Master Integrals for Four-Loop Massless Propagators up to Transcendentality Weight Twelve

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Abstract

We evaluate a Laurent expansion in dimensional regularization parameter $\epsilon = (4 - d)/2$ of all the master integrals for four-loop massless propagators up to transcendentality weight twelve, using a recently developed method of one of the present coauthors (R.L.) and extending thereby results by Baikov and Chetyrkin obtained at transcendentality weight seven. We observe only multiple zeta values in our results. Therefore, we conclude that all the four-loop massless propagator integrals, with any integer powers of numerators and propagators, have only multiple zeta values in their epsilon expansions up to transcendentality weight twelve.

Keywords:
multiloop Feynman integrals, dimensional regularization, multiple zeta values

About one year ago Baikov and Chetyrkin published their results of evaluation of all the master integrals for four-loop massless propagators in a Laurent expansion in dimensional regularization parameter $\epsilon = (4-d)/2$ up to transcendentality weight seven [1].

These integrals are associated with graphs depicted in Fig. 1. The corresponding coefficients at powers of ϵ turned out to be linear combinations of

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multiple zeta values (MZV)

$$\zeta(m_1, \ldots, m_k) = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \prod_{j=1}^{k} \frac{\text{sgn}(m_j)^{i_j}}{i_j^{|m_j|}} \tag{1}$$

(see, e.g., Ref. [2]). In our recent publication [3], we studied three four-loop non-planar massless propagator integrals (corresponding to non-planar graphs) where, according to the analysis of Brown [4], one could meet not only MZV but also Goncharov's polylogarithms [5] with sixth roots of unity as arguments. One of them is $M_{4,5}$ in Fig. 1 and the other two integrals are not master integrals and reduce to some master integrals in this figure. We performed calculations up to transcendentality weight twelve [3] and observed only MZV in results.

In this paper we continue our experimental investigation of the four-loop massless propagator diagrams and present results for *all* the master integrals up to transcendentality weight twelve. Our motivation is twofold. First, we would like to check explicitly whether there are only MZV in results. Second, we would like to demonstrate further the power of the "dimensional-recurrence-and-analyticity" (DRA) method [6] that we use. The method is based on the use of dimensional recurrence relations (DRR) [7] and analytic properties of Feynman integrals as functions of the parameter of dimensional regularization, d, and was already successfully applied in previous calculations [8, 9, 10, 11, 12, 3].

A necessary condition of the application of DRA method is the possibility to make an integration-by-parts (IBP) [13] reduction of integrals participating in DRR to master integrals. To do this, we use the `C++` version of the code `FIRE` [14]. To reveal analytic properties of the master integrals we used a sector decomposition [15, 16, 17] implemented in the code `FIESTA` [17, 18]. To fix remaining constants in the homogenous solutions of dimensional recurrence relations it was generally not sufficient (contrary to our previous work [3] on the evaluation of three non-planar diagrams) to use analytic results for the four-loop massless propagators master integrals [1] (confirmed numerically by `FIESTA` in Ref. [19] where one more term of the ϵ-expansion was obtained) up to transcendentality weight seven. To obtain additional information here we applied the method of Mellin–Barnes (MB) representation [20, 21, 22]. For each integral the number of terms calculated with the help of MB representation was sufficiently large to provide at least one consistency check of our results.

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At the final stage of the method, we applied the PSLQ algorithm [23] as well as the code HPL [24] for dealing with MZV. Since the rational coefficients at transcendental numbers turn out to be quite cumbersome we applied PSLQ with the rather high accuracy of 1500 digits.

In our results presented below, we tried to reveal as much as possible the homogeneous transcendentality. This is achieved by pulling out suitable rational functions of ϵ and by considering a linear combination, with rational coefficients, of a given master integral and some of its lower master integrals. However, for the last three integrals, the task of finding such linear combination is quite complicated because of the large number of the lower master integrals. For these three integrals we have succeeded to find their representation in the form of a sum of several homogeneous terms with rational coefficients in d. This representation allowed us to pull out a rational factor so that any transcendental number appeared only in the limited number of consecutive terms of ϵ expansion. We choose the following loop integration measure

$$\frac{\Gamma(d-2)}{\Gamma(3-d/2)\Gamma(d/2-1)^2}\frac{d^d l}{i\pi^{d/2}}, \tag{2}$$

so that $M_{3,1}(4-2\epsilon) = \epsilon^{-4}$. This choice corresponds to the normalization of Refs. [1, 3].

Below we list our results ordered by the complexity level (c.l.) [8].

Integrals with c.l.=1

$$M_{2,1} + \frac{(3-7\epsilon)(3-5\epsilon)(4-5\epsilon)(3-4\epsilon)}{(1-3\epsilon)(1-2\epsilon)^3}M_{0,1} - \frac{(3-5\epsilon)(2-3\epsilon)(1-\epsilon)}{2(1-4\epsilon)(1-2\epsilon)^2}M_{1,3}$$

$$= \frac{1}{1-4\epsilon}\left\{-\frac{5}{48\epsilon^3} - \frac{19\zeta_3}{12} - \frac{19\zeta_4\epsilon}{8} - \frac{341\zeta_5\epsilon^2}{4} - \left(-\frac{493\zeta_3^2}{6} + \frac{1255\zeta_6}{6}\right)\epsilon^3\right.$$

$$- \left(-\frac{493}{2}\zeta_3\zeta_4 + \frac{16619\zeta_7}{8}\right)\epsilon^4 - \left(-2402\zeta_3\zeta_5 + \frac{61523\zeta_8}{16} + \frac{1215\zeta_{2,6}}{2}\right)\epsilon^5$$

$$- \left(\frac{13166\zeta_3^3}{9} - 7248\zeta_4\zeta_5 - \frac{25895\zeta_3\zeta_6}{3} + \frac{3258785\zeta_9}{72}\right)\epsilon^6 - \left(6583\zeta_3^2\zeta_4\right.$$

$$- \frac{4293369\zeta_5^2}{112} - \frac{7884283\zeta_3\zeta_7}{112} + \frac{28842885\zeta_{10}}{224} - \frac{242325\zeta_{3,7}}{112}\right)\epsilon^7 - \left(49534\zeta_3^2\zeta_5\right.$$

$$- \frac{453725\zeta_5\zeta_6}{2} - \frac{1981671\zeta_4\zeta_7}{16} - 182183\zeta_3\zeta_8 + 173160\zeta_2\zeta_9 + \frac{90139333\zeta_{11}}{128}$$
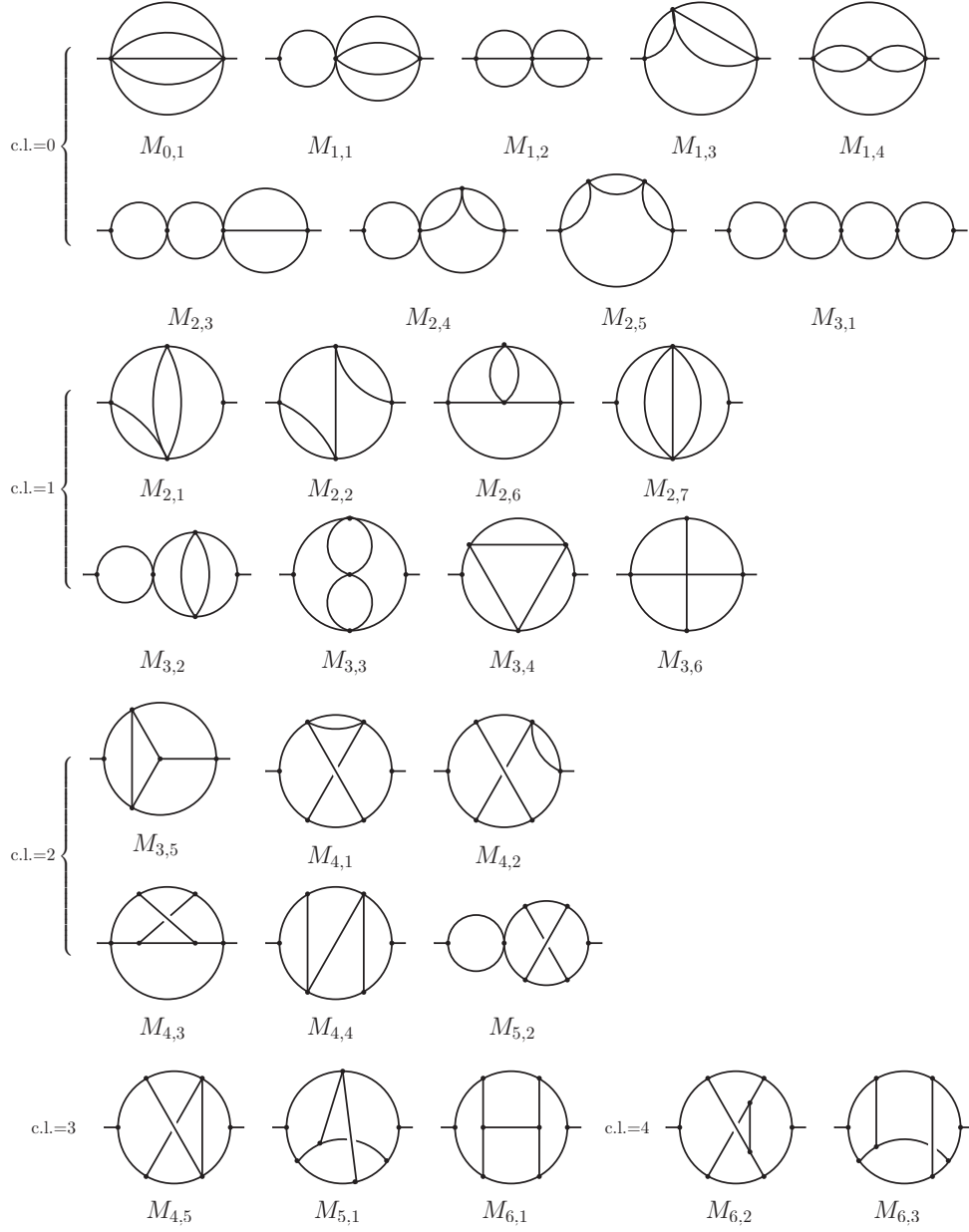
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Figure 1: Master diagrams for four-loop massless propagators. Here c.l. means the complexity level — see Ref. [8].

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$$-19080\zeta_3\zeta_{2,6} + 9360\zeta_{2,1,8}\Big)\epsilon^8 - \Big(-\frac{290335\zeta_3^4}{18} + \frac{536919}{2}\zeta_3\zeta_4\zeta_5$$

$$+\frac{2074005}{14}\zeta_2\zeta_5^2 + \frac{2107405}{12}\zeta_3^2\zeta_6 + \frac{2108145}{7}\zeta_2\zeta_3\zeta_7 - \frac{4328161\zeta_5\zeta_7}{2}$$

$$-\frac{146898725\zeta_3\zeta_9}{72} + \frac{1792356175927\zeta_{12}}{530688} - \frac{59715}{2}\zeta_4\zeta_{2,6} - \frac{76815}{7}\zeta_2\zeta_{3,7}$$

$$+\frac{270635\zeta_{3,9}}{24} + 17070\zeta_{2,1,1,8}\Big)\epsilon^9 + O\left(\epsilon^{10}\right)\bigg\}, \tag{3}$$

$$M_{2,2} - \frac{4(3-5\epsilon)(4-5\epsilon)(3-4\epsilon)}{(1-3\epsilon)(1-2\epsilon)^2}M_{0,1} + \frac{(2-3\epsilon)^2}{(2-5\epsilon)(1-2\epsilon)}M_{1,2}$$

$$= \frac{2(1-2\epsilon)}{(1-5\epsilon)(2-5\epsilon)}\bigg\{-\frac{1}{4\epsilon^3} + 10\zeta_3 + 15\zeta_4\epsilon + 185\zeta_5\epsilon^2 + \Big(-204\zeta_3^2 + \frac{875\zeta_6}{2}\Big)\epsilon^3$$

$$+ \Big(-612\zeta_3\zeta_4 + \frac{13157\zeta_7}{4}\Big)\epsilon^4 + \Big(-6830\zeta_3\zeta_5 + \frac{67767\zeta_8}{8} + 243\zeta_{2,6}\Big)\epsilon^5$$

$$+ \Big(\frac{8644\zeta_3^3}{3} - 11703\zeta_4\zeta_5 - 17270\zeta_3\zeta_6 + \frac{735227\zeta_9}{12}\Big)\epsilon^6 - \Big(-12966\zeta_3^2\zeta_4$$

$$+\frac{3770679\zeta_5^2}{56} + \frac{7170073\zeta_3\zeta_7}{56} - \frac{20316339\zeta_{10}}{112} + \frac{40851\zeta_{3,7}}{56}\Big)\epsilon^7$$

$$+ \Big(133222\zeta_3^2\zeta_5 - \frac{659003\zeta_5\zeta_6}{2} - \frac{1644819\zeta_4\zeta_7}{8} - \frac{1341627\zeta_3\zeta_8}{4} + 16317\zeta_2\zeta_9$$

$$+\frac{75373337\zeta_{11}}{64} - 9144\zeta_3\zeta_{2,6} + 882\zeta_{2,1,8}\Big)\epsilon^8 - \Big(\frac{94907\zeta_3^4}{3} - 459003\zeta_3\zeta_4\zeta_5$$

$$-\frac{143613}{7}\zeta_2\zeta_5^2 - \frac{707359}{2}\zeta_3^2\zeta_6 - \frac{291954}{7}\zeta_2\zeta_3\zeta_7 + \frac{4983229\zeta_5\zeta_7}{2} + \frac{9870127\zeta_3\zeta_9}{4}$$

$$-\frac{976579279975\zeta_{12}}{265344} + 13266\zeta_4\zeta_{2,6} + \frac{10638}{7}\zeta_2\zeta_{3,7} + \frac{19445\zeta_{3,9}}{12}$$

$$- 2364\zeta_{2,1,1,8}\Big)\epsilon^9 + O\left(\epsilon^{10}\right)\bigg\}, \tag{4}$$

$$M_{2,6} + \frac{4(3-5\epsilon)(4-5\epsilon)(3-4\epsilon)}{(1-3\epsilon)^2(1-2\epsilon)}M_{0,1} = \frac{2(1-2\epsilon)^2}{(1-5\epsilon)(2-5\epsilon)(1-3\epsilon)}\bigg\{-\frac{1}{8\epsilon^3}$$

$$+ 2\zeta_3 + 3\zeta_4\epsilon + 24\zeta_5\epsilon^2 + \Big(\frac{49\zeta_3^2}{2} + 55\zeta_6\Big)\epsilon^3 + \Big(\frac{147\zeta_3\zeta_4}{2} + \frac{2475\zeta_7}{4}\Big)\epsilon^4$$

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$$+ \left(507\zeta_3\zeta_5 + \frac{48531\zeta_8}{16} - 243\zeta_{2,6}\right)\epsilon^5 + \left(-\frac{3875\zeta_3^3}{3} + \frac{4437\zeta_4\zeta_5}{2} + 2360\zeta_3\zeta_6\right.$$

$$\left. + \frac{49652\zeta_9}{3}\right)\epsilon^6 + \left(-\frac{11625}{2}\zeta_3^2\zeta_4 + \frac{327123\zeta_5^2}{28} + \frac{239661\zeta_3\zeta_7}{28} + \frac{4104945\zeta_{10}}{56}\right.$$

$$\left. + \frac{17253\zeta_{3,7}}{28}\right)\epsilon^7 + \left(-45204\zeta_3^2\zeta_5 + \frac{269691\zeta_5\zeta_6}{4} + \frac{78381\zeta_4\zeta_7}{4} - \frac{142251\zeta_3\zeta_8}{8}\right.$$

$$\left. - \frac{80919\zeta_2\zeta_9}{2} + \frac{7592883\zeta_{11}}{16} + 10449\zeta_3\zeta_{2,6} - 2187\zeta_{2,1,8}\right)\epsilon^8 + \left(\frac{143995\zeta_3^4}{6}\right.$$

$$\left. - 188829\zeta_3\zeta_4\zeta_5 - \frac{118098}{7}\zeta_2\zeta_5^2 - 149047\zeta_3^2\zeta_6 - \frac{240084}{7}\zeta_2\zeta_3\zeta_7 + \frac{727515\zeta_5\zeta_7}{2}\right.$$

$$\left. + \frac{161044\zeta_3\zeta_9}{3} + \frac{79499888185\zeta_{12}}{44224} + 17496\zeta_4\zeta_{2,6} + \frac{8748}{7}\zeta_2\zeta_{3,7} + 675\zeta_{3,9}\right.$$

$$\left. - 1944\zeta_{2,1,1,8}\right)\epsilon^9 + O\left(\epsilon^{10}\right)\Bigg\}, \tag{5}$$

$$M_{2,7} - \frac{(3-5\epsilon)(4-5\epsilon)(3-4\epsilon)\left(7-41\epsilon+58\epsilon^2\right)}{(1-4\epsilon)(1-3\epsilon)(2-3\epsilon)(1-2\epsilon)^2}M_{0,1} = \frac{2(1-2\epsilon)}{(1-4\epsilon)(2-3\epsilon)}$$

$$\times \left\{\frac{1}{48\epsilon^3} + \frac{7\zeta_3}{6} + \frac{7\zeta_4\epsilon}{4} + \frac{221\zeta_5\epsilon^2}{4} + \left(-\frac{145\zeta_3^2}{3} + \frac{3245\zeta_6}{24}\right)\epsilon^3 + \left(\frac{11289\zeta_7}{8}\right.\right.$$

$$\left. - 145\zeta_3\zeta_4\right)\epsilon^4 + \left(-875\zeta_3\zeta_5 + \frac{7441\zeta_8}{4} + \frac{1215\zeta_{2,6}}{2}\right)\epsilon^5 + \left(\frac{8404\zeta_3^3}{9} - \frac{9915\zeta_4\zeta_5}{2}\right.$$

$$\left. - \frac{14950\zeta_3\zeta_6}{3} + \frac{286295\zeta_9}{9}\right)\epsilon^6 + \left(4202\zeta_3^2\zeta_4 - \frac{600375\zeta_5^2}{28} - \frac{1124475\zeta_3\zeta_7}{28}\right.$$

$$\left. + \frac{4628241\zeta_{10}}{56} - \frac{71685\zeta_{3,7}}{28}\right)\epsilon^7 + \left(17701\zeta_3^2\zeta_5 - \frac{618655\zeta_5\zeta_6}{4} - 82305\zeta_4\zeta_7\right.$$

$$\left. - 96139\zeta_3\zeta_8 + \frac{404595\zeta_2\zeta_9}{2} + \frac{3052633\zeta_{11}}{8} - 20655\zeta_3\zeta_{2,6} + 10935\zeta_{2,1,8}\right)\epsilon^8$$

$$+ \left(-\frac{105418\zeta_3^4}{9} + 129648\zeta_3\zeta_4\zeta_5 + \frac{590490}{7}\zeta_2\zeta_5^2 + \frac{323765}{3}\zeta_3^2\zeta_6\right.$$

$$\left. + \frac{1200420}{7}\zeta_2\zeta_3\zeta_7 - 1237050\zeta_5\zeta_7 - \frac{10667105\zeta_3\zeta_9}{9} + \frac{36344694233\zeta_{12}}{16584}\right.$$

$$\left. - 40095\zeta_4\zeta_{2,6} - \frac{43740}{7}\zeta_2\zeta_{3,7} - 11475\zeta_{3,9} + 9720\zeta_{2,1,1,8}\right)\epsilon^9 + O\left(\epsilon^{10}\right)\Bigg\}, \tag{6}$$

$$M_{3,2} + \frac{4(3-4\epsilon)(2-3\epsilon)}{(1-3\epsilon)(1-2\epsilon)}M_{1,1} = \frac{1-2\epsilon}{1-3\epsilon}\left\{ \frac{1}{3\epsilon^4} + \frac{14\zeta_3}{3\epsilon} + 7\zeta_4 + 126\zeta_5\epsilon \right.$$

$$+ \left(-\frac{226\zeta_3^2}{3} + \frac{910\zeta_6}{3}\right)\epsilon^2 + \left(-226\zeta_3\zeta_4 + 1960\zeta_7\right)\epsilon^3 + \epsilon^4\left(-612\zeta_3\zeta_5\right.$$

$$\left. + \frac{11851\zeta_8}{4} + 648\zeta_{2,6}\right) + \left(\frac{5260\zeta_3^3}{9} - 4806\zeta_4\zeta_5 - \frac{13180\zeta_3\zeta_6}{3} + \frac{247094\zeta_9}{9}\right)\epsilon^5$$

$$+ \left(2630\zeta_3^2\zeta_4 - 6912\zeta_5^2 - 11926\zeta_3\zeta_7 + \frac{843327\zeta_{10}}{20} + 5751\zeta_{2,8}\right)\epsilon^6 + \left(8244\zeta_3^2\zeta_5\right.$$

$$- 93294\zeta_5\zeta_6 - 50694\zeta_4\zeta_7 - \frac{115061\zeta_3\zeta_8}{2} + 107892\zeta_2\zeta_9 + \frac{417459\zeta_{11}}{2}$$

$$\left. - 8424\zeta_3\zeta_{2,6} + 5832\zeta_{2,1,8}\right)\epsilon^7 + \left(-\frac{25330\zeta_3^4}{9} + 50004\zeta_3\zeta_4\zeta_5 + 58320\zeta_2\zeta_5^2\right.$$

$$+ \frac{122876}{3}\zeta_3^2\zeta_6 + 111456\zeta_2\zeta_3\zeta_7 - 524340\zeta_5\zeta_7 - \frac{4054892\zeta_3\zeta_9}{9} - 17496\zeta_4\zeta_{2,6}$$

$$\left.\left. + \frac{30212886835\zeta_{12}}{33168} + 11664\zeta_2\zeta_{2,8} + 8100\zeta_{2,10} + 5184\zeta_{2,1,1,8}\right)\epsilon^8 + O\left(\epsilon^9\right)\right\}, \tag{7}$$

$$M_{3,3} + \frac{4(2-5\epsilon)(3-5\epsilon)}{(1-4\epsilon)(1-3\epsilon)}M_{1,4} = \frac{1-2\epsilon}{1-4\epsilon}\left\{ \frac{1}{6\epsilon^4} + \frac{31\zeta_3}{3\epsilon} + \frac{31\zeta_4}{2} + 449\zeta_5\epsilon \right.$$

$$+ \left(\frac{3290\zeta_6}{3} - \frac{983\zeta_3^2}{3}\right)\epsilon^2 + \left(11338\zeta_7 - 983\zeta_3\zeta_4\right)\epsilon^3 + \left(4860\zeta_{2,6} - 3914\zeta_3\zeta_5\right.$$

$$\left. + \frac{121261\zeta_8}{8}\right)\epsilon^4 + \left(\frac{47918\zeta_3^3}{9} - 35031\zeta_4\zeta_5 - \frac{97340\zeta_3\zeta_6}{3} + \frac{2293555\zeta_9}{9}\right)\epsilon^5$$

$$+ \left(23959\zeta_3^2\zeta_4 - \frac{1069773\zeta_5^2}{7} - \frac{1753546\zeta_3\zeta_7}{7} + \frac{4720473\zeta_{10}}{7} - \frac{143370\zeta_{3,7}}{7}\right)\epsilon^6$$

$$+ \left(93802\zeta_3^2\zeta_5 - 1151470\zeta_5\zeta_6 - 552282\zeta_4\zeta_7 - \frac{2732573\zeta_3\zeta_8}{4} + 1618380\zeta_2\zeta_9\right.$$

$$\left. + 3055286\zeta_{11} - 136080\zeta_3\zeta_{2,6} + 87480\zeta_{2,1,8}\right)\epsilon^7 + \left(718806\zeta_3\zeta_4\zeta_5 - \frac{499487\zeta_3^4}{9}\right.$$

$$+ \frac{4723920}{7}\zeta_2\zeta_5^2 + \frac{1842460}{3}\zeta_3^2\zeta_6 + \frac{9603360}{7}\zeta_2\zeta_3\zeta_7 - 9290256\zeta_5\zeta_7$$

$$- \frac{71396590\zeta_3\zeta_9}{9} + \frac{1180748875591\zeta_{12}}{66336} - 277020\zeta_4\zeta_{2,6} - \frac{349920}{7}\zeta_2\zeta_{3,7}$$

$$- 91800\zeta_{3,9} + 77760\zeta_{2,1,1,8}\Big)\epsilon^8 + O\left(\epsilon^9\right)\bigg\}, \tag{8}$$

$$M_{3,4} + \frac{2(2-5\epsilon)(3-5\epsilon)(1-2\epsilon)}{(1-4\epsilon)(1-3\epsilon)^2}M_{1,4} = \frac{(1-2\epsilon)^2}{(1-4\epsilon)(1-3\epsilon)}\bigg\{\frac{1}{12\epsilon^4} + \frac{25\zeta_3}{6\epsilon}$$

$$+ \frac{25\zeta_4}{4} + \frac{465\zeta_5\epsilon}{2} + \left(-\frac{1247\zeta_3^2}{6} + \frac{3425\zeta_6}{6}\right)\epsilon^2 + \left(-\frac{1247}{2}\zeta_3\zeta_4 + \frac{12503\zeta_7}{2}\right)\epsilon^3$$

$$+ \left(1944\zeta_{2,6} - 5895\zeta_3\zeta_5 + \frac{182497\zeta_8}{16}\right)\epsilon^4 + \left(\frac{37081\zeta_3^3}{9} - \frac{41013\zeta_4\zeta_5}{2}\right.$$

$$\left.- \frac{70255\zeta_3\zeta_6}{3} + \frac{2619709\zeta_9}{18}\right)\epsilon^5 + \left(\frac{37081}{2}\zeta_3^2\zeta_4 - \frac{1520763\zeta_5^2}{14} - \frac{1391503\zeta_3\zeta_7}{7}\right.$$

$$\left.+ \frac{11712423\zeta_{10}}{28} - \frac{55656\zeta_{3,7}}{7}\right)\epsilon^6 + \left(144561\zeta_3^2\zeta_5 - 680047\zeta_5\zeta_6 - \frac{741363\zeta_4\zeta_7}{2}\right.$$

$$\left.- \frac{4252007\zeta_3\zeta_8}{8} + 605616\zeta_2\zeta_9 + \frac{4622343\zeta_{11}}{2} - 59952\zeta_3\zeta_{2,6} + 32736\zeta_{2,1,8}\right)\epsilon^7$$

$$+ \left(-\frac{870911\zeta_3^4}{18} + 783699\zeta_3\zeta_4\zeta_5 + \frac{3297024}{7}\zeta_2\zeta_5^2 + \frac{1557073}{3}\zeta_3^2\zeta_6\right.$$

$$+ \frac{6702592}{7}\zeta_2\zeta_3\zeta_7 - \frac{19417073\zeta_5\zeta_7}{3} - \frac{55088251\zeta_3\zeta_9}{9} + \frac{4433462543495\zeta_{12}}{398016}$$

$$\left.- 98328\zeta_4\zeta_{2,6} - \frac{244224}{7}\zeta_2\zeta_{3,7} + \frac{94000\zeta_{3,9}}{9} + 54272\zeta_{2,1,1,8}\right)\epsilon^8 + O\left(\epsilon^9\right)\bigg\}, \tag{9}$$

$$M_{3,6} = \frac{(1-2\epsilon)^3}{1-5\epsilon}\bigg\{\frac{5\zeta_5}{\epsilon} + \left(\frac{25\zeta_6}{2} - 7\zeta_3^2\right) + \left(\frac{127\zeta_7 - 21\zeta_3\zeta_4}{2}\right)\epsilon - \left(-994\zeta_3\zeta_5\right.$$

$$\left.+ \frac{14595\zeta_8}{8} - 486\zeta_{2,6}\right)\epsilon^2 - \left(-\frac{1742\zeta_3^3}{3} + 1425\zeta_4\zeta_5 - 90\zeta_3\zeta_6 - \frac{346\zeta_9}{3}\right)\epsilon^3$$

$$- \left(-2613\zeta_3^2\zeta_4 + \frac{32958\zeta_5^2}{7} - \frac{73231\zeta_3\zeta_7}{7} + \frac{124308\zeta_{10}}{7} + \frac{11799\zeta_{3,7}}{7}\right)\epsilon^4$$

$$- \left(37843\zeta_5\zeta_6 - 334\zeta_3^2\zeta_5 + 39900\zeta_4\zeta_7 - \frac{542969\zeta_{11}}{4} + 100566\zeta_2\zeta_9 + 5436\zeta_{2,1,8}\right.$$

$$\left.- \frac{383805\zeta_3\zeta_8}{4} + 8172\zeta_3\zeta_{2,6}\right)\epsilon^5 - \left(13891\zeta_3^4 - 135750\zeta_3\zeta_4\zeta_5 - \frac{425736}{7}\zeta_2\zeta_5^2\right.$$

$$-47134\zeta_3^2\zeta_6 - \frac{865488}{7}\zeta_2\zeta_3\zeta_7 + 410096\zeta_5\zeta_7 - \frac{271438\zeta_3\zeta_9}{3} + \frac{23122877963\zeta_{12}}{66336}$$

$$-1152\zeta_4\zeta_{2,6} + \frac{31536}{7}\zeta_2\zeta_{3,7} + \frac{14840\zeta_{3,9}}{3} - 7008\zeta_{2,1,1,8}\bigg)\epsilon^6 + O\left(\epsilon^7\right)\bigg\}, \qquad (10)$$

Integrals with c.l.=2

$$M_{3,5} = \frac{(1-2\epsilon)^3}{(1-5\epsilon)(1-4\epsilon)}\bigg\{\frac{\zeta_3}{2\epsilon^2} + \frac{3\zeta_4}{4\epsilon} - \frac{23\zeta_5}{2} - \left(-\frac{29\zeta_3^2}{2} + 30\zeta_6\right)\epsilon$$

$$- \left(-\frac{87}{2}\zeta_3\zeta_4 + \frac{1105\zeta_7}{4}\right)\epsilon^2 + \left(1639\zeta_3\zeta_5 - \frac{29043\zeta_8}{16} + 243\zeta_{2,6}\right)\epsilon^3 - \bigg(967\zeta_3^3$$

$$- \frac{2001\zeta_4\zeta_5}{2} - 2810\zeta_3\zeta_6 + 5144\zeta_9\bigg)\epsilon^4 + \bigg(-\frac{8703}{2}\zeta_3^2\zeta_4 + \frac{35235\zeta_5^2}{2} + \frac{43361\zeta_3\zeta_7}{2}$$

$$- \frac{105813\zeta_{10}}{4} - 1296\zeta_{3,7}\bigg)\epsilon^5 - \bigg(86492\zeta_3^2\zeta_5 - \frac{288269\zeta_5\zeta_6}{4} + \frac{14163\zeta_4\zeta_7}{2}$$

$$- \frac{917679\zeta_3\zeta_8}{8} + \frac{118881\zeta_2\zeta_9}{2} - \frac{12109\zeta_{11}}{32} + 10881\zeta_3\zeta_{2,6} + 3213\zeta_{2,1,8}\bigg)\epsilon^6$$

$$+ \bigg(\frac{120277\zeta_3^4}{6} - 256497\zeta_3\zeta_4\zeta_5 - \frac{1055592}{7}\zeta_2\zeta_5^2 - 137128\zeta_3^2\zeta_6 - \frac{2145936}{7}\zeta_2\zeta_3\zeta_7$$

$$+ \frac{2500975\zeta_5\zeta_7}{2} + \frac{1911321\zeta_3\zeta_9}{2} - \frac{55562910103\zeta_{12}}{66336} - 24534\zeta_4\zeta_{2,6}$$

$$+ \frac{78192}{7}\zeta_2\zeta_{3,7} - \frac{291485\zeta_{3,9}}{6} - 17376\zeta_{2,1,1,8}\bigg)\epsilon^7 + O\left(\epsilon^8\right)\bigg\}, \qquad (11)$$

$$M_{4,1} = (1-2\epsilon)^2\bigg\{\frac{20\zeta_5}{\epsilon} + \left(-22\zeta_3^2 + 50\zeta_6\right) + \left(-66\zeta_3\zeta_4 + \frac{4685\zeta_7}{8}\right)\epsilon$$

$$- \left(-4835\zeta_3\zeta_5 + \frac{119987\zeta_8}{16} - \frac{4563\zeta_{2,6}}{2}\right)\epsilon^2 + \bigg(860\zeta_3^3 - \frac{12873\zeta_4\zeta_5}{2} + 790\zeta_3\zeta_6$$

$$+ \frac{102457\zeta_9}{8}\bigg)\epsilon^3 - \bigg(-3870\zeta_3^2\zeta_4 - \frac{622205\zeta_5^2}{112} - \frac{3987915\zeta_3\zeta_7}{112} + \frac{13703985\zeta_{10}}{224}$$

$$+ \frac{1210959\zeta_{3,7}}{112}\bigg)\epsilon^4 + \bigg(12996\zeta_{2,1,8} - 122760\zeta_3^2\zeta_5 - \frac{288509\zeta_5\zeta_6}{2} - \frac{2362023\zeta_4\zeta_7}{16}$$

$$+ \frac{2536537\zeta_3\zeta_8}{8} + 240426\zeta_2\zeta_9 - \frac{14723695\zeta_{11}}{128} - 71613\zeta_3\zeta_{2,6}\bigg)\epsilon^5 - \bigg(\frac{121663\zeta_3^4}{6}$$

$$+ \frac{415317}{2}\zeta_3\zeta_4\zeta_5 + \frac{3519855}{14}\zeta_2\zeta_5^2 - \frac{335005}{12}\zeta_3^2\zeta_6 + \frac{3577795}{7}\zeta_2\zeta_3\zeta_7$$

$$- \frac{4191530\zeta_5\zeta_7}{3} - \frac{140309257\zeta_3\zeta_9}{72} + \frac{3406738440035\zeta_{12}}{1592064} + 148641\zeta_4\zeta_{2,6}$$

$$- \frac{130365}{7}\zeta_2\zeta_{3,7} + \frac{14194343\zeta_{3,9}}{72} + 28970\zeta_{2,1,1,8}\bigg)\epsilon^6 + O\left(\epsilon^7\right)\bigg\}, \qquad (12)$$

$$M_{4,2} = (1-2\epsilon)^2\bigg\{\frac{20\zeta_5}{\epsilon} + \left(8\zeta_3^2 + 50\zeta_6\right) + \left(24\zeta_3\zeta_4 + 520\zeta_7\right)\epsilon + \bigg(-4120\zeta_3\zeta_5$$

$$+ 6958\zeta_8 - 1296\zeta_{2,6}\bigg)\epsilon^2 + \left(-320\zeta_3^3 + 1596\zeta_4\zeta_5 - 3860\zeta_3\zeta_6 + \frac{23753\zeta_9}{2}\right)\epsilon^3$$

$$+ \left(-1440\zeta_3^2\zeta_4 - \frac{691465\zeta_5^2}{28} - \frac{1315395\zeta_3\zeta_7}{28} + \frac{5181705\zeta_{10}}{56} + \frac{151659\zeta_{3,7}}{28}\right)\epsilon^4$$

$$+ \bigg(85950\zeta_3^2\zeta_5 - \frac{63389\zeta_5\zeta_6}{2} + \frac{97263\zeta_4\zeta_7}{4} - 234601\zeta_3\zeta_8 - 154179\zeta_2\zeta_9$$

$$+ \frac{16056145\zeta_{11}}{32} + 34902\zeta_3\zeta_{2,6} - 8334\zeta_{2,1,8}\bigg)\epsilon^5 + \bigg(3682\zeta_3^4 - 35946\zeta_3\zeta_4\zeta_5$$

$$- \frac{1844370}{7}\zeta_2\zeta_5^2 + 80045\zeta_3^2\zeta_6 - \frac{3749460}{7}\zeta_2\zeta_3\zeta_7 + \frac{387745\zeta_5\zeta_7}{2} + \frac{399371\zeta_3\zeta_9}{6}$$

$$+ \frac{153248180675\zeta_{12}}{132672} + 42084\zeta_4\zeta_{2,6} + \frac{136620}{7}\zeta_2\zeta_{3,7} + \frac{13691\zeta_{3,9}}{6}$$

$$- 30360\zeta_{2,1,1,8}\bigg)\epsilon^6 + O\left(\epsilon^7\right)\bigg\}, \qquad (13)$$

$$M_{4,3} = \frac{(1-2\epsilon)^3}{1+3\epsilon}\bigg\{-\frac{5\zeta_5}{\epsilon} - \left(17\zeta_3^2 + \frac{25\zeta_6}{2}\right) - \left(51\zeta_3\zeta_4 + \frac{225\zeta_7}{2}\right)\epsilon$$

$$- \left(\frac{40201\zeta_8}{8} - 3030\zeta_3\zeta_5 - 1134\zeta_{2,6}\right)\epsilon^2 - \bigg(-\frac{2690\zeta_3^3}{3} + 2259\zeta_4\zeta_5 - 1990\zeta_3\zeta_6$$

$$+ \frac{22009\zeta_9}{9}\bigg)\epsilon^3 + \left(4035\zeta_3^2\zeta_4 + \frac{125425\zeta_5^2}{14} + \frac{289795\zeta_3\zeta_7}{14} - \frac{1059305\zeta_{10}}{28}\right.$$

$$- \frac{42513\zeta_{3,7}}{14}\bigg)\epsilon^4 + \bigg(-99740\zeta_3^2\zeta_5 - \frac{30257\zeta_5\zeta_6}{6} - \frac{41491\zeta_4\zeta_7}{2} + \frac{930921\zeta_3\zeta_8}{4}$$

$$+ 94091\zeta_2\zeta_9 - \frac{1536475\zeta_{11}}{8} - 48178\zeta_3\zeta_{2,6} + 5086\zeta_{2,1,8}\bigg)\epsilon^5 + \bigg(-\frac{204427\zeta_3^4}{9}$$

$$- 35626\zeta_3\zeta_4\zeta_5 + \frac{234900}{7}\zeta_2\zeta_5^2 - \frac{269950}{9}\zeta_3^2\zeta_6 + \frac{1432600}{21}\zeta_2\zeta_3\zeta_7 + \frac{1200535\zeta_5\zeta_7}{9}$$

$$+ \frac{5297692\zeta_3\zeta_9}{27} - \frac{94566489365\zeta_{12}}{597024} - 76996\zeta_4\zeta_{2,6} - \frac{17400}{7}\zeta_2\zeta_{3,7} - \frac{404498\zeta_{3,9}}{27}$$

$$+ \frac{11600}{3}\zeta_{2,1,1,8}\Bigg)\epsilon^6 + O\left(\epsilon^7\right)\Bigg\}, \tag{14}$$

$$M_{4,4} = (1 - 2\epsilon)^3\Bigg\{ \frac{441\zeta_7}{8} + \epsilon\left(-216\zeta_3\zeta_5 + \frac{5733\zeta_8}{16} - \frac{81\zeta_{2,6}}{2}\right) + \Bigg(-267\zeta_3^3$$

$$- 81\zeta_4\zeta_5 - \frac{675\zeta_3\zeta_6}{2} + \frac{4583\zeta_9}{2}\Bigg)\epsilon^2 + \Bigg(-\frac{2403}{2}\zeta_3^2\zeta_4 - \frac{502287\zeta_5^2}{56} - \frac{7731\zeta_3\zeta_7}{56}$$

$$+ \frac{1324935\zeta_{10}}{112} + \frac{18441\zeta_{3,7}}{56}\Bigg)\epsilon^3 + \Bigg(-\frac{24315}{2}\zeta_3^2\zeta_5 - \frac{358023\zeta_5\zeta_6}{8} + \frac{139401\zeta_4\zeta_7}{8}$$

$$- \frac{59895\zeta_3\zeta_8}{4} + \frac{232767\zeta_2\zeta_9}{4} - \frac{402081\zeta_{11}}{32} - \frac{621}{2}\zeta_3\zeta_{2,6} + \frac{6291}{2}\zeta_{2,1,8}\Bigg)\epsilon^4$$

$$- \Bigg(-6023\zeta_3^4 + 6660\zeta_3\zeta_4\zeta_5 - \frac{650997}{7}\zeta_2\zeta_5^2 + 40507\zeta_3^2\zeta_6 - \frac{1323426}{7}\zeta_2\zeta_3\zeta_7$$

$$+ \frac{1750957\zeta_5\zeta_7}{2} + \frac{964778\zeta_3\zeta_9}{3} - \frac{104287641323\zeta_{12}}{132672} - 2853\zeta_4\zeta_{2,6} + \frac{48222}{7}\zeta_2\zeta_{3,7}$$

$$- \frac{190175\zeta_{3,9}}{6} - 10716\zeta_{2,1,1,8}\Bigg)\epsilon^5 + O\left(\epsilon^6\right)\Bigg\}, \tag{15}$$

$$M_{5,2}(4 - 2\epsilon) = (1 - 2\epsilon)^2\Bigg\{ \frac{20\zeta_5}{\epsilon} + \left(68\zeta_3^2 + 50\zeta_6\right) + \left(204\zeta_3\zeta_4 + 450\zeta_7\right)\epsilon$$

$$+ \left(-11520\zeta_3\zeta_5 + \frac{40201\zeta_8}{2} - 4536\zeta_{2,6}\right)\epsilon^2 + \Bigg(9936\zeta_4\zeta_5 - \frac{4640\zeta_3^3}{3} - 6460\zeta_3\zeta_6$$

$$+ \frac{88036\zeta_9}{9}\Bigg)\epsilon^3 + \Bigg(-6960\zeta_3^2\zeta_4 - 73022\zeta_5^2 - 142178\zeta_3\zeta_7 + \frac{6483199\zeta_{10}}{20}$$

$$- 42513\zeta_{2,8}\Bigg)\epsilon^4 + \Bigg(83120\zeta_3^2\zeta_5 + \frac{227014\zeta_5\zeta_6}{3} + 103232\zeta_4\zeta_7 - 317196\zeta_3\zeta_8$$

$$- 376364\zeta_2\zeta_9 + \frac{1536475\zeta_{11}}{2} + 56632\zeta_3\zeta_{2,6} - 20344\zeta_{2,1,8}\Bigg)\epsilon^5 + \Bigg(\frac{124708\zeta_3^4}{9}$$

$$+ 11464\zeta_3\zeta_4\zeta_5 - 174000\zeta_2\zeta_5^2 - \frac{40700}{9}\zeta_3^2\zeta_6 - \frac{997600}{3}\zeta_2\zeta_3\zeta_7 - \frac{2338736\zeta_5\zeta_7}{3}$$

$$- \frac{26211464\zeta_3\zeta_9}{27} + \frac{9482746239\zeta_{12}}{2764} + 103864\zeta_4\zeta_{2,6} - 34800\zeta_2\zeta_{2,8} - \frac{808996\zeta_{2,10}}{3}$$

$$- \frac{46400}{3}\zeta_{2,1,1,8}\bigg)\epsilon^6 + O\left(\epsilon^7\right)\bigg\}, \tag{16}$$

Integrals with c.l.=3

$$M_{4,5}(4-2\epsilon) = \frac{(1-2\epsilon)^3}{1-6\epsilon}\bigg\{ 36\zeta_3^2 - \bigg(-108\zeta_3\zeta_4 + 378\zeta_7\bigg)\epsilon + \bigg(5868\zeta_3\zeta_5$$

$$- \frac{14805\zeta_8}{2} + 1512\zeta_{2,6}\bigg)\epsilon^2 - \bigg(732\zeta_3^3 + 270\zeta_4\zeta_5 - 6930\zeta_3\zeta_6 + \frac{42458\zeta_9}{3}\bigg)\epsilon^3$$

$$+ \bigg(-3294\zeta_3^2\zeta_4 + \frac{202569\zeta_5^2}{7} + \frac{697887\zeta_3\zeta_7}{7} - \frac{895521\zeta_{10}}{7} - \frac{58563\zeta_{3,7}}{7}\bigg)\epsilon^4$$

$$- \bigg(223152\zeta_3^2\zeta_5 - 105454\zeta_5\zeta_6 + 214815\zeta_4\zeta_7 - \frac{1404105\zeta_3\zeta_8}{2} + 1002552\zeta_2\zeta_9$$

$$- \frac{4736453\zeta_{11}}{4} + 33504\zeta_3\zeta_{2,6} + 54192\zeta_{2,1,8}\bigg)\epsilon^5 + \bigg(-\frac{22816\zeta_3^4}{3} - 611040\zeta_3\zeta_4\zeta_5$$

$$- 1041984\zeta_2\zeta_5^2 - \frac{553066}{3}\zeta_3^2\zeta_6 - 2118272\zeta_2\zeta_3\zeta_7 + \frac{17993629\zeta_5\zeta_7}{3}$$

$$+ \frac{60130438\zeta_3\zeta_9}{9} - \frac{659340762619\zeta_{12}}{99504} - 59016\zeta_4\zeta_{2,6} + 77184\zeta_2\zeta_{3,7}$$

$$- \frac{3234680\zeta_{3,9}}{9} - 120064\zeta_{2,1,1,8}\bigg)\epsilon^6 + O\left(\epsilon^7\right)\bigg\}, \tag{17}$$

$$M_{5,1} = \frac{(1-2\epsilon)^3}{1+3\epsilon}\bigg\{ -\frac{5\zeta_5}{\epsilon} - \bigg(17\zeta_3^2 + \frac{25\zeta_6}{2}\bigg) - \bigg(51\zeta_3\zeta_4 + \frac{85\zeta_7}{2}\bigg)\epsilon$$

$$+ \bigg(\frac{138159\zeta_8}{8} - 11650\zeta_3\zeta_5 - 4266\zeta_{2,6}\bigg)\epsilon^2 + \bigg(8121\zeta_4\zeta_5 - \frac{4310\zeta_3^3}{3} - 7710\zeta_3\zeta_6$$

$$+ \frac{16232\zeta_9}{3}\bigg)\epsilon^3 + \bigg(-6465\zeta_3^2\zeta_4 - \frac{863970\zeta_5^2}{7} - \frac{572995\zeta_3\zeta_7}{7} + \frac{1796310\zeta_{10}}{7}$$

$$+ \frac{171531\zeta_{3,7}}{7}\bigg)\epsilon^4 + \bigg(456170\zeta_3^2\zeta_5 - 312737\zeta_5\zeta_6 + 530472\zeta_4\zeta_7 - \frac{4475049\zeta_3\zeta_8}{4}$$

$$
\begin{aligned}
&+ 580086\zeta_2\zeta_9 - \frac{2373525\zeta_{11}}{4} + 189612\zeta_3\zeta_{2,6} + 31356\zeta_{2,1,8}\bigg)\epsilon^5 + \bigg(\frac{196901\zeta_3^4}{3} \\
&+ 1041474\zeta_3\zeta_4\zeta_5 + \frac{12645720}{7}\zeta_2\zeta_5^2 + \frac{35990}{3}\zeta_3^2\zeta_6 + \frac{25707760}{7}\zeta_2\zeta_3\zeta_7 \\
&- \frac{38973620\zeta_5\zeta_7}{3} - \frac{77902346\zeta_3\zeta_9}{9} + \frac{2383042633405\zeta_{12}}{199008} + 393504\zeta_4\zeta_{2,6} \\
&- \frac{936720}{7}\zeta_2\zeta_{3,7} + \frac{6770464\zeta_{3,9}}{9} + 208160\zeta_{2,1,1,8}\bigg)\epsilon^6 + O\left(\epsilon^7\right)\bigg\},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
M_{6,1} = \frac{(1-2\epsilon)^3}{(1+3\epsilon)^3}\bigg\{ &- \frac{10\zeta_5}{\epsilon} - \left(100\zeta_5 + 10\zeta_3^2 + 25\zeta_6\right) - \bigg(210\zeta_5 + 100\zeta_3^2 \\
&+ 250\zeta_6 + 30\zeta_3\zeta_4 - \frac{19\zeta_7}{2}\bigg)\epsilon + \bigg(-66\zeta_3^2 - 525\zeta_6 - 300\zeta_3\zeta_4 + 1257\zeta_7 \\
&+ 1564\zeta_3\zeta_5 - 567\zeta_8 + 162\zeta_{2,6}\bigg)\epsilon^2 + \bigg(-198\zeta_3\zeta_4 + \frac{11151\zeta_7}{2} + 19344\zeta_3\zeta_5 \\
&+ 1043\zeta_8 + 972\zeta_{2,6} + \frac{3440\zeta_3^3}{3} + 1374\zeta_4\zeta_5 + 3150\zeta_3\zeta_6 + \frac{21637\zeta_9}{3}\bigg)\epsilon^3 \\
&+ \bigg(43092\zeta_3\zeta_5 + 28455\zeta_8 - 1782\zeta_{2,6} + \frac{40792\zeta_3^3}{3} + 23184\zeta_4\zeta_5 + 44000\zeta_3\zeta_6 \\
&+ \frac{339727\zeta_9}{3} + 5160\zeta_3^2\zeta_4 + \frac{287486\zeta_5^2}{7} - \frac{286\zeta_3\zeta_7}{7} + \frac{316935\zeta_{10}}{7} + \frac{6642\zeta_{3,7}}{7}\bigg)\epsilon^4 \\
&+ \bigg(17848\zeta_3^3 + 75330\zeta_4\zeta_5 + 116970\zeta_3\zeta_6 + \frac{1061648\zeta_9}{3} + 61188\zeta_3^2\zeta_4 \\
&+ \frac{807571\zeta_5^2}{2} + \frac{313421\zeta_3\zeta_7}{2} + \frac{2509185\zeta_{10}}{4} + \frac{17127\zeta_{3,7}}{2} - 196432\zeta_3^2\zeta_5 \\
&+ 158884\zeta_5\zeta_6 + 141174\zeta_4\zeta_7 + 133604\zeta_3\zeta_8 + 682428\zeta_2\zeta_9 - \frac{17560877\zeta_{11}}{24} \\
&- 78360\zeta_3\zeta_{2,6} + 36888\zeta_{2,1,8}\bigg)\epsilon^5 + \bigg(80316\zeta_3^2\zeta_4 + \frac{13986207\zeta_5^2}{14} + \frac{4863417\zeta_3\zeta_7}{14} \\
&+ \frac{55755309\zeta_{10}}{28} + \frac{490491\zeta_{3,7}}{14} - 1439456\zeta_3^2\zeta_5 + 1489454\zeta_5\zeta_6 + \frac{3397821\zeta_4\zeta_7}{2} \\
&+ 925922\zeta_3\zeta_8 + 7235868\zeta_2\zeta_9 - \frac{330988949\zeta_{11}}{48} - 583584\zeta_3\zeta_{2,6} + 391128\zeta_{2,1,8} \\
&- \frac{720896\zeta_3^4}{9} - 777104\zeta_3\zeta_4\zeta_5 - \frac{3772008}{7}\zeta_2\zeta_5^2 - \frac{1498208}{9}\zeta_3^2\zeta_6 + \frac{279408}{7}\zeta_2\zeta_{3,7}
\end{aligned}
$$

$$-\frac{23004592}{21}\zeta_2\zeta_3\zeta_7 + \frac{32644190\zeta_5\zeta_7}{9} + \frac{42933380\zeta_3\zeta_9}{27} + \frac{530242449679\zeta_{12}}{298512}$$

$$-219440\zeta_4\zeta_{2,6} - \frac{2444290\zeta_{3,9}}{27} - \frac{186272}{3}\zeta_{2,1,1,8}\bigg)\epsilon^6 + O\left(\epsilon^7\right)\bigg\}, \tag{19}$$

Integrals with c.l.=4

$$M_{6,2} = \frac{(1-2\epsilon)^3}{(1+3\epsilon)^3(1+4\epsilon)}\bigg\{-\frac{10\zeta_5}{\epsilon} - \left(60\zeta_5 + 10\zeta_3^2 + 25\zeta_6 + 70\zeta_7\right) - \bigg(50\zeta_5$$

$$-780\zeta_3^2 + 150\zeta_6 + 30\zeta_3\zeta_4 + \frac{1829\zeta_7}{2} - 2560\zeta_3\zeta_5 + 4655\zeta_8 - 1080\zeta_{2,6}\bigg)\epsilon$$

$$+\bigg(120\zeta_5 + 6046\zeta_3^2 - 125\zeta_6 + 2340\zeta_3\zeta_4 + 7575\zeta_7 + 29776\zeta_3\zeta_5 - 53193\zeta_8$$

$$+12258\zeta_{2,6} + \frac{4528\zeta_3^3}{3} - 2640\zeta_4\zeta_5 + 1000\zeta_3\zeta_6 - \frac{58460\zeta_9}{9}\bigg)\epsilon^2 + \bigg(10704\zeta_3^2$$

$$+300\zeta_6 + 18138\zeta_3\zeta_4 + \frac{156783\zeta_7}{2} + 214920\zeta_3\zeta_5 - 215670\zeta_8 + 59940\zeta_{2,6}$$

$$+\frac{73244\zeta_3^3}{3} - 28884\zeta_4\zeta_5 + 13200\zeta_3\zeta_6 - \frac{1510937\zeta_9}{18} + 6792\zeta_3^2\zeta_4 + \frac{143960\zeta_5^2}{7}$$

$$+\frac{495583\zeta_3\zeta_7}{7} - \frac{1038970\zeta_{10}}{7} - \frac{88260\zeta_{3,7}}{7}\bigg)\epsilon^3 + \bigg(32112\zeta_3\zeta_4 + 146088\zeta_7$$

$$+908472\zeta_3\zeta_5 - 395297\zeta_8 + 167994\zeta_{2,6} + 140784\zeta_3^3 - 37260\zeta_4\zeta_5 + 233700\zeta_3\zeta_6$$

$$+\frac{166790\zeta_9}{3} + 109866\zeta_3^2\zeta_4 + \frac{8380551\zeta_5^2}{28} + \frac{26770505\zeta_3\zeta_7}{28} - \frac{108421795\zeta_{10}}{56}$$

$$-\frac{4631493\zeta_{3,7}}{28} + 390194\zeta_3^2\zeta_5 - \frac{2438995\zeta_5\zeta_6}{6} + 488786\zeta_4\zeta_7 - \frac{1567351\zeta_3\zeta_8}{2}$$

$$+3792685\zeta_2\zeta_9 - \frac{100223975\zeta_{11}}{16} - 11366\zeta_3\zeta_{2,6} + 205010\zeta_{2,1,8}\bigg)\epsilon^4$$

$$+\bigg(1388592\zeta_3\zeta_5 - 324240\zeta_8 + 211248\zeta_{2,6} + \frac{1085348\zeta_3^3}{3} + 354744\zeta_4\zeta_5$$

$$+1400980\zeta_3\zeta_6 + \frac{45002569\zeta_9}{18} + 633528\zeta_3^2\zeta_4 + \frac{15929910\zeta_5^2}{7} + \frac{43860237\zeta_3\zeta_7}{7}$$

$$-\frac{54746115\zeta_{10}}{7} - \frac{6015150\zeta_{3,7}}{7} + 5634364\zeta_3^2\zeta_5 - \frac{16554412\zeta_5\zeta_6}{3}$$

$$+\frac{28467331\zeta_4\zeta_7}{4} - 11209448\zeta_3\zeta_8 + 53532932\zeta_2\zeta_9 - \frac{2821800989\zeta_{11}}{32}$$

$$-143668\zeta_3\zeta_{2,6} + 2893672\zeta_{2,1,8} - \frac{696554\zeta_3^4}{9} - 4241552\zeta_3\zeta_4\zeta_5 - \frac{42064920}{7}\zeta_2\zeta_5^2$$

$$+ \frac{7951810}{9}\zeta_3^2\zeta_6 - \frac{256544080}{21}\zeta_2\zeta_3\zeta_7 + \frac{254930897\zeta_5\zeta_7}{9} + \frac{714442631\zeta_3\zeta_9}{27}$$

$$- \frac{12873185340379\zeta_{12}}{597024} - 843884\zeta_4\zeta_{2,6} + \frac{3115920}{7}\zeta_2\zeta_{3,7} - \frac{43891225\zeta_{3,9}}{27}$$

$$- \frac{2077280}{3}\zeta_{2,1,1,8}\Bigg)\epsilon^5 + O\left(\epsilon^6\right)\Bigg\}, \tag{20}$$

$$M_{6,3} = \frac{(1-2\epsilon)^3}{(1+3\epsilon)(1+4\epsilon)}\Bigg\{ -\frac{5\zeta_5}{\epsilon} - \left(20\zeta_5 + 41\zeta_3^2 + \frac{25\zeta_6}{2} - \frac{161\zeta_7}{2}\right)$$

$$+ \left(-308\zeta_3^2 - 50\zeta_6 - 123\zeta_3\zeta_4 + 514\zeta_7 + 4862\zeta_3\zeta_5 - \frac{24451\zeta_8}{4} + 1566\zeta_{2,6}\right)\epsilon$$

$$+ \left(-924\zeta_3\zeta_4 - 1500\zeta_7 + 68636\zeta_3\zeta_5 - \frac{744639\zeta_8}{8} + 23220\zeta_{2,6} + \frac{1526\zeta_3^3}{3}\right.$$

$$\left. - 2103\zeta_4\zeta_5 + 4325\zeta_3\zeta_6 + \frac{111709\zeta_9}{36}\right)\epsilon^2 + \left(235200\zeta_3\zeta_5 - \frac{710311\zeta_8}{2}\right.$$

$$+ 85536\zeta_{2,6} + \frac{22048\zeta_3^3}{3} - 36366\zeta_4\zeta_5 + 55695\zeta_3\zeta_6 + \frac{237103\zeta_9}{12} + 2289\zeta_3^2\zeta_4$$

$$+ \frac{1341143\zeta_5^2}{56} + \frac{3816969\zeta_3\zeta_7}{56} - \frac{7815019\zeta_{10}}{112} - \frac{500565\zeta_{3,7}}{56}\right)\epsilon^3 + \left(\frac{61040\zeta_3^3}{3}\right.$$

$$- 160416\zeta_4\zeta_5 + 161860\zeta_3\zeta_6 - \frac{460411\zeta_9}{9} + 33072\zeta_3^2\zeta_4 + \frac{60035137\zeta_3\zeta_7}{56}$$

$$+ \frac{12859479\zeta_5^2}{56} - \frac{134815227\zeta_{10}}{112} - \frac{7724781\zeta_{3,7}}{56} - 453668\zeta_3^2\zeta_5 + \frac{280574047\zeta_{11}}{64}$$

$$+ \frac{1346777\zeta_5\zeta_6}{6} - \frac{4654793\zeta_4\zeta_7}{8} + 1309878\zeta_3\zeta_8 - 2749211\zeta_2\zeta_9 - 87752\zeta_3\zeta_{2,6}$$

$$- 148606\zeta_{2,1,8}\right)\epsilon^4 + \left(91560\zeta_3^2\zeta_4 + \frac{5319415\zeta_5^2}{14} + \frac{55472425\zeta_3\zeta_7}{14} - \frac{705626\zeta_3^4}{9}\right.$$

$$- \frac{7239597\zeta_{3,7}}{14} - 5183674\zeta_3^2\zeta_5 + \frac{3338505\zeta_5\zeta_6}{2} - 1597650\zeta_{2,1,8} - \frac{1752859}{18}\zeta_3^2\zeta_6$$

$$+ \frac{62741559\zeta_3\zeta_8}{4} - 29556525\zeta_2\zeta_9 + \frac{2990096591\zeta_{11}}{64} - \frac{29828681054659\zeta_{12}}{2388096}$$

$$- \frac{53503881\zeta_4\zeta_7}{8} - \frac{142150835\zeta_{10}}{28} - 1729457\zeta_3\zeta_4\zeta_5 - \frac{25832277}{7}\zeta_2\zeta_5^2$$

$$-\frac{157544998}{21}\zeta_2\zeta_3\zeta_7 + \frac{311533051\zeta_5\zeta_7}{18} + \frac{1792012205\zeta_3\zeta_9}{108} - 1085340\zeta_3\zeta_{2,6}$$

$$-227636\zeta_4\zeta_{2,6} + \frac{1913502}{7}\zeta_2\zeta_{3,7} - \frac{105698899\zeta_{3,9}}{108} - \frac{1275668}{3}\zeta_{2,1,1,8}\Bigg)\epsilon^5$$

$$+O\left(\epsilon^6\right)\Bigg\}. \qquad (21)$$

One may notice that rational coefficients in front of ζ_{12} involve the very big prime number 691 in the denominator. This is because the same number appears in the numerator of $\zeta_{12} = 691\pi^{12}/638512875$. If we use π^{12}, instead of ζ_{12}, this number will disappear, and the highest prime factor in the coefficients denominators will be only 13.

Our results are in the full agreement with the results of Ref. [1] up to terms considered in that paper. Partly, this is because we used some of the data from Ref. [1] for the determination of the homogeneous parts of the solution of DRR. Nevertheless, our calculation can be considered as a nontrivial check of the results of Ref. [1] because of many constraints on the terms of ϵ-expansion provided by the DRA method and fulfilled by the results of Ref. [1].

We also confirm the numerical results of Ref. [19] where one more term (as compared with Ref. [1]) of the ϵ-expansions of the most complicated thirteen master integrals was obtained using FIESTA [17, 18].

We observe that only MZV are present in our results for the four-loop massless propagator master integrals. Since any other four-loop massless propagator integral, with any integer powers of numerators and propagators, can be represented, due to an IBP reduction, as a linear combination of the master integrals, with coefficients which are rational functions of d, we come to the conclusion that any four-loop massless propagator integral has only MZV in its epsilon expansion up to transcendentality weight twelve. This means that if we want to find something beyond MZV in four-loop massless propagator diagrams we have to go to higher transcendentality weights. This is certainly possible within our approach. In fact, we have chosen weight twelve because it looks to be already a sufficiently big number. We can perform calculation up to higher weights and will do our best on demand, for example, if somebody has reasons to believe that unusual transcendental constants can appear at some specific weight.

This possibility is due to an important feature of the method that we use. Once we have a result in terms of a multiple series which always appears to be

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well convergent, going to higher powers of ϵ is an easy procedure, in contrast to other methods.[1]

Taking our results into account one obtains more motivations to try to prove that there are only MZV in massless propagator diagrams. Another alternative is to continue to look for unusual constants in higher loops. Keeping in mind the dramatic progress of the last years in the field of evaluating Feynman integrals, this also looks to be a possible scenario. All results presented here are available in computer-readable form on

`www-ttp.particle.uni-karlsruhe.de/Progdata/ttp11/ttp11-20/`.

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[1]For example, one of us (V.S.) remembers very well how painful it was to obtain higher powers of ϵ when evaluating master integrals for three-loop form factors by the method of MB representation [25, 26]. In fact, two highest coefficients were presented only in a numerical form at this time. Only one year later, they were evaluated analytically by the present method [8].

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