Prepared for submission to JHEP

TTP13-003
SFB/CPP-13-04

OPE of the pseudoscalar gluonium correlator in massless QCD to three-loop order

---

M. F. Zoller[a]

[a] *Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), D-76128 Karlsruhe, Germany*

E-mail: max.zoller@kit.edu

Abstract: In this paper analytical results are presented for higher order corrections to coefficient functions of the operator product expansion (OPE) for the correlator of two pseudoscalar gluonium operators $\tilde{O}_1 = G^{\mu\nu}\tilde{G}_{\mu\nu}$. The Wilson coefficient in front of the scalar gluon condensate operator $O_1 = -\frac{1}{4}G^{\mu\nu}G_{\mu\nu}$ is given at three-loop accuracy. The leading coefficient C_0 in front of the unity operator $O_0 = \mathbb{1}$ has been calculated up to three-loop order some time ago [1] but has been checked independently in this work. It is interesting to see that the coefficient C_1 in the pseudoscalar case is finite, whereas contact terms appear in C_0 in this case and in both coefficients C_0 and C_1 in the cases of the scalar gluonium correlator and the energy momentum tensor correlator [2]. For the corresponding Renormalization Group invariant Wilson coefficients which are also constructed the results are partially extended to four-loop accuracy. All results are given in the $\overline{\text{MS}}$-scheme at zero temperature.

Keywords: QCD, Quark-Gluon Plasma, Sum Rules

Contents

1 Motivation

Euclidian correlators of local operators are important objects in quantum field theory. Firstly, they have many important applications, e.g. in sum rules, where they are connected to physical quantities like spectral densities through dispersion relations. Secondly, they often have interesting properties in themselves, like their non-trivial renormalization, which are important for the understanding of quantum field theories. Such correlators are defined in momentum space as

$$i \int \mathrm{d}^4x \, e^{iqx} T\{\, [O](x)[O](0)\} \tag{1.1}$$

with a large Euclidian momentum q. Here and in the following the squared brackets indicate that the renormalized form of some operator O is used. Usually, we are interested in the vacuum expectation value (VEV) of the correlator

$$\Pi(Q^2) = i \int \mathrm{d}^4x \, e^{iqx} \, \langle 0|T\{\, [O](x)[O](0)]\}|0\rangle \qquad (Q^2 = -q^2) \tag{1.2}$$

which can be calculated in perturbation theory. But if we take $|0\rangle$ to be the physical vacuum state we also have to consider non-perturbative effects. Starting from the perturbative region of momentum space this is done by means of an operator product expansion (OPE). The idea is to expand the bilocal operator product (1.1) in a series of local operators with

Wilson coefficients depending on the large Euclidean momentum q [3]: [1]

$$i\int \mathrm{d}^4x\, e^{iqx}T\{\,[O](x)[O](0)\} = \sum_i C_i^B(q)(Q^2)^{\frac{2\,\dim(O)-\dim(O_i)-4}{2}}O_i^B \tag{1.3}$$

$$= \sum_i C_i(q)(Q^2)^{\frac{2\,\dim(O)-\dim(O_i)-4}{2}}[O_i]. \tag{1.4}$$

where the index B marks bare quantities and the factor $(Q^2)^{\frac{2\,\dim(O)-\dim(O_i)-4}{2}}$ constructed from the mass dimensions of the operators involved makes the Wilson coefficients $C_i(q)$ dimensionless.

In a sum rule approach to glueballs three operators are usually investigated as insertions on the lhs of (1.3) (see e.g. [5]):

$$O_1(x) = -\frac{1}{4}G^{\mu\nu}G_{\mu\nu}(x) \qquad \text{(scalar)} \tag{1.5}$$

$$\tilde{O}_1(x) = G^{\mu\nu}\tilde{G}_{\mu\nu}(x) \qquad \text{(pseudoscalar)} \tag{1.6}$$

$$O_T^{\mu\nu}(x) = T^{\mu\nu}(x) \qquad \text{(tensor)} \tag{1.7}$$

where $G_{\mu\nu}$ is the gluon field strength tensor,

$$\tilde{G}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}G^{\rho\sigma} \tag{1.8}$$

the dual gluon field strength tensor and $T^{\mu\nu}$ the energy-momentum tensor of QCD. Having discussed the correlators of O_1 and $O_T^{\mu\nu}$ in [2] the results for the correlator of (1.6)

$$X_t(q) := i\int \mathrm{d}^4x\, e^{iqx}T\{\,[\tilde{O}_1](x)[\tilde{O}_1](0)\}, \tag{1.9}$$

whose VEV $\chi_t(q) := \langle 0|X_t(q)|0\rangle$ is also known as the topological susceptibility of QCD[2], are presented here. This correlator has been connected to the mass of the η'-meson through the Witten-Veneziano formula [8–11]:

$$\left.\frac{\alpha_s^2}{32i\pi^2}\chi_t(q)\right|_{q\to 0,\,\frac{n_f}{N_c}\to 0} = \frac{m_{\eta'}^2F_\pi^2}{n_f} \quad \text{(leading order)} \tag{1.10}$$

where $F_\pi \approx 94$ MeV is the pion decay constant. An explicit sum rule calculation with an OPE at one-loop level using a Borel transformation has been done in [12]. In this work the value $m_{\eta'} \approx 1$ GeV is correctly estimated.

The correlator defined in (1.1) with renormalized operators is finite, i.e. all its matrix elements are finite, except for possible contact terms. These arise from the point where $x \equiv 0$

---

[1] Effectively this expansion separates the high energy physics, which is contained in the Wilson coefficients, from the low energy physics which is taken into account by the VEVs of the local operators, the so-called condensates [4]. These cannot be calculated in perturbation theory, but need to be derived from low energy theorems or be calculated on the lattice.

[2] For a discussion of topological effects in QCD and the significance of the operator $\tilde{O}_1$ and the correlator (1.9) in that respect see e.g. [6, 7].

and manifest themselves as divergences $\propto \delta(x)$ and derivatives of $\delta(x)$ or in momentum space terms polynomial in q. These local terms do not contribute to sum rules and can and should be subtracted with proper counterterms.

The leading term on the rhs of (1.3) is the coefficient in front of the unit operator $\mathbb{1}$ which is just the perturbative VEV of the correlator (1.1):

$$(Q^2)^2 C_0(q) = \langle 0|X_t(q)|0\rangle|_{\text{pert}}. \tag{1.11}$$

The coefficient C_0 is known for the scalar case (1.5) at four-loop level [13] and for the pseudoscalar case (1.6) [1] and the energy-momentum tensor correlator [2] at three-loop level. The next important contribution in the OPE is the coefficient of the dimension four operator $[O_1]$ (1.5).[3] The coefficient C_1 has been calculated at two-loop level for the scalar[4] and tensor cases [2]. Here we present the coefficent C_1 for the pseudoscalar case at three-loop level which so far has only been known to one-loop accuracy [16, 17].

All physical matrix elements of $[O_1] = Z_G O_1^B$ are finite and so is the renormalized coefficient C_1:[5]

$$C_1 = \frac{1}{Z_G} C_1^B. \tag{1.12}$$

The renormalization constant

$$Z_G = 1 + \alpha_s \frac{\partial}{\partial \alpha_s} \ln Z_{\alpha_s} = \left(1 - \frac{\beta(\alpha_s)}{\varepsilon}\right)^{-1} \tag{1.13}$$

has been derived in a simple way in [18] (see also an earlier work [19]). Here Z_{α_s} is the renormalization constant[6] for α_s and the β-function is defined as

$$\beta(\alpha_s) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \ln \alpha_s = -\sum_{i\geq 0} \beta_i \left(\frac{\alpha_s}{\pi}\right)^{i+1}. \tag{1.14}$$

The outline of this paper is as follows. In the next section the renormalization properties of $\tilde{O}_1$ will be discussed. In section 3 the details of the calculation will be described (section 3.1) and the results for the OPE of (1.9) will be presented (section 3.2). After that Renormalization Group invariant (RGI) operators and Wilson coefficients will be constructed (section 3.3) followed by a numerical evaluation of the main results (section 3.4). Finally, some conclusions and acknowledgements will be given.

---

[3] In the case of massive fermion flavours f we would also have contributions proportional to the dimension two operator $O^f = m_f^2\,\mathbb{1}$ and the dimension four operator $O_2^f = m_f \bar{\psi}_f \psi_f$. In the case of temperature $T \neq 0$ Lorentz variant operators like $T_0^0 \sim e+p$ with the energy density e and the pressure p have to be considered as well. At $T = 0$, however, only Lorentz and gauge invariant scalar operators contribute to the the VEV in (1.2) which is the quantity that we are ultimately interested in. For a discussion of the correlator $X_t(q)$ at finite temperature up to $\mathcal{O}(\alpha_s)$ see [14].

[4] The one-loop result for the scalar case was first derived in [15].

[5] In the massless case O_1 only mixes with unphysical operators whose matrix elements with physical external states vanish. The renormalization of O_1 including these unphysical contributions as well as the mixing with O_2^f in the massive case can be found in [18].

[6] Often in the literature Z_{α_s} is used instead of Z_G and $\alpha_s G^{\mu\nu} G_{\mu\nu}$ instead of O_1. This renormalization is only valid up to first order in α_s as the renormalization constants Z_G and Z_{α_s} coincide to this accuracy. In higher orders, however, Z_G and Z_{α_s} differ.

2 Renormalization of $\tilde{O}_1$ and its correlator

The operator $\tilde{O}_1$ forms a closed set under renormalization with the pseudoscalar fermionic operator

$$\partial_\mu J_5^\mu := \varepsilon^{\mu\mu_1\mu_1\mu_3}\partial_\mu \sum_f \bar{\Psi}_f \gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\Psi. \tag{2.1}$$

which can be written as

$$\partial_\mu J_5^\mu = \partial_\mu \sum_f \bar{\Psi}_f \gamma^\mu \gamma_5 \Psi \tag{2.2}$$

in the Larin scheme for γ_5 [20].

The ε-tensors appearing in (1.6) and (2.1) are then drawn out of the R-operation performed in dimensional regularization. In the correlators which have to be calculated there are always two ε-tensors involved which can be contracted and expressed through metric tensors:

$$\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\varepsilon_{\nu_1\nu_2\nu_3\nu_4} = -g^{[\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2} g^{\mu_3}_{\nu_3} g^{\mu_4]}_{\nu_4} \tag{2.3}$$

where $[\ldots]$ means complete antisymmetrization. These operators are renormalized like [20]

$$[\partial_\mu J_5^\mu] = Z_5^s Z_{MS}^s \partial_\mu J_5^{B\,\mu} = Z_J^s \partial_\mu J_5^{B\,\mu}, \tag{2.4}$$

$$[\tilde{O}_1] = Z_{G\tilde{G}} \tilde{O}_1^B + Z_{GJ}\partial_\mu J_5^{B\,\mu} \tag{2.5}$$

where Z_{MS}^s is an $\overline{\mathrm{MS}}$ renormalization constant, Z_5^s a finite renormalization constant fixed by the requirement that the one-loop character of the axial anomaly relation

$$[\partial_\mu J_5^\mu] = \frac{\alpha_s}{4\pi} n_f T_F [\tilde{O}_1] + \mathrm{CT} \tag{2.6}$$

is valid in dimensional regularization.[7] CT stands for contact terms of $\partial_\mu J_5^\mu$ with fermion fields. In the gluon sector these can be neglected. $Z_{G\tilde{G}}$ is an $\overline{\mathrm{MS}}$ renormalization constant again and Z_{GJ} starts at $\mathcal{O}(\alpha_s)$. In [20] Z_{MS}^s and Z_5^s are given up to $\mathcal{O}(\alpha_s^3)$ and $\mathcal{O}(\alpha_s^2)$ respectively. Furthermore it is shown that $Z_{G\tilde{G}} = Z_a$ (Z_a being the renormalization constant for α_s). The constant Z_{GJ} is only given at one-loop level in the literature [1, 20] but for the Wilson coefficient C_1 at three-loop level it is needed to two-loop accuracy. In section 3.3 we will also need the corresponding three-loop anomalous dimension. The simplest way to determine Z_{GJ} is by constructing the matrix elements of $\tilde{O}_1$ and $\partial_\mu J_5^{B\,\mu}$ with two external fermions (see Fig. 1) using a projector

$$P(q) := q^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\varepsilon_{\mu_1\mu_2\mu_3\mu_4} \tag{2.7}$$

on the external fermion line. From this we get

---

[7] In Pauli-Villars regularization for example this relation is automatically fulfilled. In $d \neq 4$ dimensions, however, the operators $\partial_\mu J_5^\mu$ and $\tilde{O}_1$ become linearly independent.

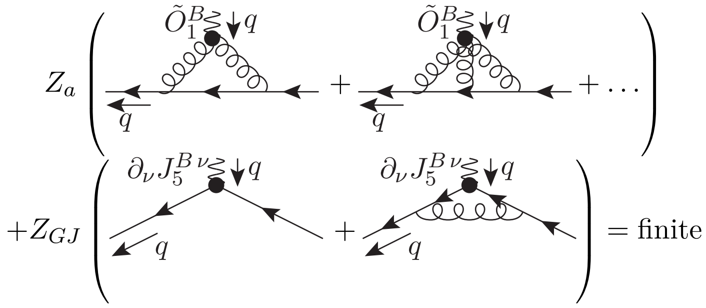


Figure 1. Diagrams for the calculation of Z_{GJ}

$$
\begin{aligned}
Z_{GJ} = {} & \frac{\alpha_s}{4\pi\varepsilon} 12 C_F + \frac{\alpha_s^2}{(4\pi)^2\varepsilon}\left\{\frac{142 C_A C_F}{3} - 42 C_F^2 - \frac{8}{3} C_F n_f T_F\right\} \\
& + \frac{\alpha_s^2}{(4\pi)^2\varepsilon^2}\left\{16 C_F n_f T_F - 44 C_A C_F\right\} \\
& + \frac{\alpha_s^3}{(4\pi)^3\varepsilon^3}\left\{\frac{484}{3} C_A^2 C_F - \frac{352}{3} n_f C_A T_F C_F + \frac{64}{3} n_f^2 T_F^2 C_F\right\} \\
& + \frac{\alpha_s^3}{(4\pi)^3\varepsilon^2}\left\{\frac{550}{3} C_A C_F^2 - \frac{2378}{9} C_A^2 C_F - \frac{32}{3} n_f T_F C_F^2 \right. \\
& \left. + \frac{1136}{9} n_f C_A T_F C_F - \frac{32}{9} n_f^2 T_F^2 C_F\right\} \\
& + \frac{\alpha_s^3}{(4\pi)^3\varepsilon}\left\{178 C_F^3 - \frac{2947}{9} C_A C_F^2 + \frac{1607}{9} C_A^2 C_F - \frac{1096}{9} n_f T_F C_F^2 \right. \\
& \left. + \frac{328}{9} n_f C_A T_F C_F - \frac{208}{9} n_f^2 T_F^2 C_F + 192\zeta_3 n_f T_F C_F^2 - 192\zeta_3 n_f C_A T_F C_F\right\}.
\end{aligned}
\tag{2.8}
$$

An interesting additional application of this result is to check the connection between the anomalous dimensions of the operator set $\{\tilde{O}_1, \partial_\nu J_5^\nu\}$. In [20] the following relations have been motivated:

$$\gamma_{G\tilde{G}} = -\frac{\beta(\alpha_s)}{\alpha_s} \tag{2.9}$$

$$\gamma_{GJ} = \left(\frac{\alpha_s}{4\pi} n_f T_F\right)^{-1} \gamma_J^s \tag{2.10}$$

with

$$\gamma_{ij} = \left(\mu^2 \frac{d}{d\mu^2} Z_{ik}\right)\left(Z^{-1}\right)_{kj}, \quad Z = \begin{pmatrix} Z_{G\tilde{G}} & Z_{GJ} \\ 0 & Z_J^s \end{pmatrix} \tag{2.11}$$

The first relation (2.9) has been explicitly checked to three-loop level in [20] the second one (2.10) only to one-loop accuracy. Now we can check this equation with γ_{GJ} at two-loop level and γ_J^s at three-loop level and it turns out to hold there as well. Using (2.8) and the renormalization constants Z_J^s and Z_a [20, 21] the following anomalous dimension is derived:[8]

$$
\begin{aligned}
\gamma_{GJ} = & -12C_F\left(\frac{\alpha_s}{4\pi}\right)+\left(\frac{\alpha_s}{4\pi}\right)^2\left\{-\frac{284}{3}C_AC_F+36C_F^2+\frac{16}{3}C_Fn_fT_F\right\} \\
& +\left(\frac{\alpha_s}{4\pi}\right)^3\left\{-\frac{1607}{3}C_A^2C_F+461C_AC_F^2+576C_AC_Fn_fT_F\zeta_3-\frac{328}{3}C_AC_Fn_fT_F\right. \\
& \left.-126C_F^3-576C_F^2n_fT_F\zeta_3+428C_F^2n_fT_F+\frac{208}{3}C_Fn_f^2T_F^2\right\}
\end{aligned}
\tag{2.12}
$$

Now we can write the correlator $X_t(q)$ as

$$
\begin{aligned}
& i\int \mathrm{d}^4x\, e^{iqx}T\{\,[\tilde{O}_1](x)[\tilde{O}_1](0)\} \\
=& i\int \mathrm{d}^4x\, e^{iqx}T\left\{Z_{G\tilde{G}}^2\tilde{O}_1^B(x)\tilde{O}_1^B(0)+2Z_{G\tilde{G}}Z_{GJ}\tilde{O}_1^B(x)\partial_\mu J_5^{B\,\mu}(0)+Z_{GJ}^2\partial_\mu J_5^{B\,\mu}(x)\partial_\nu J_5^{B\,\nu}(0)\right\}.
\end{aligned}
\tag{2.13}
$$

In [2] it has been discovered that there are contact terms at two-loop level in the coefficient C_1 for the correlator of O_1. The coefficient C_0 also has contact terms for the correlator of two operators O_1 or two operators $T^{\mu\nu}$. For the operator $\tilde{O}_1$ we can make an important restriction on possible contact terms due to the fact that it can be exactly expressed as the divergence of the Chern-Simons current:

$$
\tilde{O}_1 = \partial_\mu K^\mu \tag{2.14}
$$

with

$$
K^\mu = \varepsilon^{\mu\nu\rho\sigma}\left\{4G_\nu^a\partial_\rho G_\sigma^a+\frac{4}{3}g_sf^{abc}G_\nu^aG_\rho^bG_\sigma^c\right\}. \tag{2.15}
$$

From this follows for (2.13)

$$
\begin{aligned}
& i\int \mathrm{d}^4x\, e^{iqx}T\{[\tilde{O}_1](x)[\tilde{O}_1](0)\} \\
=& q_\mu q_\nu\, i\int \mathrm{d}^4x\, e^{iqx}T\left\{Z_{G\tilde{G}}^2K^{B\,\mu}(x)K^{B\,\nu}(0)+2Z_{G\tilde{G}}Z_{GJ}K^{B\,\mu}(x)J_5^{B\,\nu}(0)+Z_{GJ}^2J_5^{B\,\mu}(x)J_5^{B\,\nu}(0)\right\} \\
\to& q_\mu q_\nu\left\{q^2C_0^{\mu\nu}(q^2)+\frac{1}{q^2}C_1^{\mu\nu}(q^2)+\ldots\right\} \text{ for } q^2\to-\infty \text{ (OPE)}
\end{aligned}
\tag{2.16}
$$

with dimensionless coefficients $C_0^{\mu\nu}(q^2)$ and $C_1^{\mu\nu}(q^2)$. Because of the non-local factor $\frac{1}{q^2}$ the coefficient $C_1^{\mu\nu}(q^2)$ cannot contain any contact terms. This makes the Wilson coefficent $C_1(q^2) = \frac{q_\mu q_\nu}{q^2}C_1^{\mu\nu}(q^2)$ for the correlator (2.13) finite and unambiguous due to the absence of contact terms.

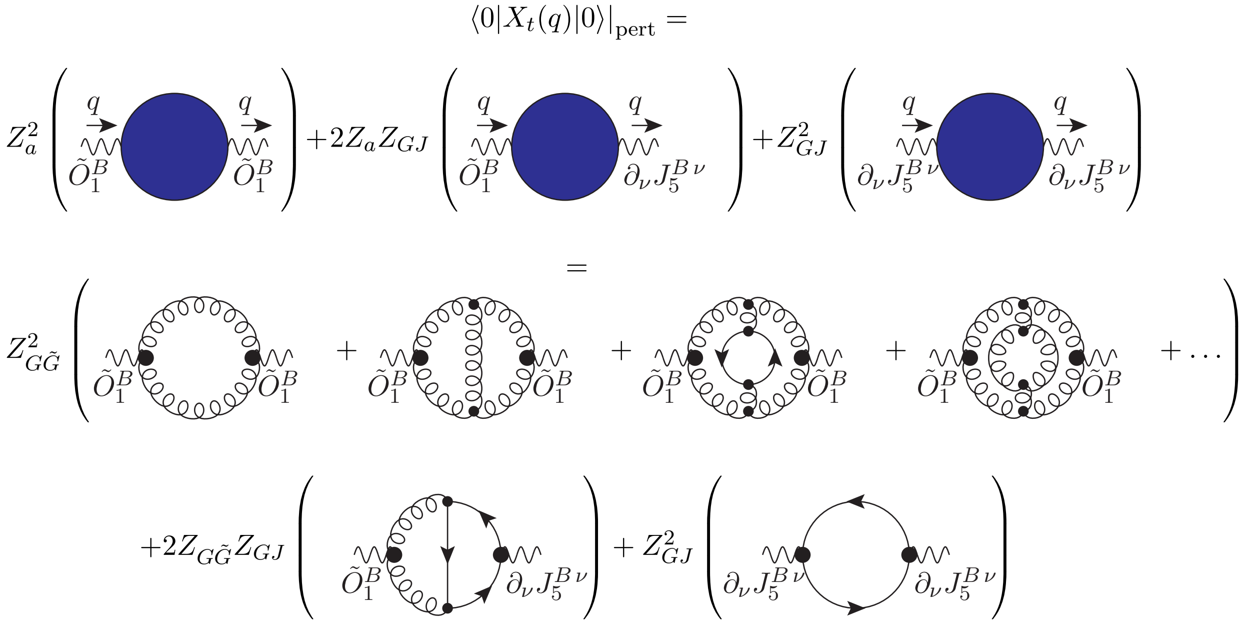


Figure 2. Diagrams for the calculation of the coefficient $C_0(Q^2)$

3 Calculation and results

3.1 Details of the calculation

The leading coefficient C_0 is just the perturbative VEV of the correlator eq. (2.13)

$$(Q^2)^2 C_0(q) = \langle 0|i\int \mathrm{d}^4x\, e^{iqx} T\{[\tilde{O}_1](x)[\tilde{O}_1](0)\}|0\rangle\Big|_{\text{pert}} \tag{3.1}$$

which has been computed up to order α_s^2 (three loops). In Figure (2) some sample Feynman diagrams contributing to this calculation are shown. The operators $\tilde{O}_1^B$ and $\partial_\mu J_5^{B\,\mu}$ play the roles of external currents. The Feynman diagrams have been produced with the program QGRAF [22]. As all diagrams in this problem are propagator-like the relevant integrals can be computed with the FORM package MINCER [23–25]. For the colour part of the diagrams the FORM package COLOR [26] has been used.

In order to compute the coefficient $C_1(Q^2)$ the method of projectors [27, 28] has been applied, which allows to express coefficient functions for any OPE of two operators in terms of massless propagator type diagrams only. The method is based the fact that in dimensional regularization every massless tadpole-like Feynman integral is set to zero.
We apply a projector to both sides of (1.3) which sets every operator on the rhs to zero

---

[8] $\gamma_{G\tilde{G}}$ and γ_J^s can be found in [20, 21] at three-loop level. All renormalization constants and anomalous dimensions are available at **http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp13/ttp13-003/**

except for O_1^B:

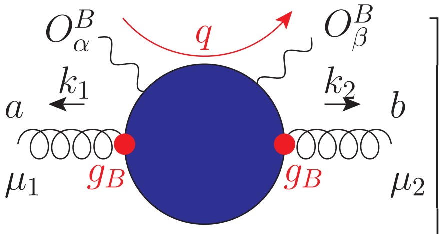
$$\mathbf{P}\{X_t(q)\} = \sum_i (Q^2)^{\frac{4-\dim(O_i)}{2}}\, C_i^{B,(r)}(Q^2)\, \mathbf{P}\{O_i^B\} \tag{3.2}$$

with $\mathbf{P}\{O_1^B\} = 1$ and $\mathbf{P}\{O_{i\neq 1}^B\} = 0$. This is done in the same way as described in [2] leading to

$$C_{1,B}(Q^2) = Z^2_{G\tilde{G}} C_{1,B}^{(\tilde{O}_1^B,\tilde{O}_1^B)}(Q^2) + 2 Z_{G\tilde{G}} Z_{GJ} C_{1,B}^{(\tilde{O}_1^B,\partial_\nu J_5^{B\,\nu})}(Q^2) + Z^2_{GJ} C_{1,B}^{(\partial_\nu J_5^{B\,\nu},\partial_\nu J_5^{B\,\nu})}(Q^2) \tag{3.3}$$

with

$$C_{1,B}^{(O_\alpha^B O_\beta^B)}(Q^2) = \frac{\delta^{ab}}{n_g}\,\frac{g^{\mu_1\mu_2}}{(D-1)}\,\frac{1}{D}\,\frac{\partial}{\partial k_1}\cdot\frac{\partial}{\partial k_2}\left[\quad\right]\Bigg|_{k_i=0}, \tag{3.4}$$



where the blue circle represents the the sum of all (bare) Feynman diagrams which become 1PI after formal gluing of the two external lines representing the operators on the lhs of the OPE.

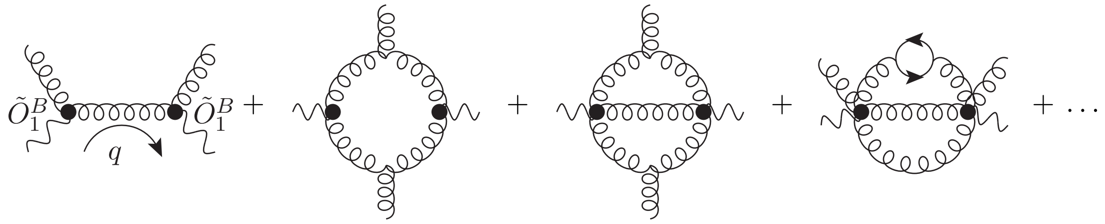


Figure 3. Diagrams for the calculation of $C_{1,B}^{(\tilde{O}_1^B,\tilde{O}_1^B)}$.

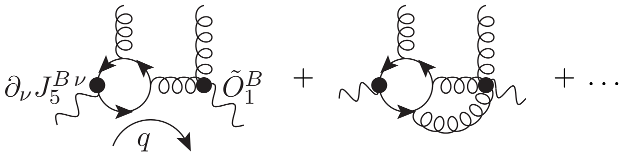


Figure 4. Diagrams for the calculation of $C_{1,B}^{(\tilde{O}_1^B,\partial_\nu J_5^{B\,\nu})}$.

Table (1) shows the number of diagrams generated for the different contributions to C_0 and C_1 All results are given in the $\overline{\text{MS}}$ scheme with $a_s = \frac{\alpha_s}{\pi}$, $\alpha_s = \frac{g_s^2}{4\pi}$ and the abbreviation $l_{\mu q} = \ln\left(\frac{\mu^2}{Q^2}\right)$ where μ is the $\overline{\text{MS}}$ renormalization scale. They can be retrieved from **http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp13/ttp13-003/**

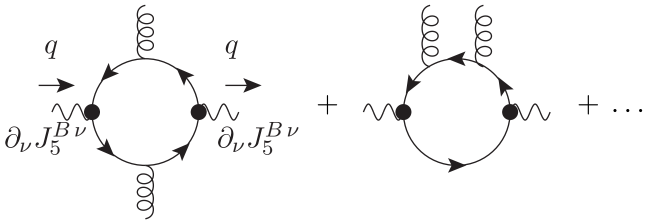


Figure 5. Diagrams for the calculation of $C_{1,B}^{(\partial_\nu J_5^{B\,\nu},\partial_\nu J_5^{B\,\nu})}$.

Correlator	0 loop	1 loop	2 loop	3 loop
$\langle 0\|\tilde{O}_1^B(x)\tilde{O}_1^B(0)\|0\rangle_{\text{pert}}$	0	1	12	215
$\langle 0\|\tilde{O}_1^B(x)\partial_\nu J_5^{B\,\nu}(0)\|0\rangle_{\text{pert}}$	0	0	1	
$\langle 0\|\partial_\nu J_5^{B\,\nu}(x)\partial_\nu J_5^{B\,\nu}(0)\|0\rangle_{\text{pert}}$	0	1		
$\mathbf{P_1}(\tilde{O}_1^B(x)\tilde{O}_1^B(0))$	2	75	2567	94964
$\mathbf{P_1}(\tilde{O}_1^B(x)\partial_\nu J_5^{B\,\nu}(0))$	0	8	345	
$\mathbf{P_1}(\partial_\nu J_5^{B\,\nu}(x)\partial_\nu J_5^{B\,\nu}(0))$	0	8		

Table 1. Number of diagrams needed for C_0 and C_1

The gauge group factors are defined in the usual way: C_F and C_A are the quadratic Casimir operators of the quark and the adjoint representation of the corresponding Lie algebra, d_R is the dimension of the quark representation, n_g is the number of gluons (dimension of the adjoint representation), T_F is defined so that $T_F\delta^{ab} = \mathbf{Tr}\left(T^aT^b\right)$ is the trace of two group generators of the quark representation.[9] For QCD (colour gauge group SU(3)) we have $C_F = 4/3$, $C_A = 3$, $T_F = 1/2$ and $d_R = 3$. By n_f we denote the number of active quark flavours.

3.2 Results

As we have seen from (2.13) contact terms in C_0 are possible and it turns out that they appear starting from one loop. Because of these contact terms an unambiguous result for C_0 can only be given up to local (that is q-independent) contributions. To avoid the

---

[9] For an SU(N) gauge group these are $d_R = N$, $C_A = 2T_FN$ and $C_F = T_F\left(N - \frac{1}{N}\right)$.

ambiguity the Q^2-derivative is presented:

$$\begin{aligned}
Q^2\frac{d}{dQ^2}\,C_0 =& \frac{n_g}{\pi^2}\Bigg[-1+a_s\left(-\frac{97}{12}C_A+\frac{7}{3}n_fT_F\right)+a_sl_{\mu q}\left(-\frac{11}{6}C_A+\frac{2}{3}n_fT_F\right)\\
&+a_s^2\left(-\frac{51959}{864}C_A^2+\frac{107}{12}n_fT_FC_F+\frac{3793}{108}n_fC_AT_F-\frac{251}{54}n_f^2T_F^2\right.\\
&\left.+\frac{55}{8}\zeta_3C_A^2-3\zeta_3n_fT_FC_F+\frac{1}{2}\zeta_3n_fC_AT_F\right)\\
&+a_s^2l_{\mu q}\left(-\frac{1135}{48}C_A^2+2n_fT_FC_F+\frac{46}{3}n_fC_AT_F-\frac{7}{3}n_f^2T_F^2\right)\\
&+a_s^2l_{\mu q}^2\left(-\frac{121}{48}C_A^2+\frac{11}{6}n_fC_AT_F-\frac{1}{3}n_f^2T_F^2\right)\Bigg].
\end{aligned}\tag{3.5}$$

This result has been derived before [1] which serves as a nice check for the setup. As discussed above the coefficient C_1 is unambiguous and is therefore given in full:

$$\begin{aligned}
C_1 =& 64\Bigg\{1+a_s\left(\frac{157}{36}C_A-\frac{5}{9}n_fT_F+\frac{11}{12}l_{\mu q}C_A-\frac{1}{3}l_{\mu q}n_fT_F\right)\\
&+a_s^2\left(\frac{25945}{1296}C_A^2-\frac{11}{2}n_fT_FC_F-\frac{4355}{648}n_fC_AT_F+\frac{25}{81}n_f^2T_F^2+\frac{1727}{216}l_{\mu q}C_A^2\right.\\
&-\frac{3}{2}l_{\mu q}n_fT_FC_F-\frac{106}{27}l_{\mu q}n_fC_AT_F+\frac{10}{27}l_{\mu q}n_f^2T_F^2+\frac{121}{144}l_{\mu q}^2C_A^2-\frac{11}{18}l_{\mu q}^2n_fC_AT_F\\
&\left.+\frac{1}{9}l_{\mu q}^2n_f^2T_F^2-\frac{33}{8}\zeta_3C_A^2+3\zeta_3n_fT_FC_F-\frac{3}{2}\zeta_3n_fC_AT_F\right)\\
&+a_s^3\left(\frac{19360399}{186624}C_A^3+\frac{461}{144}n_fT_FC_F^2-\frac{614501}{10368}n_fC_AT_FC_F\right.\\
&-\frac{1857805}{31104}n_fC_A^2T_F+\frac{28981}{2592}n_f^2T_F^2C_F+\frac{126415}{15552}n_f^2C_AT_F^2-\frac{125}{729}n_f^3T_F^3\\
&+\frac{594247}{10368}l_{\mu q}C_A^3+\frac{35}{32}l_{\mu q}n_fT_FC_F^2-\frac{1623}{64}l_{\mu q}n_fC_AT_FC_F-\frac{68935}{1728}l_{\mu q}n_fC_A^2T_F\\
&+\frac{105}{16}l_{\mu q}n_f^2T_F^2C_F+\frac{6661}{864}l_{\mu q}n_f^2C_AT_F^2-\frac{25}{81}l_{\mu q}n_f^3T_F^3+\frac{9779}{864}l_{\mu q}^2C_A^3\\
&-\frac{275}{96}l_{\mu q}^2n_fC_AT_FC_F-\frac{2795}{288}l_{\mu q}^2n_fC_A^2T_F+\frac{25}{24}l_{\mu q}^2n_f^2T_F^2C_F+\frac{61}{24}l_{\mu q}^2n_f^2C_AT_F^2\\
&-\frac{5}{27}l_{\mu q}^2n_f^3T_F^3+\frac{1331}{1728}l_{\mu q}^3C_A^3-\frac{121}{144}l_{\mu q}^3n_fC_A^2T_F+\frac{11}{36}l_{\mu q}^3n_f^2C_AT_F^2\\
&-\frac{1}{27}l_{\mu q}^3n_f^3T_F^3+\frac{55}{8}\zeta_5C_A^3-15\zeta_5n_fT_FC_F^2+\frac{15}{2}\zeta_5n_fC_AT_FC_F+5\zeta_5n_fC_A^2T_F\\
&-\frac{6893}{144}\zeta_3C_A^3+\frac{145}{12}\zeta_3n_fT_FC_F^2+\frac{1291}{48}\zeta_3n_fC_AT_FC_F-\frac{349}{144}\zeta_3n_fC_A^2T_F\\
&-\frac{13}{2}\zeta_3n_f^2T_F^2C_F+\frac{121}{36}\zeta_3n_f^2C_AT_F^2-\frac{363}{32}\zeta_3l_{\mu q}C_A^3+\frac{33}{4}\zeta_3l_{\mu q}n_fC_AT_FC_F\\
&\left.-3\zeta_3l_{\mu q}n_f^2T_F^2C_F+\frac{3}{2}\zeta_3l_{\mu q}n_f^2C_AT_F^2\right)\Bigg\}.
\end{aligned}\tag{3.6}$$

The cancellation of all divergences is a strong check for this result. Another important check is the independence of the gauge parameter ξ as all calculations have been done for an arbitrary R_ξ gauge. The leading term of (3.6) is in agreement with [12] and the part

$\propto a_s l_{\mu q}$ has been derived in [17] if we set the colour factors to their QCD values.[10] For QCD colour factors we get

$$\begin{aligned}C_1 =&64\Bigg\{1+a_s\left(\frac{157}{12}-\frac{5}{18}n_f+\frac{11}{4}l_{\mu q}-\frac{1}{6}l_{\mu q}n_f\right)\\&+a_s^2\Bigg(\frac{25945}{144}-\frac{5939}{432}n_f+\frac{25}{324}n_f^2+\frac{1727}{24}l_{\mu q}\\&-\frac{62}{9}l_{\mu q}n_f+\frac{5}{54}l_{\mu q}n_f^2+\frac{121}{16}l_{\mu q}^2\\&-\frac{11}{12}l_{\mu q}^2n_f+\frac{1}{36}l_{\mu q}^2n_f^2-\frac{297}{8}\zeta_3-\frac{1}{4}\zeta_3n_f\Bigg)\\&+a_s^3\Bigg(\frac{19360399}{6912}-\frac{7972411}{20736}n_f+\frac{611093}{62208}n_f^2-\frac{125}{5832}n_f^3\\&+\frac{594247}{384}l_{\mu q}-\frac{264113}{1152}l_{\mu q}n_f+\frac{9181}{1152}l_{\mu q}n_f^2-\frac{25}{648}l_{\mu q}n_f^3\\&+\frac{9779}{32}l_{\mu q}^2-\frac{9485}{192}l_{\mu q}^2n_f+\frac{649}{288}l_{\mu q}^2n_f^2-\frac{5}{216}l_{\mu q}^2n_f^3\\&+\frac{1331}{64}l_{\mu q}^3-\frac{121}{32}l_{\mu q}^3n_f+\frac{11}{48}l_{\mu q}^3n_f^2-\frac{1}{216}l_{\mu q}^3n_f^3\\&+\frac{1485}{8}\zeta_5+\frac{145}{6}\zeta_5n_f-\frac{20679}{16}\zeta_3+\frac{46333}{864}\zeta_3n_f\\&+\frac{17}{48}\zeta_3n_f^2-\frac{9801}{32}\zeta_3l_{\mu q}+\frac{33}{2}\zeta_3l_{\mu q}n_f+\frac{1}{8}\zeta_3l_{\mu q}n_f^2\Bigg)\Bigg\}.\end{aligned}\tag{3.7}$$

A nice consistency check for these results is to perform an OPE of the correlator

$$i\int \mathrm{d}^4x\, e^{iqx}T\{[\partial_\mu J_5^\mu](x)[\partial_\mu J_5^\mu](0)\}=(Q^2)^2C_0^{JJ}+C_1^{JJ}[O_1]+\ldots\tag{3.8}$$

and then see that (2.6) is fulfilled (except for possible contact terms):

$$Q^2\frac{d}{dQ^2}\,C_0^{JJ}=\left(\tfrac{\alpha_s}{4\pi}n_fT_F\right)^2Q^2\tfrac{d}{dQ^2}\,C_0,\tag{3.9}$$

$$C_1^{JJ}=\left(\tfrac{\alpha_s}{4\pi}n_fT_F\right)^2C_1.\tag{3.10}$$

Indeed we find

$$Q^2\frac{d}{dQ^2}\,C_0^{JJ}=\frac{n_g}{\pi^2}\left[-\frac{a_s^2}{16}n_f^2T_F^2\right]\tag{3.11}$$

and

$$C_1^{JJ}=4a_s^2n_f^2T_F^2\left\{1+a_s\left(\frac{157}{36}C_A-\frac{5}{9}n_fT_F+\frac{11}{12}l_{\mu q}C_A-\frac{1}{3}l_{\mu q}n_fT_F\right)\right\}\tag{3.12}$$

satisfying (3.9) and (3.10) up to the calculated accuracy of $\mathcal{O}(a_s^2)$ and $\mathcal{O}(a_s^3)$ respectively.

---

[10] In [17], however, the leading term differs from this result and the one derived in [12] by a minus sign and the non-logarithmic term of $\mathcal{O}(a_s)$ is also missing there.

3.3 RGI operators and Wilson coefficients

Note that the coefficients (3.5) and (3.6) are not Renormalization Group invariant (RGI). In this section we take RGI versions of all operators and construct RGI Wilson coefficients. For an operator that is renormalized multiplicatively like $\partial_\mu J_5^\mu$ in (2.5) constructing a finite and RGI operator is straightforward (see e.g. [29]). Because of

$$\mu^2 \frac{d}{d\mu^2}[\partial_\mu J_5^\mu] = \gamma_J^s(a_s(\mu))[\partial_\mu J_5^\mu] \tag{3.13}$$

we can define

$$[\partial_\mu J_5^\mu]^{\mathrm{RGI}} := \underbrace{\exp\left\{-\int^{a_s(\mu)} \frac{\gamma_J^s(a)}{a\,\beta(a)} da\right\}}_{=:E_2(a_s)}[\partial_\mu J_5^\mu] \tag{3.14}$$

which fulfills $\mu^2 \frac{d}{d\mu^2}[\partial_\mu J_5^\mu]^{\mathrm{RGI}} = 0$. A remarkable feature of the operator (3.14) is its renormalization scheme independence (see also [30]). If we start with a different renormalized operator

$$[\partial_\mu J_5^\mu]' := Z(a_s)[\partial_\mu J_5^\mu] \tag{3.15}$$

we get

$${\gamma_J^s}'(a_s) = \gamma_J^s(a_s) + \mu^2 \frac{d}{d\mu^2}\ln(Z(a_s)) \tag{3.16}$$

which leads to

$$E_2'(a_s) = \frac{E_2(a_s)}{Z(a_s)} \tag{3.17}$$

and therefore to the same RGI operator

$$[\partial_\mu J_5^\mu]^{\mathrm{RGI}} = E_2'(a_s)[\partial_\mu J_5^\mu]' = E_2(a_s)[\partial_\mu J_5^\mu]. \tag{3.18}$$

If we apply the same procedure to the non-diagonal operator $\tilde{O}_1$ we get an RG *variant* operator

$$[\tilde{O}_1]^{\mathrm{RGV}} := \underbrace{\exp\left\{-\int^{a_s(\mu)} \frac{\gamma_{G\tilde{G}}(a)}{a\,\beta(a)} da\right\}}_{=:E_1(a_s)}[\tilde{O}_1] \tag{3.19}$$

where $E_1(a_s) = a_s$ because of (2.9). Taking the derivative wrt the renormalization scale we find

$$\mu^2 \frac{d}{d\mu^2}[\tilde{O}_1]^{\mathrm{RGV}} = E_1(a_s)\gamma_{GJ}(a_s)[\partial_\mu J_5^\mu] = \frac{E_1(a_s)}{E_2(a_s)}\gamma_{GJ}(a_s)[\partial_\mu J_5^\mu]^{\mathrm{RGI}} \tag{3.20}$$

which leads to the definition of the RGI operator

$$\begin{aligned} [\tilde{O}_1]^{\mathrm{RGI}} :=& [\tilde{O}_1]^{\mathrm{RGV}} - \underbrace{\int^{a_s(\mu)} \frac{E_1(a)}{E_2(a)}\gamma_{GJ}(a)\frac{da}{a\,\beta(a)}}_{=:a_s\tilde{Z}(a_s)}[\partial_\mu J_5^\mu]^{\mathrm{RGI}} \\ =& a_s\left\{Z_{G\tilde{G}}(a_s)\tilde{O}_1^B + \Big(Z_{GJ}(a_s) - E_2(a_s)\tilde{Z}(a_s)Z_J^s(a_s)\Big)\partial_\mu J_5^{B\,\mu}\right\} \end{aligned} \tag{3.21}$$

fulfilling $\mu^2 \frac{d}{d\mu^2}[\tilde{O}_1]^{\text{RGI}} = 0$. In similar way as for (3.14) it can be shown that (3.21) is invariant under transformations $[\tilde{O}_1] \to [\tilde{O}_1]' = Z_1(a_s)[\tilde{O}_1]$. Even if we allow for redefinitons of the kind $[\tilde{O}_1] \to [\tilde{O}_1]' = Z_1(a_s)[\tilde{O}_1] + Z_2(a_s)[\partial_\mu J_5^\mu]$ the RGI operator derived with this method is the same:

$$[\tilde{O}_1]^{\text{RGV}\prime} = [\tilde{O}_1]^{\text{RGV}} + \frac{E_1(a_s)Z_2(a_s)}{E_2(a_s)Z_1(a_s)}[\partial_\mu J_5^\mu]^{\text{RGI}} \tag{3.22}$$

$$\Rightarrow \mu^2 \frac{d}{d\mu^2}[\tilde{O}_1]^{\text{RGV}\prime} = \left[\frac{E_1(a_s)}{E_2(a_s)}\gamma_{GJ}(a_s) + \mu^2 \frac{d}{d\mu^2}\left(\frac{E_1(a_s)Z_2(a_s)}{E_2(a_s)Z_1(a_s)}\right)\right][\partial_\mu J_5^\mu]^{\text{RGI}} \tag{3.23}$$

$$\Rightarrow [\tilde{O}_1]^{\text{RGI}\prime} = [\tilde{O}_1]^{\text{RGV}\prime} - \underbrace{\left[\left(\int^{a_s(\mu)} \frac{E_1(a)}{E_2(a)}\gamma_{GJ}(a)\frac{da}{a\,\beta(a)}\right) + \frac{E_1(a_s)Z_2(a_s)}{E_2(a_s)Z_1(a_s)}\right]}_{=a_s\tilde{Z}'(a_s)}[\partial_\mu J_5^\mu]^{\text{RGI}}$$

$$= [\tilde{O}_1]^{\text{RGV}} - a_s\tilde{Z}(a_s)[\partial_\mu J_5^\mu]^{\text{RGI}} = [\tilde{O}_1]^{\text{RGI}}. \tag{3.24}$$

The leading RGI Wilson coefficient

$$C_0^{\text{RGI}}(q) = \frac{1}{(Q^2)^2}\,\langle 0|X_t^{\text{RGI}}(q)|0\rangle\Big|_{\text{pert}} \tag{3.25}$$

in an OPE of the RGI correlator

$$X_t^{\text{RGI}}(q) := i\int \mathrm{d}^4x\, e^{iqx}T\{[\tilde{O}_1]^{\text{RGI}}(x)[\tilde{O}_1]^{\text{RGI}}(0)\} \tag{3.26}$$

can now be calculated from the same three bare correlators as C_0 and the result for its Q^2-derivative is

$$\begin{aligned} Q^2\frac{d}{dQ^2}\,C_0^{\text{RGI}} =& \frac{a_s^2 n_g}{\pi^2}\Bigg[-1 + a_s\left(-\frac{97}{12}C_A + \frac{7}{3}n_fT_F - \frac{11}{6}l_{\mu q}C_A + \frac{2}{3}l_{\mu q}n_fT_F\right) \\ &+\frac{a_s}{(11C_A - 4n_fT_F)}18n_fT_FC_F + a_s^2\Bigg(-\frac{51959}{864}C_A^2 + \frac{107}{12}n_fT_FC_F \\ &+\frac{3793}{108}n_fC_AT_F - \frac{251}{54}n_f^2T_F^2 - \frac{1135}{48}l_{\mu q}C_A^2 + 2l_{\mu q}n_fT_FC_F \\ &+\frac{46}{3}l_{\mu q}n_fC_AT_F - \frac{7}{3}l_{\mu q}n_f^2T_F^2 - \frac{121}{48}l_{\mu q}^2C_A^2 + \frac{11}{6}l_{\mu q}^2n_fC_AT_F \\ &-\frac{1}{3}l_{\mu q}^2n_f^2T_F^2 + \frac{55}{8}\zeta_3C_A^2 - 3\zeta_3n_fT_FC_F + \frac{1}{2}\zeta_3n_fC_AT_F\Bigg) \\ &+\frac{a_s^2}{(11C_A - 4n_fT_F)}\Bigg(\frac{291}{2}n_fC_AT_FC_F - 42n_f^2T_F^2C_F \\ &+33l_{\mu q}n_fC_AT_FC_F - 12l_{\mu q}n_f^2T_F^2C_F\Bigg) \\ &+\frac{a_s^2}{(11C_A - 4n_fT_F)^2}\Bigg(-\frac{297}{4}n_fC_AT_FC_F^2 + \frac{475}{4}n_fC_A^2T_FC_F \\ &-108n_f^2T_F^2C_F^2 - 37n_f^2C_AT_F^2C_F + 4n_f^3T_F^3C_F\Bigg)\Bigg]. \end{aligned} \tag{3.27}$$

An explicit calculation confirms that indeed $\mu^2\frac{d}{d\mu^2}\left(Q^2\frac{d}{dQ^2}\,C_0^{\text{RGI}}\right) = 0$.

As explained in [2] a finite and RGI version of O_1 can be defined as

$$O_1^{\mathrm{RGI}} := \hat{\beta}(a_s)\,[O_1], \quad \hat{\beta}(a_s) := \frac{-\beta(a_s)}{\beta_0} = a_s \left(1 + \sum_{i\geq 1} \frac{\beta_i}{\beta_0} a_s^i \right). \tag{3.28}$$

The RGI Wilson coefficient

$$\begin{aligned} C_1^{\mathrm{RGI}}(Q^2) =& \frac{a_s^2}{\hat{\beta}(a_s)} \Big\{ Z_{G\tilde{G}}^2 C_{1,B}^{(\tilde{O}_1^B,\tilde{O}_1^B)}(Q^2) \\ &+ (2Z_{G\tilde{G}} Z_{GJ} - 2E_2 Z_{G\tilde{G}} Z_J^s \tilde{Z}) C_{1,B}^{(\tilde{O}_1^B,\partial_\nu J_5^{B\,\nu})}(Q^2) \\ &+ (Z_{GJ}^2 - 2E_2 Z_{GJ} Z_J^s \tilde{Z} + (E_2 Z_J^s \tilde{Z})^2) C_{1,B}^{(\partial_\nu J_5^{B\,\nu},\partial_\nu J_5^{B\,\nu})}(Q^2) \Big\} \end{aligned} \tag{3.29}$$

which satisfies

$$C_1^{\mathrm{RGI}} [O_1]^{\mathrm{RGI}} = C_1 [O_1] \tag{3.30}$$

in the OPE of (3.26). The result is

$$\begin{aligned} C_1^{\mathrm{RGI}} =& 64 a_s \Big\{ 1 + a_s \left(\frac{157}{36} C_A - \frac{5}{9} n_f T_F + \frac{11}{12} l_{\mu q} C_A - \frac{1}{3} l_{\mu q} n_f T_F \right) \\ &+ \frac{a_s}{(11C_A - 4n_f T_F)} \left(-\frac{17}{2} C_A^2 - 15 n_f T_F C_F + 5 n_f C_A T_F \right) \\ &+ a_s^2 \Big(\frac{25945}{1296} C_A^2 - \frac{11}{2} n_f T_F C_F - \frac{4355}{648} n_f C_A T_F + \frac{25}{81} n_f^2 T_F^2 + \frac{1727}{216} l_{\mu q} C_A^2 \\ &- \frac{3}{2} l_{\mu q} n_f T_F C_F - \frac{106}{27} l_{\mu q} n_f C_A T_F + \frac{10}{27} l_{\mu q} n_f^2 T_F^2 + \frac{121}{144} l_{\mu q}^2 C_A^2 - \frac{11}{18} l_{\mu q}^2 n_f C_A T_F \\ &+ \frac{1}{9} l_{\mu q}^2 n_f^2 T_F^2 - \frac{33}{8} \zeta_3 C_A^2 + 3 \zeta_3 n_f T_F C_F - \frac{3}{2} \zeta_3 n_f C_A T_F \Big) \\ &+ \frac{a_s^2}{(11C_A - 4n_f T_F)} \Big(-\frac{2669}{72} C_A^3 - \frac{785}{12} n_f C_A T_F C_F + \frac{955}{36} n_f C_A^2 T_F \\ &+ \frac{25}{3} n_f^2 T_F^2 C_F - \frac{25}{9} n_f^2 C_A T_F^2 - \frac{187}{24} l_{\mu q} C_A^3 - \frac{55}{4} l_{\mu q} n_f C_A T_F C_F + \frac{89}{12} l_{\mu q} n_f C_A^2 T_F \\ &+ 5 l_{\mu q} n_f^2 T_F^2 C_F - \frac{5}{3} l_{\mu q} n_f^2 C_A T_F^2 \Big) + \frac{a_s^2}{(11C_A - 4n_f T_F)^2} \Big(-\frac{10619}{288} C_A^4 \\ &+ \frac{561}{8} n_f C_A T_F C_F^2 + \frac{1451}{48} n_f C_A^2 T_F C_F + \frac{3013}{48} n_f C_A^3 T_F + \frac{129}{2} n_f^2 T_F^2 C_F^2 \\ &- \frac{301}{6} n_f^2 C_A T_F^2 C_F - \frac{211}{8} n_f^2 C_A^2 T_F^2 - \frac{1}{3} n_f^3 T_F^3 C_F + \frac{79}{18} n_f^3 C_A T_F^3 \Big) \\ &+ a_s^3 \Big(\frac{19360399}{186624} C_A^3 + \frac{461}{144} n_f T_F C_F^2 - \frac{614501}{10368} n_f C_A T_F C_F \\ &- \frac{1857805}{31104} n_f C_A^2 T_F + \frac{28981}{2592} n_f^2 T_F^2 C_F + \frac{126415}{15552} n_f^2 C_A T_F^2 - \frac{125}{729} n_f^3 T_F^3 \\ &+ \frac{594247}{10368} l_{\mu q} C_A^3 + \frac{35}{32} l_{\mu q} n_f T_F C_F^2 - \frac{1623}{64} l_{\mu q} n_f C_A T_F C_F - \frac{68935}{1728} l_{\mu q} n_f C_A^2 T_F \\ &+ \frac{105}{16} l_{\mu q} n_f^2 T_F^2 C_F + \frac{6661}{864} l_{\mu q} n_f^2 C_A T_F^2 - \frac{25}{81} l_{\mu q} n_f^3 T_F^3 + \frac{9779}{864} l_{\mu q}^2 C_A^3 \\ &- \frac{275}{96} l_{\mu q}^2 n_f C_A T_F C_F - \frac{2795}{288} l_{\mu q}^2 n_f C_A^2 T_F + \frac{25}{24} l_{\mu q}^2 n_f^2 T_F^2 C_F + \frac{61}{24} l_{\mu q}^2 n_f^2 C_A T_F^2 \end{aligned} \tag{3.31}$$

$$
\begin{aligned}
&-\frac{5}{27}l_{\mu q}^{2}n_f^3T_F^3+\frac{1331}{1728}l_{\mu q}^{3}C_A^3-\frac{121}{144}l_{\mu q}^{3}n_fC_A^2T_F+\frac{11}{36}l_{\mu q}^{3}n_f^2C_AT_F^2\\
&-\frac{1}{27}l_{\mu q}^{3}n_f^3T_F^3+\frac{55}{8}\zeta_5C_A^3-15\zeta_5n_fT_FC_F^2+\frac{15}{2}\zeta_5n_fC_AT_FC_F+5\zeta_5n_fC_A^2T_F\\
&-\frac{6893}{144}\zeta_3C_A^3+\frac{145}{12}\zeta_3n_fT_FC_F^2+\frac{1291}{48}\zeta_3n_fC_AT_FC_F-\frac{349}{144}\zeta_3n_fC_A^2T_F\\
&-\frac{13}{2}\zeta_3n_f^2T_F^2C_F+\frac{121}{36}\zeta_3n_f^2C_AT_F^2-\frac{363}{32}\zeta_3l_{\mu q}C_A^3+\frac{33}{4}\zeta_3l_{\mu q}n_fC_AT_FC_F\\
&-3\zeta_3l_{\mu q}n_f^2T_F^2C_F+\frac{3}{2}\zeta_3l_{\mu q}n_f^2C_AT_F^2\Bigg)+\frac{a_s^3}{(11C_A-4n_fT_F)}\Bigg(-\frac{441065}{2592}C_A^4\\
&-\frac{109529}{432}n_fC_A^2T_FC_F+\frac{1415}{9}n_fC_A^3T_F+\frac{165}{2}n_f^2T_F^2C_F^2+\frac{15835}{216}n_f^2C_AT_F^2C_F\\
&-\frac{7825}{216}n_f^2C_A^2T_F^2-\frac{125}{27}n_f^3T_F^3C_F+\frac{125}{81}n_f^3C_AT_F^3-\frac{29359}{432}l_{\mu q}C_A^4\\
&-\frac{7717}{72}l_{\mu q}n_fC_A^2T_FC_F+\frac{5281}{72}l_{\mu q}n_fC_A^3T_F+\frac{45}{2}l_{\mu q}n_f^2T_F^2C_F^2+\frac{925}{18}l_{\mu q}n_f^2C_AT_F^2C_F\\
&-\frac{205}{9}l_{\mu q}n_f^2C_A^2T_F^2-\frac{50}{9}l_{\mu q}n_f^3T_F^3C_F+\frac{50}{27}l_{\mu q}n_f^3C_AT_F^3-\frac{2057}{288}l_{\mu q}^{2}C_A^4\\
&-\frac{605}{48}l_{\mu q}^{2}n_fC_A^2T_FC_F+\frac{451}{48}l_{\mu q}^{2}n_fC_A^3T_F+\frac{55}{6}l_{\mu q}^{2}n_f^2C_AT_F^2C_F-4l_{\mu q}^{2}n_f^2C_A^2T_F^2\\
&-\frac{5}{3}l_{\mu q}^{2}n_f^3T_F^3C_F+\frac{5}{9}l_{\mu q}^{2}n_f^3C_AT_F^3+\frac{561}{16}\zeta_3C_A^4+\frac{291}{8}\zeta_3n_fC_A^2T_FC_F\\
&-\frac{63}{8}\zeta_3n_fC_A^3T_F-45\zeta_3n_f^2T_F^2C_F^2+\frac{75}{2}\zeta_3n_f^2C_AT_F^2C_F-\frac{15}{2}\zeta_3n_f^2C_A^2T_F^2\Bigg)\\
&+\frac{a_s^3}{(11C_A-4n_fT_F)^2}\Bigg(-\frac{1667183}{10368}C_A^5+\frac{29359}{96}n_fC_A^2T_FC_F^2+\frac{227807}{1728}n_fC_A^3T_FC_F\\
&+\frac{1525313}{5184}n_fC_A^4T_F+\frac{727}{3}n_f^2C_AT_F^2C_F^2-\frac{33923}{144}n_f^2C_A^2T_F^2C_F-\frac{129511}{864}n_f^2C_A^3T_F^2\\
&-\frac{215}{6}n_f^3T_F^3C_F^2+\frac{317}{12}n_f^3C_AT_F^3C_F+\frac{10949}{324}n_f^3C_A^2T_F^3+\frac{5}{27}n_f^4T_F^4C_F-\frac{395}{162}n_f^4C_AT_F^4\\
&-\frac{116809}{3456}l_{\mu q}C_A^5+\frac{2057}{32}l_{\mu q}n_fC_A^2T_FC_F^2+\frac{15961}{576}l_{\mu q}n_fC_A^3T_FC_F+\frac{120667}{1728}l_{\mu q}n_fC_A^4T_F\\
&+\frac{143}{4}l_{\mu q}n_f^2C_AT_F^2C_F^2-\frac{897}{16}l_{\mu q}n_f^2C_A^2T_F^2C_F-\frac{12989}{288}l_{\mu q}n_f^2C_A^3T_F^2-\frac{43}{2}l_{\mu q}n_f^3T_F^3C_F^2\\
&+\frac{197}{12}l_{\mu q}n_f^3C_AT_F^3C_F+\frac{346}{27}l_{\mu q}n_f^3C_A^2T_F^3+\frac{1}{9}l_{\mu q}n_f^4T_F^4C_F-\frac{79}{54}l_{\mu q}n_f^4C_AT_F^4\Bigg)\\
&+\frac{a_s^3}{(11C_A-4n_fT_F)^3}\Bigg(-\frac{7623}{16}n_fC_A^2T_FC_F^3+\frac{22121}{32}n_fC_A^3T_FC_F^2+\frac{31207}{32}n_fC_A^4T_FC_F\\
&-\frac{2079}{4}n_f^2C_AT_F^2C_F^3+\frac{13533}{8}n_f^2C_A^2T_F^2C_F^2-\frac{29647}{12}n_f^2C_A^3T_F^2C_F-45n_f^3T_F^3C_F^3\\
&-\frac{1911}{2}n_f^3C_AT_F^3C_F^2+1443n_f^3C_A^2T_F^3C_F+178n_f^4T_F^4C_F^2-384n_f^4C_AT_F^4C_F+\frac{104}{3}n_f^5T_F^5C_F\\
&-2178\zeta_3n_f^2C_A^2T_F^2C_F^2+2178\zeta_3n_f^2C_A^3T_F^2C_F+1584\zeta_3n_f^3C_AT_F^3C_F^2-1584\zeta_3n_f^3C_A^2T_F^3C_F\\
&-288\zeta_3n_f^4T_F^4C_F^2+288\zeta_3n_f^4C_AT_F^4C_F\Bigg)\Bigg\}\,.
\end{aligned}
$$

Again an explicit calculation confirms that indeed $\mu^2\frac{d}{d\mu^2}\,C_1^{\rm RGI}=0$. This result can now be used to obtain the logarithmic pieces of $Q^2\frac{d}{dQ^2}C_0^{\rm RGI}$ and $C_1^{\rm RGI}$ at four-loop level. If a generic RGI quantitiy has the structure

$$\begin{aligned} Q^{\mathrm{RGI}} &= a_s(\mu)A_1 + a_s(\mu)^2(A_2 + l_{\mu q}B_2) + a_s(\mu)^3(A_3 + l_{\mu q}B_3 + l_{\mu q}^2 C_3) \\ &\quad + a_s(\mu)^4(A_4 + l_{\mu q}B_4 + l_{\mu q}^2 C_4 + l_{\mu q}^3 D_4) \\ &\quad + a_s(\mu)^5(A_5 + l_{\mu q}B_5 + l_{\mu q}^2 C_5 + l_{\mu q}^3 D_5 + l_{\mu q}^4 E_5) + \mathcal{O}(a_s^6) \end{aligned} \tag{3.32}$$

with scale independent coefficients $(A_i, B_i, \dots)$ the requirement $\mu^2 \frac{d}{d\mu^2} Q^{\mathrm{RGI}} \overset{!}{=} 0$ leads to the conditions

$$\begin{aligned} &B_2 = A_1\beta_0, \\ &C_3 = B_2\beta_0, \quad B_3 = A_1\beta_1 + 2A_2\beta_0, \\ &D_4 = C_3\beta_0, \quad C_4 = \frac{1}{2}\left(3B_3\beta_0 + 2B_2\beta_1\right), \quad B_4 = A_1\beta_2 + 2A_2\beta_1 + 3A_3\beta_0 \end{aligned}$$

which in the cases of $Q^2 \frac{d}{dQ^2} C_0^{\mathrm{RGI}}$ and C_1^{RGI} can be used as checks and

$$\begin{aligned} E_5 &= D_4\beta_0, \\ D_5 &= \frac{1}{3}\left(4C_4\beta_0 + 3C_3\beta_1\right), \\ C_5 &= \frac{1}{2}\left(2B_2\beta_2 + 3B_3\beta_1 + 4B_4\beta_0\right), \\ B_5 &= A_1\beta_3 + 2A_2\beta_2 + 3A_3\beta_1 + 4A_4\beta_0. \end{aligned}$$

Using the four-loop β-function[11] of QCD [37, 38] the following four-loop contributions (for QCD colour factors) are derived:

$$\begin{aligned} Q^2 \frac{d}{dQ^2} C_0^{\mathrm{RGI,\ 4loop}} &= \frac{a_s^5 n_g}{\pi^2} \Bigg\{ \left(\frac{n_f^3}{54} - \frac{11 n_f^2}{12} + \frac{121 n_f}{8} - \frac{1331}{16} \right) l_{\mu q}^3 \\ &+ \left(\frac{7 n_f^3}{36} - \frac{1783 n_f^2}{144} + \frac{21647 n_f}{96} - \frac{19569}{16} \right) l_{\mu q}^2 \\ &+ \frac{1}{(33 - 2n_f)^2} \Bigg(\frac{251 n_f^5}{81} + \frac{10 n_f^4 \zeta_3}{3} - \frac{147169 n_f^4}{432} - 330 n_f^3 \zeta_3 \\ &+ \frac{108663 n_f^3}{8} + 10890 n_f^2 \zeta_3 - \frac{48109321 n_f^2}{192} - \frac{299475 n_f \zeta_3}{2} \\ &+ \frac{138470387 n_f}{64} + \frac{5929605 \zeta_3}{8} - \frac{450379545}{64} \Bigg) l_{\mu q} + \mathrm{const.} \Bigg\}, \end{aligned} \tag{3.33}$$

---

[11] The one-loop, two-loop and three-loop results are known from [21, 31–36].

$$
\begin{aligned}
C_1^{\text{RGI, 4loop}} = 64\, a_s^5 \Bigg\{ & \left(\frac{n_f^4}{1296} - \frac{11 n_f^3}{216} + \frac{121 n_f^2}{96} - \frac{1331 n_f}{96} + \frac{14641}{256} \right) l_{\mu q}^4 \\
& + \left(\frac{5 n_f^4}{972} - \frac{1595 n_f^3}{2592} + \frac{4355 n_f^2}{216} - \frac{293975 n_f}{1152} + \frac{424105}{384} \right) l_{\mu q}^3 \\
& + \left(\frac{25 n_f^4}{1944} - \frac{n_f^3 \zeta_3}{24} - \frac{6937 n_f^3}{2304} - \frac{77 n_f^2 \zeta_3}{16} + \frac{1812625 n_f^2}{13824} \right. \\
& \left. + \frac{6171 n_f \zeta_3}{32} - \frac{954133 n_f}{512} - \frac{107811 \zeta_3}{64} + \frac{12658057}{1536} \right) l_{\mu q}^2 \\
& + \frac{1}{(33 - 2n_f)^2} \left(\frac{125 n_f^6}{2187} - \frac{17 n_f^5 \zeta_3}{18} - \frac{457613 n_f^5}{15552} - \frac{4237 n_f^4 \zeta_3}{54} \right. \\
& - \frac{580 n_f^4 \zeta_5}{9} + \frac{13206877 n_f^4}{5184} + \frac{38583 n_f^3 \zeta_3}{4} + 2695 n_f^3 \zeta_5 \\
& - \frac{905734235 n_f^3}{10368} - \frac{1172479 n_f^2 \zeta_3}{4} - \frac{56265 n_f^2 \zeta_5}{2} + \frac{6551159345 n_f^2}{4608} \\
& + \frac{113749075 n_f \zeta_3}{32} - \frac{459195 n_f \zeta_5}{4} - \frac{16816549087 n_f}{1536} \\
& \left. - \frac{486694791 \zeta_3}{32} + \frac{17788815 \zeta_5}{8} + \frac{48864828943}{1536} \right) l_{\mu q} + \text{const.} \Bigg\}
\end{aligned}
\tag{3.34}
$$

For completeness we also give the RGI Wilson coefficients for the correlator

$$
i \int \mathrm{d}^4 x\, e^{iqx} T\{[\partial_\mu J_5^\mu]^{\text{RGI}}(x) [\partial_\mu J_5^\mu]^{\text{RGI}}(0)\} = (Q^2)^2 C_0^{JJ,\text{RGI}} + C_1^{JJ,\text{RGI}} [O_1]^{\text{RGI}} + \dots . \tag{3.35}
$$

The results read

$$
Q^2 \frac{d}{dQ^2}\, C_0^{JJ,\text{RGI}} = \frac{n_g}{\pi^2} \left[-\frac{a_s^2}{16} n_f^2 T_F^2 \right] \tag{3.36}
$$

and

$$
\begin{aligned}
C_1^{JJ,\text{RGI}} = 4 a_s n_f^2 T_F^2 \Bigg\{ & 1 + a_s \left(\frac{157}{36} C_A - \frac{5}{9} n_f T_F + \frac{11}{12} l_{\mu q} C_A - \frac{1}{3} l_{\mu q} n_f T_F \right) \\
& + \frac{a_s}{(11 C_A - 4 n_f T_F)} \left(-\frac{17}{2} C_A^2 - 15 n_f T_F C_F + 5 n_f C_A T_F \right) \Bigg\} .
\end{aligned}
\tag{3.37}
$$

The four-loop extension of these results with QCD colour factors are given by

$$
Q^2 \frac{d}{dQ^2}\, C_0^{JJ,\text{RGI, 4loop}} = \frac{a_s^3 n_g}{\pi^2} \left[l_{\mu q} \frac{n_f^2(-33 + 2 n_f)}{384} + \text{const.} \right] \tag{3.38}
$$

and

$$
\begin{aligned}
C_1^{JJ,\text{RGI, 4loop}} = 4 a_s^3 \Bigg\{ & l_{\mu q} \frac{1}{864} n_f^2 (14166 - 1533 n_f + 20 n_f^2) \\
& + l_{\mu q}^2 \left(\frac{121 n_f^2}{64} - \frac{11 n_f^3}{48} + \frac{n_f^4}{144} \right) + \text{const.} \Bigg\} .
\end{aligned}
\tag{3.39}
$$

3.4 Numerics

We now consider the two cases $n_f = 0$ (pure gluodynamics) and $n_f = 3$ which are most important for applications. Furthermore we set $Q^2 = \mu^2$, i.e. $l_{\mu q} = 0$. The numerical results for C_1 and C_1^{RGI} are then

$$C_1(Q^2 = \mu^2, n_f = 0) = 64\{1 + 13.0833 a_s + 135.547 a_s^2 + 1439.88 a_s^3\}, \tag{3.40}$$

$$C_1(Q^2 = \mu^2, n_f = 3) = 64\{1 + 12.25 a_s + 94.0971 a_s^2 + 646.69 a_s^3\}, \tag{3.41}$$

$$C_1^{\mathrm{RGI}}(Q^2 = \mu^2, n_f = 0) = 64 a_s\{1 + 10.7652 a_s + 102.475 a_s^2 + 1089.78 a_s^3\}, \tag{3.42}$$

$$C_1^{\mathrm{RGI}}(Q^2 = \mu^2, n_f = 3) = 64 a_s\{1 + 9.13889 a_s + 55.9532 a_s^2 + 361.615 a_s^3\}. \tag{3.43}$$

In order to estimate the numerical significance of the higher order corrections we evaluate C_1 at $\mu = M_Z$, $\mu = 3.5$ GeV and $\mu = 2$ GeV with

$$\alpha_s^{(n_f=5)}(M_Z) \approx 0.118 \text{ , } \alpha_s^{(n_f=3)}(3.5\mathrm{GeV}) \approx 0.31 \text{ and } \alpha_s^{(n_f=3)}(2\mathrm{GeV}) \approx 0.47 \text{ [39]} \tag{3.44}$$

for the cases $n_f = 5$ and $n_f = 3$ respectively.

$$C_1(Q^2 = \mu^2 = M_Z^2, n_f = 5) = 64\ (\underbrace{0.0116}_{\text{3 loop}} + \underbrace{0.0949}_{\text{2 loop}} + \underbrace{0.4393}_{\text{1 loop}} + 1), \tag{3.45}$$

$$C_1(Q^2 = \mu^2 = (3.5\ \mathrm{GeV})^2, n_f = 3) = 64\ (\underbrace{0.6213}_{\text{3 loop}} + \underbrace{0.9162}_{\text{2 loop}} + \underbrace{1.2088}_{\text{1 loop}} + 1), \tag{3.46}$$

$$C_1(Q^2 = \mu^2 = (2\ \mathrm{GeV})^2, n_f = 3) = 64\ (\underbrace{2.1654}_{\text{3 loop}} + \underbrace{2.1061}_{\text{2 loop}} + \underbrace{1.8327}_{\text{1 loop}} + 1). \tag{3.47}$$

At the scale $\mu^2 = M_Z$ the two and three-loop contributions are about 9% and 1% wrt tree-level, whereas at a scale $\mu^2 = (2\ \mathrm{GeV})^2$ these contributions become so large that perturbation theory stops to work (as is expected). From this evaluation we can assume that in the case of $Q^2 = \mu^2$ the Wilson coefficient to this accuracy in perturbation theory is a valid approximation down to a scale of about $\mu^2 = (3.5\ \mathrm{GeV})^2$.

4 Discussion and Conclusions

I have presented higher order corrections for the coefficient function C_1 of the OPE of the correlator X_t of two pseudoscalar gluonium operators. This result extends the previously known accuracy by two loops. It is also worth of notice that no contact terms can appear in this coefficient due to the relation between the operator $\tilde{O}_1$ and the Chern-Simons current, a fact that has been explicitly checked and verified up to $\mathcal{O}(\alpha_s^3)$ by this calculation. The OPE of the correlator of two operators $\partial_\mu J_5^\mu$ which mixes with $\tilde{O}_1$ under renormalization has been performed as well and the corresponding coefficients C_0^{JJ} and C_1^{JJ} have been given at three-loop level. In addition the construction of RGI operators and Wilson coefficients has been discussed, the coefficients C_0^{RGI}, C_1^{RGI}, $C_0^{JJ,\mathrm{RGI}}$ and $C_1^{JJ,\mathrm{RGI}}$ have been presented and their logarithmic part has been derived at four-loop level from the principle of scale invariance. Finally, a numerical evaluation shows the validity of the OPE for the important case $n_f = 3$ and the choice $Q^2 = \mu^2$ down to a scale $\mu^2 = (3.5\ \mathrm{GeV})^2$.

Acknowledgements

I am grateful to K. G. Chetyrkin for initiating this project, many interesting discussions and constant support. I also thank J. H. Kühn for valuable discussions and support. Finally, I would like to thank M. Jamin, Y. Schröder and A. Vuorinen for useful comments.

All calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers using the thread-based [40] version of FORM [23]. The Feynman diagrams have been drawn with the Latex package Axodraw [41].

This work has been supported by the Deutsche Forschungsgemeinschaft in the Sonderforschungsbereich/Transregio SFB/TR-9 "Computational Particle Physics" and the Graduiertenkolleg "Elementarteilchenphysik bei höchsten Energien und höchster Präzision"

References

[1] K. Chetyrkin, B. A. Kniehl, M. Steinhauser, and W. A. Bardeen, *Effective QCD interactions of CP odd Higgs bosons at three loops*, *Nucl.Phys.* **B535** (1998) 3–18, [hep-ph/9807241].

[2] M. Zoller and K. Chetyrkin, *OPE of the energy-momentum tensor correlator in massless QCD*, arXiv:1209.1516.

[3] K. G. Wilson, *Non-lagrangian models of current algebra*, *Phys. Rev.* **179** (1969), no. 5 1499–1512.

[4] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *QCD and Resonance Physics. Sum Rules*, *Nucl. Phys.* **B147** (1979) 385–447.

[5] H. Forkel, *Direct instantons, topological charge screening, and qcd glueball sum rules*, *Phys. Rev. D* **71** (2005), no. 5 054008.

[6] M. Luscher, *Topological effects in QCD and the problem of short distance singularities*, *Phys.Lett.* **B593** (2004) 296–301, [hep-th/0404034].

[7] L. Del Debbio, L. Giusti, and C. Pica, *Topological susceptibility in su(3) gauge theory*, *Phys. Rev. Lett.* **94** (Jan, 2005) 032003.

[8] E. Witten, *Current Algebra Theorems for the U(1) Goldstone Boson*, *Nucl.Phys.* **B156** (1979) 269.

[9] G. Veneziano, *U(1) Without Instantons*, *Nucl.Phys.* **B159** (1979) 213–224.

[10] E. Seiler, *Some more remarks on the Witten-Veneziano formula for the eta-prime mass*, *Phys.Lett.* **B525** (2002) 355–359, [hep-th/0111125].

[11] L. Giusti, G. Rossi, M. Testa, and G. Veneziano, *The U(A)(1) problem on the lattice with Ginsparg-Wilson fermions*, *Nucl.Phys.* **B628** (2002) 234–252, [hep-lat/0108009].

[12] V. Novikov, M. A. Shifman, A. Vainshtein, and V. I. Zakharov, *eta-prime Meson as Pseudoscalar Gluonium*, *Phys.Lett.* **B86** (1979) 347.

[13] P. A. Baikov and K. G. Chetyrkin, *Higgs decay into hadrons to order* α_s^5, *Phys. Rev. Lett.* **97** (2006) 061803, [hep-ph/0604194].

[14] M. Laine, M. Vepsalainen, and A. Vuorinen, *Ultraviolet asymptotics of scalar and pseudoscalar correlators in hot Yang-Mills theory*, *JHEP* **1010** (2010) 010, [arXiv:1008.3263].

[15] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *In search of scalar gluonium*, *Nuclear Physics B* **165** (1980), no. 1 67 – 79.

[16] V. Novikov, M. Shifman, A. Vainshtein, and V. Zakharov, *eta-prime meson as a pseudoscalar gluonium*, *Physics Letters B* **86** (1979), no. 3-4 347 – 350.

[17] A.-l. Zhang and T. G. Steele, *Instanton and higher loop perturbative contributions to the QCD sum rule analysis of pseudoscalar gluonium*, *Nucl.Phys.* **A728** (2003) 165–181, [hep-ph/0304208].

[18] V. Spiridonov, *Anomalous dimension of $g^2_{\mu\nu}$ and β-function*, *Preprint IYAI-P-0378* (1984) [http://www-lib.kek.jp/cgi-bin/img_index?8601315].

[19] N. Nielsen, *Gauge Invariance and Broken Conformal Symmetry*, *Nucl.Phys.* **B97** (1975) 527.

[20] S. Larin, *The Renormalization of the axial anomaly in dimensional regularization*, *Phys.Lett.* **B303** (1993) 113–118, [hep-ph/9302240].

[21] S. Larin and J. Vermaseren, *The Three loop QCD Beta function and anomalous dimensions*, *Phys.Lett.* **B303** (1993) 334–336, [hep-ph/9302208].

[22] P. Nogueira, *Automatic Feynman graph generation*, *J. Comput. Phys.* **105** (1993) 279–289.

[23] J. A. M. Vermaseren, *New features of FORM*, math-ph/0010025.

[24] S. A. Larin, F. V. Tkachov, and J. A. M. Vermaseren, *The form version of mincer*, . NIKHEF-H-91-18.

[25] S. G. Gorishnii, S. A. Larin, L. R. Surguladze, and F. V. Tkachov, *MINCER: Program for multiloop calculations in quantum field theory for the SCHOONSCHIP system*, *Comput. Phys. Commun.* **55** (1989) 381–408.

[26] T. van Ritbergen, A. N. Schellekens, and J. A. M. Vermaseren, *Group theory factors for feynman diagrams*, *Int. J. Mod. Phys.* **A14** (1999) 41–96, [hep-ph/9802376].

[27] S. G. Gorishny, S. A. Larin, and F. V. Tkachov, *The Algorithm For OPE Coefficient Functions In The MS Scheme*, *Phys. Lett.* **B124** (1983) 217–220.

[28] S. G. Gorishny and S. A. Larin, *Coefficient Functions Of Asymptotic Operator Expansions In Minimal Subtraction Scheme*, *Nucl. Phys.* **B283** (1987) 452.

[29] M. Bos, *Explicit calculation of the renormalized singlet axial anomaly*, *Nucl.Phys.* **B404** (1993) 215–244, [hep-ph/9211319].

[30] K. G. Chetyrkin and J. H. Kühn, *Neutral current in the heavy top quark limit and the renormalization of the singlet axial current*, *Z. Phys.* **C60** (1993) 497–502.

[31] D. Gross and F. Wilczek, *Ultraviolet Behavior of Nonabelian Gauge Theories*, *Phys.Rev.Lett.* **30** (1973) 1343–1346.

[32] H. D. Politzer, *Reliable Perturbative Results for Strong Interactions?*, *Phys.Rev.Lett.* **30** (1973) 1346–1349.

[33] D. Jones, *Two Loop Diagrams in Yang-Mills Theory*, *Nucl.Phys.* **B75** (1974) 531.

[34] E. Egorian and O. Tarasov, *Two loop renormalization of the QCD in an arbitrary gauge*, *Teor.Mat.Fiz.* **41** (1979) 26–32.

[35] W. E. Caswell, *Asymptotic Behavior of Nonabelian Gauge Theories to Two Loop Order*, *Phys.Rev.Lett.* **33** (1974) 244.

[36] O. Tarasov, A. Vladimirov, and A. Y. Zharkov, *The Gell-Mann-Low Function of QCD in the Three Loop Approximation*, *Phys.Lett.* **B93** (1980) 429–432.

[37] T. van Ritbergen, J. Vermaseren, and S. Larin, *The Four loop beta function in quantum chromodynamics*, *Phys.Lett.* **B400** (1997) 379–384, [`hep-ph/9701390`].

[38] M. Czakon, *The Four-loop QCD beta-function and anomalous dimensions*, *Nucl.Phys.* **B710** (2005) 485–498, [`hep-ph/0411261`].

[39] K. Chetyrkin, J. H. Kuhn, and M. Steinhauser, *RunDec: A Mathematica package for running and decoupling of the strong coupling and quark masses*, *Comput.Phys.Commun.* **133** (2000) 43–65, [`hep-ph/0004189`].

[40] M. Tentyukov and J. A. M. Vermaseren, *The multithreaded version of FORM*, `hep-ph/0702279`.

[41] J. A. M. Vermaseren, *Axodraw*, *Comput. Phys. Commun.* **83** (1994) 45–58.