TTP14-030,
SFB/CPP-14-87

Standard Model beta-functions to three-loop order and vacuum stability

Max F. Zoller [a,1],

[a] Institut für Theoretische Teilchenphysik (TTP), Karlsruhe Institute for Technology

1: max.zoller@kit.edu

Abstract

Since the discovery of a Higgs particle [1, 2] the effective Higgs potential of the Standard Model or extensions and the stability of the ground state corresponding to its minimum at the electroweak scale have been subject to a lot of investigation. The vacuum expectation value of the scalar SU(2) doublet field in the Standard Model, which is responsible for the masses of elementary particles, may in fact not be at the global minimum of the effective Higgs potential. The question whether there is a deeper minimum at some large scale is closely linked to the behaviour of the running quartic Higgs self-interaction $\lambda(\mu)$. In this talk an update on the analysis of the evolution of this coupling is given. We use three-loop beta-functions for the Standard Model couplings, two-loop matching between on-shell and $\overline{\text{MS}}$ quantities and compare the theoretical precision achieved in this way to the precision in the latest experimental values for the key parameters.

1 Introduction: The stability of the Standard Model ground state

In the Standard Model (SM) of particle physics fermions interact via the exchange of gauge bosons. The strength of these interactions is given by the coupling constants g_s for the QCD part and g_2, g_1 for the electroweak part. Furthermore, a scalar SU(2) doublet $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ is introduced which couples to the SU(2) gauge bosons via the coupling g_2 and to the fermions via Yukawa couplings, the top-Yukawa coupling y_t being the strongest. The quartic self-coupling λ of the field Φ appears in the classical Higgs potential

$$V(\Phi) = m^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi\right)^2 . \tag{1}$$

For $m^2 < 0$ the doublet Φ aquires a vacuum expectation value (VEV) $\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$ in the minimum of the classical Higgs potential (Fig. 1 (a)). The masses of the quarks, leptons, massive gauge bosons and the Higgs boson are then proportional to the value $v \approx 246.2$ GeV [3].

If we assume the SM to be valid up to the Planck scale $\Lambda \sim 10^{19}$ GeV – a reasonable scenario in the absence of new physics – we have to include quantum corrections which change the shape of the effective Higgs potential [4] significantly as compared to the classical potential at high scales.

The effective potential is best introduced in the path integral approach to quantum field theory. Consider the generating functional $W[J]$ defined by the path integral

$$e^{i\,W[J]} := \int \mathrm{d}\Phi \mathrm{d}A\, e^{i\int \mathrm{d}^4x[\mathcal{L}(\Phi(x),\partial_\mu\Phi(x),A(x),\partial_\mu A(x))+J(x)\Phi(x)]} = \langle 0^+|0^-\rangle|_J \tag{2}$$

which describes the transition from the vacuum state at $t \to -\infty$ to the one at $t \to +\infty$ in the presence of the external current $J = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}$ coupling to the scalar doublet Φ. All other fields in the Lagrangian $\mathcal{L}$ are denoted by A. We can eliminate the external current J by a Legendre transformation introducing the effective action

$$\Gamma[\Phi_{\mathrm{cl}}] := W[J] - \int \mathrm{d}^4x J(x)\Phi_{\mathrm{cl}}(x) \tag{3}$$

and the classical field strength

$$\Phi_{\mathrm{cl}}(x) := \frac{\delta W[J]}{\delta J(x)} = \left.\frac{\langle 0^+|\Phi(x)|0^-\rangle}{\langle 0^+|0^-\rangle}\right|_J . \tag{4}$$

The effective Higgs potential V_{eff} is a function of Φ_{cl} and can be defined as the first term in an expansion of the effective action around the point where all fields have zero momentum:

$$\Gamma[\Phi_{\mathrm{cl}}] = \int \mathrm{d}^4x \left(-V_{\mathrm{eff}}(\Phi_{\mathrm{cl}}) + \frac{1}{2}(\partial_\mu\Phi_{\mathrm{cl}})^2 Z(\Phi_{\mathrm{cl}}) + \ldots\right). \tag{5}$$

The so-defined effective potential, which in general depends on all parameters of the theory, contains two main pieces of information. On the one hand the nth derivative wrt Φ_{cl} gives the effective strength of the interaction of n external scalar fields, e.g.

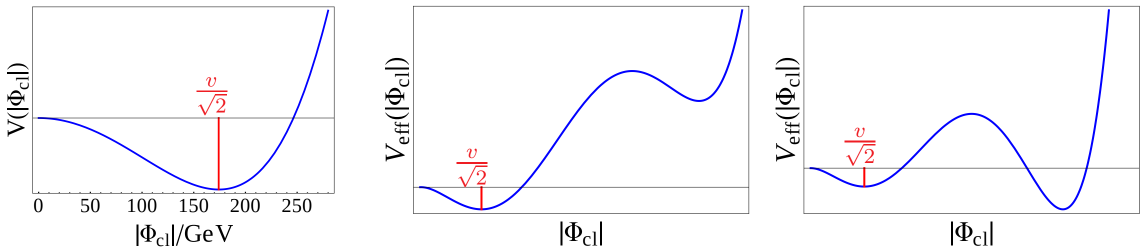
$$\frac{\mathrm{d}^2 V_{\mathrm{eff}}}{\mathrm{d}\Phi_{\mathrm{cl}}^2} = m_{\mathrm{eff}}^2, \qquad \frac{\mathrm{d}^4 V_{\mathrm{eff}}}{\mathrm{d}\Phi_{\mathrm{cl}}^4} = \lambda_{\mathrm{eff}}. \tag{6}$$

On the other hand the requirement

$$\frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}\Phi_{\mathrm{cl}}} = 0. \tag{7}$$

yields all the candidates for VEVs of the scalar field for $J = 0$, which correspond to local minima in the effective potential.

Generic shapes of this effective potential are shown in Fig. 1 for the cases of a Higgs mass larger (b) and smaller (c) than a critical value m_{min}, the minimal stability bound



(a) $\boxed{\text{tree level}}$ (b) $\boxed{\lambda(\mu) > 0 \, \forall \, \mu \leq \Lambda}$ (c) $\boxed{\exists \, \mu \leq \Lambda : \, \lambda(\mu) < 0}$

Figure 1: Classical and effective Higgs potential as a function of $|\Phi_{\rm cl}| := \sqrt{\Phi^\dagger \Phi}$.

(see also [5]). If the second minimum at large scales is deeper than the first at the electroweak scale the latter is not stable against tunneling to this global minimum[1].

For large field strengths $\Phi_{\rm cl} \sim \Lambda \gg v$ we can use the approximation [7]

$$V_{\rm eff}(\Phi_{\rm cl}) \approx \lambda(\Phi_{\rm cl}) \, \Phi_{\rm cl}^4 \left({\rm e}^{-\frac{1}{2}\int\limits_0^t {\rm d}t' \gamma_\Phi(t')} \right)^4 \tag{8}$$

with $t := \ln\left(\frac{\Phi_{\rm cl}^2}{v^2}\right)$ and the running coupling $\lambda(\mu)$ evolved to the scale $\mu = \Phi_{\rm cl}$. From this it has been demonstrated that the stability of the SM vacuum is in good approximation equivalent to the question whether the running coupling $\lambda(\mu)$ stays positive up to the scale Λ [7, 8, 9]. It is this requirement which will be investigated at high precision in this talk.

The vacuum stability problem has been subject to a lot of investigation over the last years [10, 11, 12, 13, 14, 15, 5, 16, 17, 18]. For a recent discussion of the vacuum stability problem in the MSSM see [19].

2 Calculations: β-functions and matching relations

The evolution of the Higgs self-coupling λ with the energy scale μ is given by the β-function

$$\beta_\lambda(\lambda, y_t, g_s, g_2, g_1, \ldots) = \mu^2 \frac{d}{d\mu^2} \lambda(\mu). \tag{9}$$

This power series in the couplings of the SM is computed in perturbation theory and is available up to three-loop order [13, 20, 21, 22] as well as the β-functions for the gauge [23, 24, 25] and Yukawa [13, 26] couplings, which are also needed in order to solve

[1] Note that the effective potential is a gauge dependent quantity (as are the renormalized field Φ and its anomalous dimension γ_Φ) and hence the exact location of the second minimum is also gauge dependent. The existence of a second minimum and the fact whether it is lower or higher then the first, however, does not depent on the gauge parameters. For a recent discussion of this topic see [6].

eq. (9) numerically. The second ingredient to the solution of eq. (9) is a set of initial conditions for each coupling, e.g. their values at the scale of the top mass M_t. These values are needed in the $\overline{\rm MS}$-scheme in which the β-functions are computed. Matching relations between experimentally accessible on-shell quantities, such as the top quark pole mass M_t and Higgs pole mass $M_{\rm H}$, and $\overline{\rm MS}$ parameters have been calculated at two-loop level [14, 27, 5, 28, 29, 30]. For the key parameters

$$M_{\rm t} = (173.34 \pm 0.76)\ \text{GeV [31]}, \tag{10}$$

$$M_{\rm H} = (125.7 \pm 0.4)\ \text{GeV [3]}, \tag{11}$$

$$\alpha_s^{\overline{\rm MS}}(M_Z) = 0.1185 \pm 0.0006\ \text{[3]} \tag{12}$$

we find the following best values for the $\overline{\rm MS}$ parameters:

$g_s(M_{\rm t})$	1.1671
$g_2(M_{\rm t})$	0.6483
$g_1(M_{\rm t})$	0.3587
$y_t(M_{\rm t})$	$0.9369 \pm 0.00050_{\rm (th,match)}$
$\lambda(M_{\rm t})$	$0.1272 \pm 0.00030_{\rm (th,match)}$

Table 1: SM couplings in the $\overline{\rm MS}$-scheme at $\mu = M_{\rm t}$, theoretical uncertainties for y_t and λ stem from the on-shell to $\overline{\rm MS}$ matching [14].

3 Analysis: The evolution of $\lambda(\mu)$

Applying three-loop β-functions for λ, y_t, g_s, g_2 and g_1 as well as the initial conditions from Tab. 2 we evaluate $\lambda(\mu)$ numerically up to $\mu = \Lambda \sim 10^{19}$ GeV.

Fig. 2 shows the results for one-loop, two-loop and three-loop β-functions which demonstrates the excellent convergence of the perturbation series. The difference between the two and three-loop curve can be taken as an estimate for the theoretical uncertainty stemming from the β-functions. From this we see that the SM ground state is no longer stable if we extend it to scales $\gtrsim 10^{10}$ GeV. This could be mended by some new physics appearing between this scale and the electroweak scale. On the other hand – as λ stays close to zero – the two minima of the effective Higgs potential are almost degenerate in energy which leads to a lifetime of the electroweak ground state much longer than the age of the universe [14, 5, 16, 17], and such a metastable scenario does not contradict our observations.

But in order to give a definitive answer to the question whether the SM is stable up to large scales we have to consider all sources of uncertainty. Apart from the small uncertainty stemming from the β-functions there is also a matching uncertainty of which the two main contributions, the initial value for λ and for y_t, are given in Tab. 2.

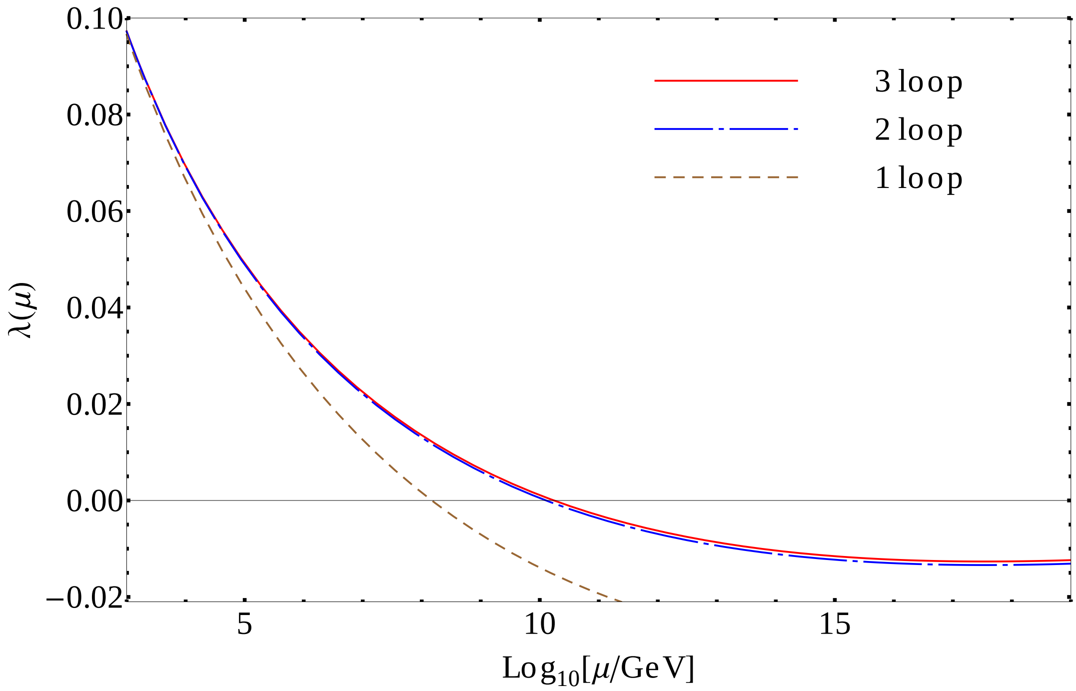


Figure 2: Evolution of λ: One-, two- and three-loop beta-functions

The effect of varying these two parameters by one σ_{matching} for the three-loop curve is shown in Fig. 3. From this we can estimate that the matching precision is comparable to the precision in the β-functions.

By comparison the experimental uncertainties are significantly larger. The dashed (dotted) lines in Fig. 4 show the behaviour of λ evolved using three-loop β-functions but with M_{t}, M_{H} and α_s increased (decreased) by one standard deviation.

While the uncertainties originating from α_s and M_{H} are approximately of the same size and at the Planck scale about a factor 2 larger than the difference between the two-loop and three-loop curves, the uncertainty stemming from the top mass measurement is about an order of magnitude larger than the theoretical one at $\mu \sim 10^{19}$ GeV. Within one σ we are clearly in the metastable scenario for the SM but the large uncertainty in the top mass measurement does not allow for a final answer to the question of vacuum stability. It is interesting to investigate the possibilities offered by a precision determination of the top mass, e.g. at the ILC, where an uncertainty of $\sigma_{M_{\text{t}}} \sim 30$ MeV is within reach [32]. Similarly, an uncertainty $\sigma_{M_{\text{t}}} < 100$ MeV is anticipated for CLIC [33]. In Fig. 5 the evolution of λ is shown for values of the top mass varied by the prospective $\sigma_{M_{\text{t}}} = 30$ MeV which leads to an experimental uncertainty for this parameter which would be competitive with the theory uncertainty for the evolution of the Higgs self-coupling.

Such a precision measurement of M_{t} or the appearance of new physics between the electroweak and the Planck scale will hopefully lead to an answer to the question of vacuum stability in the not too distant future.

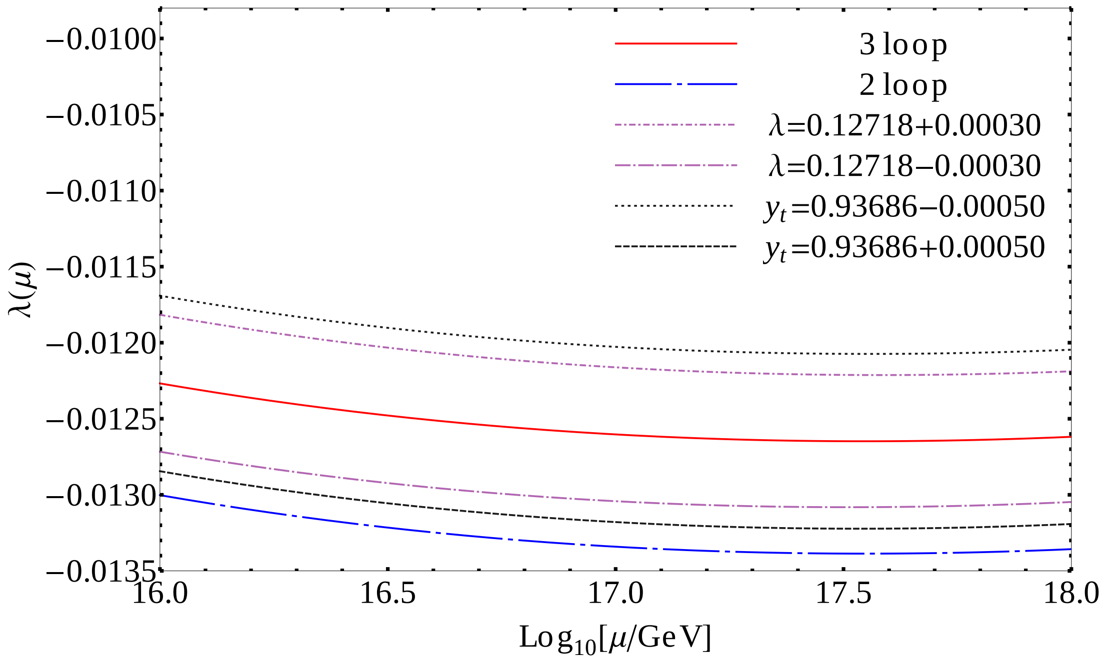


Figure 3: Evolution of λ: Matching uncertainties

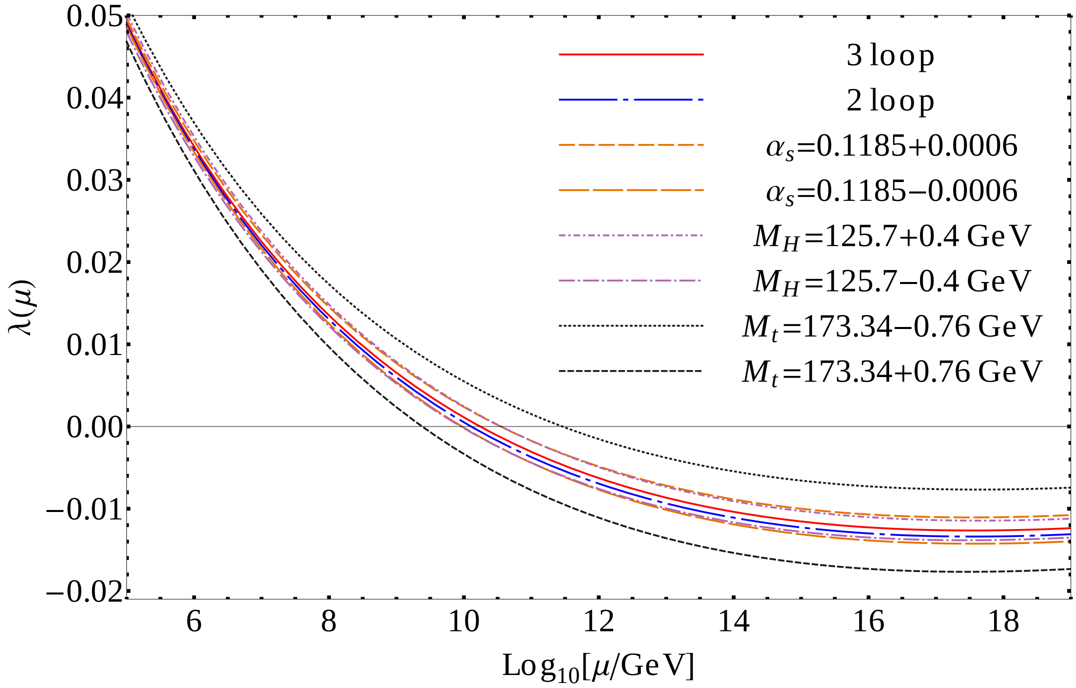


Figure 4: Evolution of λ: Experimental uncertainties

Acknowledgements

I thank K. G. Chetyrkin for his collaboration on the project and many invaluable discussions and J. H. Kühn for his support and useful comments. This work has

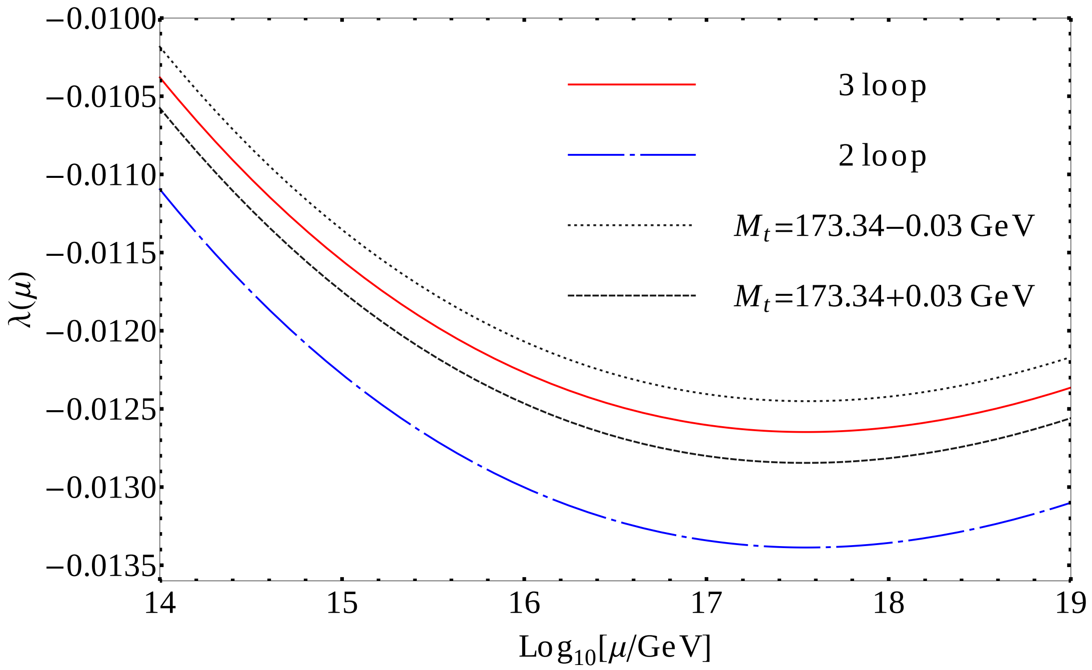


Figure 5: Evolution of λ: Top mass uncertainty at the ILC

been supported by the Deutsche Forschungsgemeinschaft in the Sonderforschungsbereich/Transregio SFB/TR-9 "Computational Particle Physics" and the Graduiertenkolleg 1694 "Elementarteilchenphysik bei höchsten Energien und höchster Präzission".

References

[1] **ATLAS** Collaboration, G. Aad *et al.*, *Combined search for the Standard Model Higgs boson using up to 4.9 fb-1 of pp collision data at sqrt(s) = 7 TeV with the ATLAS detector at the LHC*. Phys.Lett. **B710** (2012) 49–66, `arXiv:1202.1408 [hep-ex]`.

[2] **CMS** Collaboration, S. Chatrchyan *et al.*, *Combined results of searches for the standard model Higgs boson in pp collisions at $\sqrt{s} = 7$ TeV*. Phys.Lett. **B710** (2012) 26–48, `arXiv:1202.1488 [hep-ex]`.

[3] **Particle Data Group** Collaboration, K. Olive *et al.*, *Review of Particle Physics*. Chin.Phys. **C38** (2014) 090001.

[4] S. Coleman and E. Weinberg, *Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*. Phys. Rev. D **7** (1973) 1888–1910.

[5] F. Bezrukov, M. Y. Kalmykov, B. A. Kniehl, and M. Shaposhnikov, *Higgs Boson Mass and New Physics*. JHEP **1210** (2012) 140, `arXiv:1205.2893 [hep-ph]`.

[6] L. Di Luzio and L. Mihaila, *On the gauge dependence of the Standard Model vacuum instability scale*. arXiv:1404.7450 [hep-ph].

[7] G. Altarelli and G. Isidori, *Lower limit on the Higgs mass in the standard model: An update*. Physics Letters B **337** (1994) no. 1-2, 141–144.

[8] N. Cabibbo, L. Maiani, G. Parisi, and R. Petronzio, *Bounds on the Fermions and Higgs Boson Masses in Grand Unified Theories*. Nucl. Phys. **B158** (1979) 295–305.

[9] C. Ford, D. Jones, P. Stephenson, and M. Einhorn, *The Effective potential and the renormalization group*. Nucl. Phys. **B395** (1993) 17–34, arXiv:hep-lat/9210033 [hep-lat].

[10] M. Zoller, *Three-loop beta function for the Higgs self-coupling*. PoS **LL2014** (2014) 014, arXiv:1407.6608 [hep-ph].

[11] M. Zoller, *Beta-function for the Higgs self-interaction in the Standard Model at three-loop level*. PoS **(EPS-HEP 2013)** (2013) 322, arXiv:1311.5085 [hep-ph].

[12] M. Zoller, *Vacuum stability in the SM and the three-loop β-function for the Higgs self-interaction*. arXiv:1209.5609 [hep-ph].

[13] K. Chetyrkin and M. Zoller, *Three-loop β-functions for top-Yukawa and the Higgs self-interaction in the Standard Model*. JHEP **1206** (2012) 033, arXiv:1205.2892 [hep-ph].

[14] D. Buttazzo, G. Degrassi, P. P. Giardino, G. F. Giudice, F. Sala, *et al.*, *Investigating the near-criticality of the Higgs boson*. arXiv:1307.3536 [hep-ph].

[15] I. Masina, *Higgs boson and top quark masses as tests of electroweak vacuum stability*. Phys.Rev. **D87** (2013) no. 5, 053001, arXiv:1209.0393 [hep-ph].

[16] G. Degrassi, S. Di Vita, J. Elias-Miro, J. R. Espinosa, G. F. Giudice, *et al.*, *Higgs mass and vacuum stability in the Standard Model at NNLO*. JHEP **1208** (2012) 098, arXiv:1205.6497 [hep-ph].

[17] J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori, A. Riotto, *et al.*, *Higgs mass implications on the stability of the electroweak vacuum*. Phys. Lett. **B709** (2012) 222–228, arXiv:1112.3022 [hep-ph].

[18] M. Holthausen, K. S. Lim, and M. Lindner, *Planck scale Boundary Conditions and the Higgs Mass*. JHEP **1202** (2012) 037, arXiv:1112.2415 [hep-ph].

[19] M. Bobrowski, G. Chalons, W. G. Hollik, and U. Nierste, *Vacuum stability of the effective Higgs potential in the Minimal Supersymmetric Standard Model*. Phys.Rev. **D90** (2014) 035025, arXiv:1407.2814 [hep-ph].

[20] K. Chetyrkin and M. Zoller, *β-function for the Higgs self-interaction in the Standard Model at three-loop level*. JHEP **1304** (2013) 091, `arXiv:1303.2890 [hep-ph]`.

[21] A. Bednyakov, A. Pikelner, and V. Velizhanin, *Higgs self-coupling beta-function in the Standard Model at three loops*. Nucl.Phys. **B875** (2013) 552–565, `arXiv:1303.4364`.

[22] A. Bednyakov, A. Pikelner, and V. Velizhanin, *Three-loop Higgs self-coupling beta-function in the Standard Model with complex Yukawa matrices*. `arXiv:1310.3806 [hep-ph]`.

[23] L. N. Mihaila, J. Salomon, and M. Steinhauser, *Gauge coupling beta functions in the standard model to three loops*. Phys. Rev. Lett. **108** (2012) 151602.

[24] L. N. Mihaila, J. Salomon, and M. Steinhauser, *Renormalization constants and beta functions for the gauge couplings of the Standard Model to three-loop order*. Phys. Rev. D **86** (2012) 096008, `arXiv:1208.3357 [hep-ph]`.

[25] A. Bednyakov, A. Pikelner, and V. Velizhanin, *Anomalous dimensions of gauge fields and gauge coupling beta-functions in the Standard Model at three loops*. JHEP **1301** (2013) 017, `arXiv:1210.6873 [hep-ph]`.

[26] A. Bednyakov, A. Pikelner, and V. Velizhanin, *Yukawa coupling beta-functions in the Standard Model at three loops*. Phys.Lett. **B722** (2013) 336–340, `arXiv:1212.6829`.

[27] F. Jegerlehner, M. Y. Kalmykov, and B. A. Kniehl, *On the difference between the pole and the MSbar masses of the top quark at the electroweak scale*. Phys.Lett. **B722** (2013) 123–129, `arXiv:1212.4319 [hep-ph]`.

[28] J. Espinosa, G. Giudice, and A. Riotto, *Cosmological implications of the Higgs mass measurement*. JCAP **0805** (2008) 002, `arXiv:0710.2484 [hep-ph]`.

[29] R. Hempfling and B. A. Kniehl, *On the relation between the fermion pole mass and MS Yukawa coupling in the standard model*. Phys.Rev. **D51** (1995) 1386–1394, `arXiv:hep-ph/9408313 [hep-ph]`.

[30] A. Sirlin and R. Zucchini, *Dependence of the Higgs coupling hMS(M) on mH and the possible onset of new physics*. Nucl. Phys. B **266** (1986) no. 2, 389–409.

[31] ATLAS, CDF, CMS, and D0, *First combination of Tevatron and LHC measurements of the top-quark mass*. `arXiv:1403.4427 [hep-ex]`.

[32] M. Martinez and R. Miquel, *Multiparameter fits to the t anti-t threshold observables at a future e+ e- linear collider*. Eur.Phys.J. **C27** (2003) 49–55, `arXiv:hep-ph/0207315 [hep-ph]`.

[33] K. Seidel, F. Simon, M. Tesar, and S. Poss, *Top quark mass measurements at and above threshold at CLIC*. Eur.Phys.J. **C73** (2013) 2530, arXiv:1303.3758 [hep-ex].