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HQET heavy–heavy vertex diagram with two velocities

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Abstract. The one-loop HQET heavy–heavy vertex diagram with arbitrary powers of all three denominators and arbitrary residual energies is investigated. Various particular cases in which the result becomes simpler are considered.

1 Introduction

The heavy–heavy quark current in HQET (see, e.g., [1, 2]) transforms a heavy quark with velocity v_1 into a heavy quark with velocity v_2. Loop diagrams with a velocity-changing vertex appear, e.g., when the anomalous dimension of the heavy-heavy current [3] or correlators involving this current (see [1]) are considered. They can be reduced to a set of master integrals. Some of them have the form of the one-loop vertex diagram with various ε-dependent powers of the denominators.

Here we consider this one-loop diagram (Fig. 1) in dimensional regularization in the most general case. It is

$$\begin{aligned}
&I(n_1,n_2,n_3;\vartheta;\omega_1,\omega_2) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D_1^{n_1} D_2^{n_2} D_3^{n_3}}\,,\\
&D_1 = -2(k+p_1)\cdot v_1\,,\quad D_2 = -2(k+p_2)\cdot v_2\,,\\
&D_3 = -k^2\,.
\end{aligned} \tag{1}$$

It depends on the residual energies $\omega_{1,2} = p_{1,2}\cdot v_{1,2}$ and the Minkowski angle $\cosh\vartheta = v_1\cdot v_2$. It is symmetric:

$$I(n_1,n_2,n_3;\vartheta;\omega_1,\omega_2) = I(n_2,n_1,n_3;\vartheta;\omega_2,\omega_1)\,; \tag{2}$$

in the following, we shall not explicitly write down relations obtained by this obvious symmetry. This diagram vanishes at integer $n_3 \le 0$. For integer $n_2 \le 0$, it becomes the self-energy diagram with a numerator; if, in addition, n_1 is integer and $n_1 \le 0$, the diagram vanishes. The diagram has cuts in $\omega_{1,2}$ from 0 to $+\infty$. We shall consider $\omega_{1,2} < 0$, other cases can be treated by analytical continuation.

In physical applications, $n_{1,2} = m_{1,2} + 2l_{1,2}\varepsilon$, $n_3 = m_3 + l_3\varepsilon$, where m_i, l_i are integer, and $l_i \ge 0$. All integrals with given l_i can be reduced, using integration by parts [4], to 3 master integrals

$$\begin{aligned}
&I(1+2l_1\varepsilon, 1+2l_2\varepsilon, 1+l_3\varepsilon;\vartheta;\omega_1,\omega_2)\,,\\
&I(2l_1\varepsilon, 1+2l_2\varepsilon, 1+l_3\varepsilon;\vartheta;\omega_1,\omega_2)\,,\\
&I(1+2l_1\varepsilon, 2l_2\varepsilon, 1+l_3\varepsilon;\vartheta;\omega_1,\omega_2)\,.
\end{aligned} \tag{3}$$

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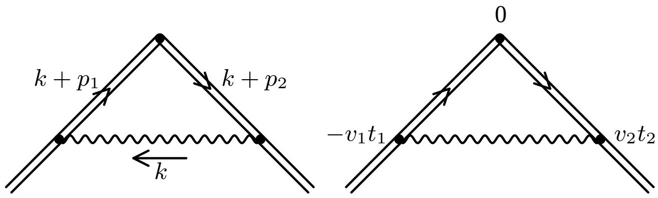

Fig. 1. The one-loop HQET heavy–heavy vertex diagram in momentum and coordinate space

If $l_1 = 0$, the second integral is trivial:

$$\begin{aligned}
&I(0,n_2,n_3;\vartheta;\omega_1,\omega_2) = I(n_2,n_3)(-2\omega_2)^{d-n_2-2n_3}\,,\\
&I(n_1,n_2) = \frac{\Gamma(n_1+2n_2-d)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)}\,,
\end{aligned} \tag{4}$$

see [5] (similarly for the third integral if $l_2 = 0$). Reduction algorithms for all cases has been constructed and implemented in `REDUCE`, and can be downloaded from [6].

Using HQET Feynman parametrization (see, e.g., [2]), we can write the diagram (1) as an integral in 2 parameters $y_{1,2}$ which have dimensionality of energy and vary from 0 to ∞. After the substitution $y_1 = yx$, $y_2 = y(1-x)$, the integral in y can be calculated:

$$\begin{aligned}
&I(n_1,n_2,n_3;\vartheta;\omega_1,\omega_2) = I(n_1+n_2,n_3)\frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)}\\
&\times \int_0^1 A^{-l}(-2\Omega)^{-r} x^{n_1-1}(1-x)^{n_2-1} dx\,,\\
&A = x^2 + (1-x)^2 + 2x(1-x)\cosh\vartheta\\
&\quad = \left[1-(1-e^{\vartheta})x\right]\left[1-(1-e^{-\vartheta})x\right]\,,\\
&\Omega = \omega_1 x + \omega_2(1-x)\,,\\
&l = \frac{d}{2} - n_3\,,\quad r = n_1 + n_2 - 2l\,.
\end{aligned} \tag{5}$$

This integral depends on d and n_3 only via l; therefore, shifting d by ± 2 [7] is equivalent to shifting n_3 by ± 1. It is Lauricella function F_D [8]

$$I(n_1,n_2,n_3;\vartheta;\omega_1,\omega_2) = I(n_1+n_2,n_3)(-2\omega_2)^{-r}$$

$$\times F_D(n_1;l,l,r;n_1+n_2;1-e^{\vartheta},1-e^{-\vartheta},\xi_-)\,, \qquad (6)$$

where

$$\xi_\pm = 1\pm\xi\,,\quad \xi=\frac{\omega_1}{\omega_2}\,. \qquad (7)$$

This result can also be obtained in coordinate space (Fig. 1). Substituting $t_1=tx$, $t_2=t(1-x)$, we can calculate the integral in t and reproduce (5). Another one-dimensional integral representation can be obtained by separating the k space into the 2-dimensional longitudinal and $(d-2)$-dimensional transverse subspaces [9], but this representation seems more complicated.[1]

The case $v_1=v_2$ ($\vartheta=0$) has been considered in [13]:

$$I(n_1,n_2,n_3;0;\omega_1,\omega_2) = I(n_1+n_2,n_3)(-2\omega_2)^{-r} \times {}_2F_1\left(\left.\begin{array}{c}n_1,r\\ n_1+n_2\end{array}\right|\xi_-\right)\,. \qquad (8)$$

If $n_{1,2}$ are integer, it reduces to trivial cases (4).

In the single-scale case $\omega_1=\omega_2$, the integral (5) gives the Appell function F_1 [8]

$$I(n_1,n_2,n_3;\vartheta;\omega,\omega) = I(n_1+n_2,n_3)(-2\omega)^{-r} \times F_1(n_1;l,l;n_1+n_2;1-e^{\vartheta},1-e^{-\vartheta})\,. \qquad (9)$$

Also,

$$I(n_1,n_2,n_3;\vartheta;\omega,0) = I(d-n_2-2n_3,n_2,n_3;\vartheta;\omega,\omega) \qquad (10)$$

(similarly for $\omega_1=0$); so, these cases reduce to $\omega_1=\omega_2$.

2 Exact results

In the Euclidean region $\vartheta=i\vartheta_E$, let's consider

$$I=\frac{I(n_1,n_2,n_3;i\vartheta_E;\omega_1,\omega_2)}{I(n_1+n_2,n_3)(-2\omega_2)^{-r}}\,. \qquad (11)$$

Let's denote

$$A=1-2tz+z^2\,,\quad t=\sin\frac{\vartheta_E}{2}\,,\quad z=2tx\,. \qquad (12)$$

If $t<1/2$, we can expand

$$A^{-l}=\sum_{k=0}^{\infty} z^k C_k^l(t)\,, \qquad (13)$$

where $C_k^l(t)$ are Gegenbauer polynomials; results for $t>1/2$ can be obtained by analytical continuation. Expanding also Ω/ω_2 in x, we obtain from (5)

$$I=\sum_{s=0}^{\infty}\frac{(r)_s}{s!}\xi_-^s J_s\,, \qquad (14)$$

$$J_s=\sum_{k=0}^{\infty}\frac{(n_1)_{k+s}}{(n_1+n_2)_{k+s}}(2t)^k C_k^l(t)\,.$$

[1] In the ordinary relativistic theory, the one-loop vertex diagram in the most general case is given by F_D [10]; the massless one via Appell F_4 functions [11]; and the one with any masses but all powers of the denominators equal to 1 via Appell F_1 [12].

Substituting the explicit form of Gegenbauer polynomials

$$C_k^l(t)=\sum_{m=0}^{[k/2]}\frac{(-1)^m(l)_{k-m}}{m!\,(k-2m)!}(2t)^{k-2m}$$

and interchanging the order of summations, we get

$$J_s=\sum_{m=0}^{\infty}\sum_{k=2m}^{\infty}\frac{(-1)^m(l)_{k-m}(n_1)_{k+s}}{m!\,(k-2m)!\,(n_1+n_2)_{k+s}}(2t)^{2(k-m)}\,.$$

Now we substitute $k=p+m$ and interchange the order of summations again:

$$J_s=\sum_{p=0}^{\infty}(l)_p(2t)^{2p}\sum_{m=0}^{p}\frac{(-1)^m(n_1)_{m+p+s}}{m!\,(p-m)!\,(n_1+n_2)_{m+p+s}}\,.$$

Here the inner sum is a terminating hypergeometric series

$$\frac{(n_1)_{p+s}}{p!\,(n_1+n_2)_{p+s}}\,{}_2F_1\left(\left.\begin{array}{c}-p,p+s+n_1\\ p+s+n_1+n_2\end{array}\right|1\right)\,;$$

expressing it via Γ functions, we have

$$J_s=\sum_{p=0}^{\infty}\frac{(n_1)_{p+s}(n_2)_p(l)_p}{p!\,(n_1+n_2)_{2p+s}}(2t)^{2p} = \frac{(n_1)_s}{(n_1+n_2)_s}\,{}_3F_2\left(\left.\begin{array}{c}s+n_1,n_2,l\\ \frac{s+n_1+n_2}{2},\frac{s+n_1+n_2+1}{2}\end{array}\right|t^2\right)\,. \qquad (15)$$

Alternatively, interchanging the order of summations in I (14), we can write it as

$$I=\sum_{p=0}^{\infty}\frac{(n_1)_p(n_2)_p(l)_p}{p!\,(n_1+n_2)_{2p}}(2t)^{2p}\,{}_2F_1\left(\left.\begin{array}{c}r,p+n_1\\ 2p+n_1+n_2\end{array}\right|\xi_-\right)\,.$$

Unfortunately, these sums seem not to be expressible via any well-known special functions.

The result simplifies if $n_1=n_2$. Using the quadratic transformation

$${}_2F_1\left(\left.\begin{array}{c}a,b\\ 2b\end{array}\right|z\right)=\left(1-\frac{z}{2}\right)^{-a}{}_2F_1\left(\left.\begin{array}{c}\frac{a}{2},\frac{a+1}{2}\\ b+\frac{1}{2}\end{array}\right|\left(\frac{z}{2-z}\right)^2\right)$$

and expanding the new hypergeometric function, we can express I via the Appell function F_3 [8]:

$$\left(\frac{\xi_+}{2}\right)^r I=\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{\left(\frac{r}{2}\right)_q\left(\frac{r+1}{2}\right)_q(n_1)_p(l)_p}{q!\,p!\,\left(n_1+\frac{1}{2}\right)_{q+p}}\left(\frac{\xi_-}{\xi_+}\right)^{2q}t^{2p} = F_3\left(\frac{r+1}{2},l;\frac{r}{2},n_1;n_1+\frac{1}{2};\frac{\xi_-^2}{\xi_+^2},t^2\right)\,. \qquad (16)$$

Here $(r+1)/2+l=n_1+1/2$, and we can reduce this function to the Appell F_1 using

$$F_3(a_1,a_2;b_1,b_2;a_1+a_2;z_1,z_2)= (1-z_1)^{-b_1}F_1\left(a_2;b_1,b_2;a_1+a_2;\frac{z_1}{z_1-1},z_2\right)\,.$$

Finally, we arrive at the result

$$I(n_1,n_1,n_3;\vartheta;\omega_1,\omega_2) = I(2n_1,n_3)\left(-2\sqrt{\omega_1\omega_2}\right)^{-r} \times F_1\left(l;\frac{r}{2},n_1;n_1+\frac{1}{2};-\frac{(\omega_1-\omega_2)^2}{4\omega_1\omega_2},\frac{1-\cosh\vartheta}{2}\right) \quad (17)$$

At $\vartheta=0$ it reduces to (8), due to the quadratic transformation

$${}_2F_1\left(\left.\begin{array}{c}a,b\\2b\end{array}\right|z\right) = (1-z)^{-a/2}\,{}_2F_1\left(\left.\begin{array}{c}\frac{a}{2},b-\frac{a}{2}\\b+\frac{1}{2}\end{array}\right|-\frac{z^2}{4(1-z)}\right).$$

Another case when the result simplifies is $\omega_1=\omega_2$. Then the only non-vanishing term in the sum (14) is $s=0$, and from (15) we obtain the result

$$I(n_1,n_2,n_3;\vartheta;\omega,\omega) = I(n_1+n_2,n_3)(-2\omega)^{-r} \times {}_3F_2\left(\left.\begin{array}{c}n_1,n_2,l\\\frac{n_1+n_2}{2},\frac{n_1+n_2+1}{2}\end{array}\right|\frac{1-\cosh\vartheta}{2}\right). \quad (18)$$

This form is much simpler than (9). Due to (10), we also know $I(n_1,n_2,n_3;\vartheta;\omega,0)$. If $n_1=n_2$, this hypergeometric function reduces to ${}_2F_1$, which agrees with (17).

A simple result for $n_1=n_2=1$ can be derived directly from (5). Substituting $x=\left[-1+(e^\vartheta+1)z\right]/(e^\vartheta-1)$, we get

$$I = (e^{2\vartheta}-1)^{1-2l}e^{\vartheta l}(e^\vartheta-\xi)^{2l-2} \times\left[F\left(\frac{e^\vartheta}{e^\vartheta+1},y\right)-F\left(\frac{1}{e^\vartheta+1},y\right)\right],$$

$$F(x,y) = \int_0^x z^{-l}(1-z)^{-l}(1-yz)^{2l-2}dz\,, \quad (19)$$

where

$$y=\frac{(e^\vartheta+1)(1-\xi)}{e^\vartheta-\xi}.$$

Expanding the two brackets in (19) and integrating, we obtain

$$F(x,y) = \frac{x^{1-l}}{1-l}\sum_{n,m=0}^\infty \frac{(l)_n(2-2l)_m(1-l)_{n+m}}{n!\,m!\,(2-l)_{n+m}}x^n(xy)^m = \frac{x^{1-l}}{1-l}F_1(1-l;l,2-2l;2-l;x,xy).$$

This Appell function can be reduced using

$$F_1(a;b_1,b_2;b_1+b_2;z_1,z_2) = (1-z_2)^{-a}\,{}_2F_1\left(\left.\begin{array}{c}a,b_1\\b_1+b_2\end{array}\right|\frac{z_1-z_2}{1-z_2}\right),$$

and so

$$F(x,y) = \frac{(x^{-1}-y)^{l-1}}{1-l}\,{}_2F_1\left(\left.\begin{array}{c}1-l,l\\2-l\end{array}\right|\frac{1-y}{x^{-1}-y}\right).$$

At last, we arrive at

$$I(1,1,n;\vartheta;\omega_1,\omega_2) = I(2,n)\left(-2\sqrt{\omega_1\omega_2}\right)^{-r} \times\frac{1}{l-1}(e^\vartheta-e^{-\vartheta})^{-l}(e^{\vartheta/2}\xi^{-1/2}-e^{-\vartheta/2}\xi^{1/2})^{l-1} \times\left[(e^{-\vartheta}\xi)^{(1-l)/2}\,{}_2F_1\left(\left.\begin{array}{c}1-l,l\\2-l\end{array}\right|\frac{\xi-e^{-\vartheta}}{e^\vartheta-e^{-\vartheta}}\right) - (e^{-\vartheta}\xi)^{(l-1)/2}\,{}_2F_1\left(\left.\begin{array}{c}1-l,l\\2-l\end{array}\right|\frac{e^\vartheta-\xi^{-1}}{e^\vartheta-e^{-\vartheta}}\right)\right]. \quad (20)$$

At $\omega_1=\omega_2$ this should be equivalent to (18).

3 Expansions in ε

The hypergeometric $\omega_1=\omega_2$ result (18) can be expanded in ε. Expansion of such hypergeometric functions was considered in a number of papers [14]. It can be performed automatically using the `Mathematica` package `HypExp` [15]. Coefficients are expressed via harmonic polylogarithms [16], and can be simplified by the package `HPL` [17]. It is better to work in Euclidean space, where arguments of logarithms and polylogarithms are away from branch cuts, and to do analytical continuation to Minkowski space at the end.

All integrals reduce to 3 master ones. Their expansions have the form

$$\frac{I(1+2l_1\varepsilon,1+2l_2\varepsilon,1+l_3\varepsilon;\vartheta;\omega,\omega)}{I(2+2(l_1+l_2)\varepsilon,1+l_3\varepsilon)(-2\omega)^{-2(1+l_1+l_2+l_3)\varepsilon}} = \frac{1}{\sinh\vartheta}\left[\vartheta+a_1(\vartheta)\varepsilon+a_2(\vartheta)\varepsilon^2+\cdots\right], \quad (21)$$

$$\frac{I(1+2l_1\varepsilon,2l_2\varepsilon,1+l_3\varepsilon;\vartheta;\omega,\omega)}{I(2+2(l_1+l_2)\varepsilon,1+l_3\varepsilon)(-2\omega)^{1-2(1+l_1+l_2+l_3)\varepsilon}} = 1-2l_2\varepsilon\left[\tau\vartheta+b_1(\vartheta)\varepsilon+b_2(\vartheta)\varepsilon^2+\cdots\right], \quad (22)$$

where $\tau=\tanh(\vartheta/2)$ (expansion of $I(2l_1\varepsilon,1+2l_2\varepsilon,1+l_3\varepsilon;\vartheta;\omega,\omega)$ is given by the formula symmetric to (22)). At $l_2=0$, the right-hand side of (22) is exactly 1; $a_n(\vartheta)$ are odd functions ($\sim\vartheta^3$ at $\vartheta\to0$); $b_n(\vartheta)$ are even functions ($\sim\vartheta^2$ at $\vartheta\to0$).

The first corrections are

$$a_1(\vartheta) = c_1(\vartheta)+2l_{12}\vartheta\,, \quad (23)$$

$$b_1(\vartheta) = \tau c_1(\vartheta)-\frac{1}{2}j'\vartheta^2\,, \quad (24)$$

$$c_1 = -2l_{12}L_{2a}-jL_{2b}\,,$$

where $l_{12}=l_1+l_2$, $j_3=1+l_3$, $j=j_3-l_{12}$, $j'=j_3-l_1+l_2$,

$$L_{2a} = \mathrm{Li}_2(\tau)-\mathrm{Li}_2(-\tau)\,,$$

$$L_{2b} = \mathrm{Li}_2\left(\frac{1+\tau}{2}\right)-\mathrm{Li}_2\left(\frac{1-\tau}{2}\right)-L\vartheta\,,$$

$$L = -\frac{1}{2}\log\frac{1-\tau^2}{4} = \log\left(2\cosh\frac{\vartheta}{2}\right).$$

Two dilogarithms are not independent:

$$\mathrm{Li}_2\left(\frac{1+\tau}{2}\right)+\mathrm{Li}_2\left(\frac{1-\tau}{2}\right) = -L^2+\frac{\vartheta^2}{4}+\frac{\pi^2}{6};$$

L_{2b} is written here in the manifestly odd form.

The second corrections are

$$a_2(\vartheta) = c_2(\vartheta)-4l_{12}^2L_{2a}-2l_{12}jL_{2b}\,, \quad (25)$$

$$b_2(\vartheta) = \tau c_2(\vartheta)+4j'j_3L_{3d} - 2\left[(3l_1+l_2)j_3-l_1^2+l_2^2\right]L_{3e} - j_3\vartheta\left[4l_1L_{2a}+(j_3-3l_1+l_2)L_{2b}+j'L\vartheta\right], \quad (26)$$

$$c_2(\vartheta) = 4l_{12}j_3L_{3a}+2jj_3L_{3b}-2l_{12}jL_{3c} - \frac{1}{12}(j_3^2-5l_{12}j_3+8l_1l_2)\vartheta^3\,,$$

where

$$
\begin{aligned}
&L_{3a} = \mathrm{Li}_3(\tau) - \mathrm{Li}_3(-\tau)\,,\\
&L_{3b} = \mathrm{Li}_3\left(\frac{1+\tau}{2}\right) - \mathrm{Li}_3\left(\frac{1-\tau}{2}\right) + \frac{\vartheta}{2}\left(L^2 - \frac{\pi^2}{6}\right)\,,\\
&L_{3c} = \mathrm{Li}_3\left(\frac{2\tau}{1+\tau}\right) - \mathrm{Li}_3\left(\frac{-2\tau}{1-\tau}\right)\,,\\
&L_{3d} = \mathrm{Li}_3\left(\frac{1+\tau}{2}\right) + \mathrm{Li}_3\left(\frac{1-\tau}{2}\right)\\
&\quad - \frac{1}{3}L^3 + \frac{\pi^2}{6}L - \frac{7}{4}\zeta(3)\,,\\
&L_{3e} = \mathrm{Li}_3\left(\frac{2\tau}{1+\tau}\right) + \mathrm{Li}_3\left(\frac{-2\tau}{1-\tau}\right)
\end{aligned}
$$

(L_{3d} can be written via a single trilogarithm $\mathrm{Li}_3(-(1\pm\tau)/(1\mp\tau))$).

The next correction can be derived by `HypExp`; it contains harmonic polylogarithms which cannot be reduced to ordinary polylogarithms.

If $n_1 = n_2 = 1$, I (11) depends on n_3 and ε only via the product $j_3\varepsilon$, see (5). The coefficients (23), (25) satisfy this requirement, and coincide with the expansion of (20) at $\omega_1 = \omega_2$.

At small ϑ, keeping only the leading ϑ^3 terms in $a_{1,2}(\vartheta)$ and the leading ϑ^2 terms in $b_{1,2}(\vartheta)$, we reproduce the ϑ^2 term in (18) expanded in ε.

4 Conclusion

In the single-scale case $\omega_1 = \omega_2$, the vertex diagram with arbitrary powers of all denominators is given by the surprisingly simple formula (18). The cases $\omega_{1,2} = 0$ reduce to this one (10). Expansions in ε has been derived in Sect. 3. If $\omega_1 \neq \omega_2$, we were able to obtain relatively simple formulas for $n_1 = n_2$ (17) and especially $n_1 = n_2 = 1$ (20).

The integral (1) with $n_1 = n_2 = n_3 = 1$ was considered in [18] for general $\omega_{1,2}$ (in fact, even a more general integral with a nonzero mass m of the light line was calculated up to $\mathcal{O}(\varepsilon^0)$). Our a_1 (23) with $l_1 = l_2 = l_3 = 0$ reproduces the formula (37) from this paper.

Integrals with $n_1 = n_2 = n_3 = 1$; $n_1 = n_2 = 1$, $n_3 = 1+\varepsilon$; $n_1 = 1+2\varepsilon$, $n_2 = n_3 = 1$ were considered in the Appendix of [19] (the formulas (69), (71), (72)). It is difficult to make sense of these formulas, because all integrals in the left-hand sides of equations in this Appendix are equal to 0, and the right-hand sides have dimensionality different from the left-hand sides. Evidently, the author assumed some different integrals in the left-hand sides, and some dimensional factors in the right-hand sides, but it seems impossible to guess their real meaning. In particular, these formulas cannot be interpreted as our integrals with $\omega_1 = \omega_2$ with appropriate powers of -2ω inserted into the right-hand sides, because already their values at $\vartheta = 0$ differ. It would be interesting to compare the formulas (69), (71) (containing ${}_2F_1$ hypergeometric functions) and (72) (containing the Appell F_1 function) to our results. The $\mathcal{O}(\varepsilon^0)$ term in (69) contains dilogarithms similar to those in a_1 (23) (at $l_1 = l_2 = l_3 = 0$), but the leading $\mathcal{O}(\varepsilon^{-1})$ terms differ.

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