level is given by

$$
\begin{aligned}
\gamma_2^{\phi\,(4)} =& y_t^2 g_s^6\, d_R \left(\frac{4651}{12}C_F^3 - \frac{1282}{3}C_A C_F^2 + \frac{267889}{972}C_A^2 C_F - 125 n_f T_F C_F^2\right.\\
&-\frac{631}{243}n_f C_A T_F C_F - \frac{6500}{243}n_f^2 T_F^2 C_F - 360\zeta_5 C_F^3 - 180\zeta_5 C_A C_F^2\\
&+180\zeta_5 C_A^2 C_F - 108\zeta_4 C_F^3 - 78\zeta_4 C_A C_F^2 + 66\zeta_4 C_A^2 C_F\\
&+96\zeta_4 n_f T_F C_F^2 - 72\zeta_4 n_f C_A T_F C_F + 232\zeta_3 C_F^3 + 518\zeta_3 C_A C_F^2\\
&\left.-\frac{950}{3}\zeta_3 C_A^2 C_F + 16\zeta_3 n_f T_F C_F^2 - \frac{160}{3}\zeta_3 n_f C_A T_F C_F + \frac{64}{3}\zeta_3 n_f^2 T_F^2 C_F\right)\\
&+\mathcal{O}(y_t^4) + \mathcal{O}(\lambda) + \mathcal{O}(g_2) + \mathcal{O}(g_1).
\end{aligned}
\tag{3.6}
$$

For the lower loop contributions we refer to [9, 28–31].

4 Comparison with available QCD results

All diagrams discussed in the previous section are special in one aspect: they comprise just the minimal number of non-QCD vertexes, that is one for β_{y_t}, two for γ_2^{Φ} and four for β_λ. Even more, these non-QCD vertexes are of one and the same type, namely the insertion of the *scalar* top-quark current $\bar{t}\,t$. This means that the corresponding anomalous dimensions should be related to some RG functions in pure QCD describing the QCD evolution of the scalar current(s). The corresponding "effective" Lagrangian

$$
\mathcal{L}_{eff} = \mathcal{L}_{QCD} - \frac{y_t}{\sqrt{2}}H\bar{t}t - \frac{\lambda}{4}H^4 \tag{4.1}
$$

implies, obviously, the following identities **valid in all orders** in α_s (dots below stand for terms which have a different dependence on the SM coupling constants than $y_t\,\alpha_s^n$ (first line) and $y_t^2\,\alpha_s^n$ (second line) correspondingly):

$$
\beta_{y_t} = y_t\,\gamma_m(\alpha_s) \ + \ \ldots, \tag{4.2}
$$

$$
\gamma_2^{\Phi} = \frac{y_t^2}{2}\,\gamma_q^{SS}(\alpha_s) \ + \ \ldots. \tag{4.3}
$$

Here $\gamma_m(\alpha_s)$ is the quark mass anomalous dimension and the function $\gamma_q^{SS}(\alpha_s)$ appears in the evolution equation

$$
\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\Pi^S = -2\gamma_m\,\Pi^S + \gamma_q^{SS}Q^2 + \gamma_m^{SS}m_t^2
$$

for the scalar correlator (B marking bare quantities)

$$
\Pi_B^S(Q^2, m_t) = i\int \mathrm{d}^D x\, e^{iqx}\langle 0|T\,[j_s^B(x) j_s^B(0)]\,|0\rangle, \quad j_s^B = \bar{t}_B t_B, \quad Q^2 = -q^2.
$$

which is renormalized as

$$
\Pi^S(q^2, m_t) = Z_m^2\Pi_B^S(q^2, m_t^B) + \left(Z_2^{SS}Q^2 + Z_m^{SS}m_t^2\right)\mu^{-2\varepsilon}. \tag{4.4}
$$

The anomalous dimensions are found to be

$$\gamma_m = -\frac{d\ln Z_m}{d\ln\mu^2}, \tag{4.5}$$

$$\gamma_q^{SS} = \frac{dZ_2^{SS}}{d\ln\mu^2} + (2\gamma_m - \varepsilon)\, Z_2^{SS}, \tag{4.6}$$

$$\gamma_m^{SS} = \frac{dZ_m^{SS}}{d\ln\mu^2} + (4\gamma_m - \varepsilon)\, Z_m^{SS}. \tag{4.7}$$

Needless to say that a comparison with available results for $\gamma_m(\alpha_s)$ [45, 46] and $\gamma_q^{SS}(\alpha_s)$ [47] confirms the relations (4.2) and (4.3) at four-loop order[3].

Finally, let us consider in some detail the last (and somewhat more complicated) case, viz. the β-function for the Higgs self-coupling β_λ. The corresponding renormalization constant coincides up to a factor $\frac{y_t^4}{4\cdot 4!}$ to that which renormalizes the (1PI) Green's function of the T-product of **four** scalar currents:

$$(2\pi)^D \delta\,(p_1+p_2+p_3+p_4)\;\Gamma(\{p_i\},\alpha_s,m_t,\mu) \tag{4.8}$$

$$= Z_m^4\, Z_2^2 \int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_4 \left(\prod_i e^{ip_i\cdot x_i}\right) \langle 0|T\,[j_s(x_1)\,j_s(x_2)\,j_s(x_3)\,j_s(x_4)]\,|0\rangle. \tag{4.9}$$

With $m_t \neq 0$ we could nullify all external momenta p_i in eq. (4.9) and, thus, consider all j_s-operators on the rhs of (4.9) as insertions of the scalar quark current *at zero momentum* transfer.

As is well-known such insertions can be generated by multiple differentiations of QCD Green functions wrt a quark mass[4]. This means that the corresponding anomalous dimensions should be related to some pure QCD RG functions. The well-known way to construct the corresponding relations is to use the (renormalized) Quantum Action Principle [54, 55]. Let us briefly outline the main points.

The Quantum Action Principle relates properties of (regularized) Lagrangian and the full Green's functions. Consider the generating functional of (connected) Green's functions

$$W(J) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\, G^{(n)}(x_1,\ldots,x_n)\, J(x_1)\ldots J(x_n) \tag{4.10}$$

defined in

$$Z(\mathcal{L},J) = e^{iW(J)} = \int \mathcal{D}\,\Phi e^{iS(\Phi)+\int\Phi\cdot J\mathrm{d}^4x}, \quad S(\Phi) = \int \mathcal{L}(\Phi)\mathrm{d}^4x. \tag{4.11}$$

The Action Principle states (in particular) that

$$\frac{\partial}{\partial\lambda}W(J) \equiv \left(\int \mathcal{D}\,\Phi e^{iS(\Phi)+\int\Phi\cdot J\mathrm{d}^4x}\frac{\partial}{\partial\lambda}S(\Phi)\right)/Z(\mathcal{L},J), \tag{4.12}$$

[3] In fact, both quantities are currently also known to **five** loops from [48, 49].

[4] For the Higgs decay via heavy top loops such relations have been known as *low-energy* theorems for a long time [50–53]

where λ is a any parameter in the Lagrangian $\mathcal{L}$. The action principle works for DR Green functions [56] (modulo axial anomalies).

An example: the (renormalized) QCD Lagrangian with n_l massless quarks and a massive (top) one is customarily written as

$$\mathcal{L}_{QCD} = -\frac{1}{4}Z_3\,(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}g\,Z_1^{3g}\,(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\,(A_\mu \times A_\nu)^a$$
$$- \frac{1}{4}g^2\,Z_1^{4g}\,(A_\mu \times A_\nu)^2 + Z_2\sum_{i=1}^{n_l}\bar{\psi}_i\,(\mathrm{i}\partial\!\!\!/ + gZ_1^{\psi\psi g}Z_2^{-1}A\!\!\!/)\,\psi_i \tag{4.13}$$
$$+ Z_2\,\bar{t}\,(\mathrm{i}\partial\!\!\!/ + gZ_1^{\psi\psi g}Z_2^{-1}A\!\!\!/ - Z_m m_t)\,t.$$

With properly chosen renormalization constants Z_i (4.13) should produce *finite* Green's functions. If one differentiates $W(J)$ wrt the (renormalized) top quark mass m_t the functional

$$\left(\int \mathcal{D}\,\Phi e^{iS(\Phi)+\Phi\cdot J}\left(\int Z_m Z_2\,\bar{t}t(x)dx\right)\right)/G(\mathcal{L},J) \tag{4.14}$$

should also be finite. However, let us consdider eq. (4.14) at $J = 0$. It corresponds, obviously, to the VEV of the operator $Z_m Z_2\,\bar{t}t(x)$, which is *not* finite already at order α_s^0 (a couple of typical diagrams contributing to (4.14) at $J = 0$ are shown on Fig. 4).

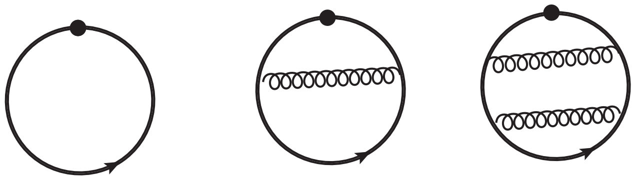

Figure 4: Sample diagrams contributing to the VEV of $Z_m Z_2\,\bar{t}t(x)$

This means that our QCD Lagrangian (4.13) is not *full*: the term responsible for the renormalization of the *vacuum energy* is missing. The full QCD Lagrangian reads [57]

$$\mathcal{L}^{full}_{QCD} = \mathcal{L}_{QCD} - E_0^B, \quad E_0^B = \mu^{-2\epsilon}(E_0(\mu) - Z_0(\alpha_s)m_t^4(\mu)), \qquad \epsilon = (4-D)/2, \tag{4.15}$$

here $E_0(\mu)$ is the (renormalized) vacuum energy and Z_0 is the corresponding renormalization constant.

Now, a four-fold differentiation of the generating functional (4.11) (with $\mathcal{L} = \mathcal{L}^{full}_{QCD}$) wrt m_t immediately leads us to the conclusion that the combination

$$\Gamma(\{p_i\}, \alpha_s, m_t, \mu) - i\,4!Z_0 \tag{4.16}$$

should be finite. As a result, we arrive at the following identity **valid in all orders** in α_s:

$$\beta_\lambda = y_t^4\,\gamma_0(\alpha_s)\ +\ \ldots, \tag{4.17}$$

where the dots stand for terms which have a different dependence on the SM coupling constants than $y_t^4\,\alpha_s^n$ and[5]

$$\gamma_0 = (4\gamma_m - \varepsilon)\, Z_0 + (\beta - \varepsilon)\,\alpha_s \frac{\partial Z_0}{\partial \alpha_s} \tag{4.18}$$

is the anomalous dimension of the vaccuum energy

$$\frac{dE_0}{d\ln\mu^2} = \gamma_0(\alpha_s)\, m_t^4. \tag{4.19}$$

The vacuum anomalous dimension plays an important role in the description of the renormalization mixing of all three scalar gauge-invariant operators with (mass) dimension four:

$$O_1 = -\frac{1}{4}(G_{\mu\nu})^2, \;\; O_2 = m_t \bar{t}\, t, \;\; O_3 = m_t^4, \tag{4.20}$$

$$\mu^2 \frac{d}{d\mu^2} O_i = \sum_{j=1}^{3} \gamma_{ij} O_j. \tag{4.21}$$

It was proven in [57, 58] that the matrix of anomalous dimensions in (4.21) reads:

$$\gamma_{ij} = \left(\mu^2 \frac{d}{d\mu^2} Z_{ik}\right)\left(Z^{-1}\right)_{kj} = \begin{pmatrix} -\alpha_s \frac{\partial\beta}{\partial\alpha_s} & -\alpha_s \frac{\partial\gamma_m}{\partial\alpha_s} & -\alpha_s \frac{\partial\gamma_0}{\partial\alpha_s} \\ 0 & 0 & -4\gamma_0 \\ 0 & 0 & 4\gamma_m \end{pmatrix}. \tag{4.22}$$

In addition, the two-point scalar correlator (4.3) at $q = 0$ is obviosly related (via the action principle) to the four-point correlator (4.9), as the latter can be obtained from the former by a double differentiation wrt m_t. This, in turn, leads to the following remarkable relation (again valid in all orders in α_s):

$$\gamma_0 = \frac{1}{12}\gamma_m^{SS}. \tag{4.23}$$

Thus, one could compute γ_0 in a few different ways.

1. Direct renormalization of the vacuum energy diagrams. This was done for two and three loops in the papers [57] and [53] respectively. At foor loops it was first found in this way for a space-time dimension $D = 3$ [59, 60] (in the process of computing the free energy in the effective high temperature QCD) and (implicitly, via eq. (4.17)) in the present paper.

2. By renormalizing the 4-loop scalar correlator [61] in the limit of small quark mass[6]. The result was later used in [62] to compute the quartic mass corrections to R_{had} at $\mathcal{O}(\alpha_s^3)$.

3. By computing the lowest moment of the scalar correlator at 4 loops [63].

4. By computing the lowest low-energy moment of the axial-vector correlator (related via a Ward identity to the VEV of the scalar current) [64].

[5] We define the QCD β-function as $\beta(\alpha_s) = \frac{d\alpha_s}{d\ln\mu^2}$.

[6] That is the scalar corelator was expanded at the large momentum limit and the γ_m^{SS} was found by renormalizing the term of order m_t^2/Q^2.

Note, finally, that the last two evaluations dealt with massive tadpole diagrams and that all calculations of γ_0 at the four-loop level described above are in mutual agreement as well as the two results for $\beta_\lambda^{(4)}$: the one displayed in (3.4) and the one obtained via relation (4.17).

5 Numerical analysis

For the experimental values $M_t \approx 173.34 \pm 0.76$ GeV [65], $M_H \approx 125.09 \pm 0.24$ GeV[66] and $\alpha_s(M_Z) = 0.1184 \pm 0.0007$ [67] we get the couplings in the $\overline{\text{MS}}$-scheme at the scale of the top mass using two-loop matching relations [15]

$$
\begin{aligned}
g_s(M_t) &= 1.1666 \pm 0.0035(\text{exp}), \\
y_t(M_t) &= 0.9369 \pm 0.0046(\text{exp}) \pm 0.0005(\text{theo}), \\
\lambda(M_t) &= 0.1259 \pm 0.0005(\text{exp}) \pm 0.0003(\text{theo}), \\
g_2(M_t) &= 0.6483, \\
g_1(M_t) &= 0.3587,
\end{aligned} \tag{5.1}
$$

where the experimental uncertainty (exp) stems from M_t, M_H and $\alpha_s(M_Z)$ and the theoretical one (theo) from the matching of on-shell to $\overline{\text{MS}}$ parameters [15]. Evaluating β_{y_t} at the scale M_t we find

$$
\begin{aligned}
\frac{\beta_{y_t}^{(1)}(\mu = M_t)}{(16\pi^2)} &= -2.4 \times 10^{-2}, \\
\frac{\beta_{y_t}^{(2)}(\mu = M_t)}{(16\pi^2)^2} &= -2.9 \times 10^{-3}, \\
\frac{\beta_{y_t}^{(3)}(\mu = M_t)}{(16\pi^2)^3} &= -1.2 \times 10^{-4}, \\
\frac{\beta_{y_t}^{(4)}(\mu = M_t)}{(16\pi^2)^4} &= +6.0 \times 10^{-6},
\end{aligned} \tag{5.2}
$$

which shows the expected suppression of higher order contributions. The four-loop terms are significantly smaller than the lower loop terms. For β_λ however the picture is different. Already in previous works [9, 28] we found a slow convergence of the perturbation series up to three-loop order at the electroweak scale and this is also found true at four-loop level:

$$
\begin{aligned}
\frac{\beta_{\lambda}^{(1)}(\mu = M_t)}{(16\pi^2)} &= -1.0 \times 10^{-2}, \\
\frac{\beta_{\lambda}^{(2)}(\mu = M_t)}{(16\pi^2)^2} &= +2.4 \times 10^{-5}, \\
\frac{\beta_{\lambda}^{(3)}(\mu = M_t)}{(16\pi^2)^3} &= +1.1 \times 10^{-5}, \\
\frac{\beta_{\lambda}^{(4)}(\mu = M)_t}{(16\pi^2)^4} &= +1.3 \times 10^{-5}.
\end{aligned} \tag{5.3}
$$

At higher scales, where g_s and y_t become smaller the convergence of the perturbative series is of course better. At $\mu = 10^9\text{GeV}$ we find

$$\begin{aligned}
\frac{\beta_\lambda^{(1)}(\mu = 10^9\text{GeV})}{(16\pi^2)} &= -1.34 \times 10^{-3}, \\
\frac{\beta_\lambda^{(2)}(\mu = 10^9\text{GeV})}{(16\pi^2)^2} &= +3.17 \times 10^{-7}, \\
\frac{\beta_\lambda^{(3)}(\mu = 10^9\text{GeV})}{(16\pi^2)^3} &= +3.05 \times 10^{-7}, \\
\frac{\beta_\lambda^{(4)}(\mu = 10^9\text{GeV})}{(16\pi^2)^4} &= +6.83 \times 10^{-8},
\end{aligned} \tag{5.4}$$

where the leading four-loop contribution is a factor $4-5$ smaller than the three-loop result but still not completely negligible. We will now check what this means for the evolution of the Higgs self-coupling λ up to the Planck scale.

6 Evolution of λ and vacuum stability

We evolve λ from the scale M_t using the initial conditions (5.1) and the full SM β-functions (including $g_s, y_t, g_2, g_1, \lambda$) up to three-loop order and at four-loop level $\beta_{g_s}^{(4)}(g_s, y_t, \lambda)$ [1, 2] and the leading contributions to $\beta_{y_t}^{(4)}(g_s, y_t)$ and $\beta_\lambda^{(4)}(g_s, y_t)$, as given in (3.5) and (3.4). Fig. 5 shows the result compared to the largest remaining uncertainties on the theory and on the experimental side. The smaller error band marks the 1σ uncertainty stemming from the top matching, i.e. we vary y_t by ± 0.0005 (see (5.1)). The larger error band marks the 1σ uncertainty stemming from the measured top mass (assuming that this parameter is close to the top pole mass).

The difference between the evolution of λ with three-loop β-functions and including the leading four-loop terms should give some indication on the uncertainty stemming from the truncation of the perturbative series for the β-functions. In order to see this difference we have to zoom in. We choose to do this at the scale where λ becomes negative, which is shown in Fig. 6

The conlusion for vacuum stability remains the same as in our previous works [9, 10, 12–14]. It looks as if λ becomes negative at $\log_{10}\left(\frac{\mu}{\text{GeV}}\right) \sim 9.8$ rendering the SM not stable if extended up to scales above, but a definitive answer is pending on a more precise extraction of $y_t(\mu_0)$ from experimental data. It is worth noting, however, that due to the reduction in the top mass uncertainty since the combined LHC and TEVATRON analysis [65] a stable SM up to the Planck scale is strongly disfavoured.

7 Conclusions

In this work we have presented analytical results for the leading four-loop contributions to the β-function for the Higgs self-coupling λ and the top-Yukawa coupling as well as to the

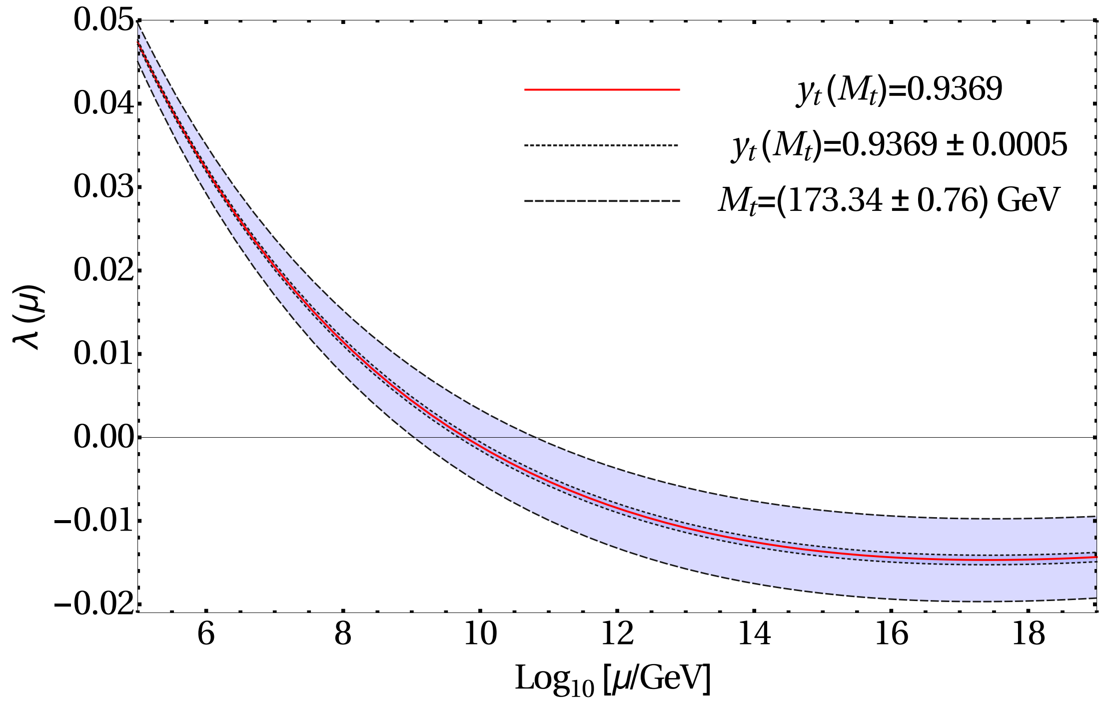

Figure 5: Evolution of λ: top matching and top measurement uncertainties

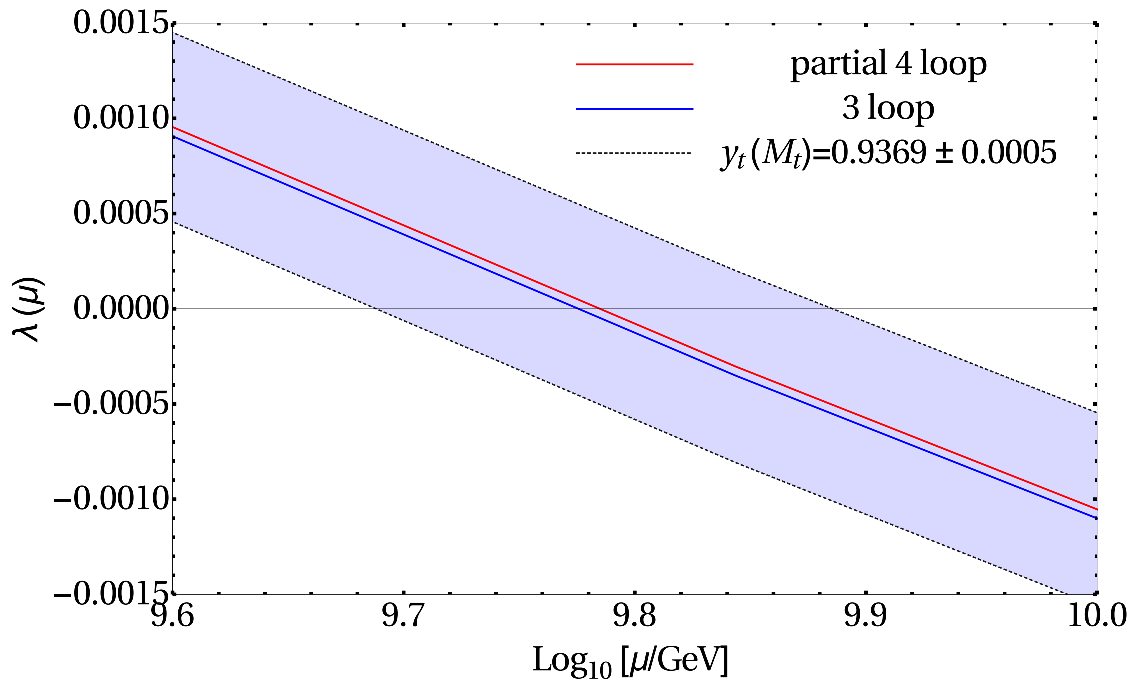

Figure 6: Evolution of λ: 3 loop and partial 4 loop β-functions, top matching uncertainty

anomalous dimension of the Higgs field. We have performed an analysis of the evolution of the Higgs self-coupling updating the analyses presented in previous works [9, 10, 12–14] and establishing a nice hirarchy between the diefferent sources of uncertainty.

With the computation of the leading four-loop terms to $\beta_\lambda, \beta_{y_t}$ and β_{g_s} the β-function uncertainty to the question of vacuum stability becomes significantly smaller than the matching uncertainty (before the two were comparable) which is in turn significantly smaller than the experimental top mass uncertainty. We expect that a full calculation of four-loop β-functions in the SM will confirm this conclusion rendering the remaining β-function uncertainty almost negligible in comparison to the other sources of uncertainty for a vacuum stability analysis.

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