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On Epsilon Expansions of Four-loop Non-planar Massless Propagator Diagrams

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Abstract. We evaluate three typical four-loop non-planar massless propagator diagrams in a Taylor expansion in dimensional regularization parameter $\epsilon = (4-d)/2$ up to transcendentality weight twelve, using a recently developed method of one of the present coauthors (R.L.). We observe only multiple zeta values in our results.

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1 Introduction

Analytic results for one-scale multiloop Feynman integrals in a Laurent expansion in $\epsilon = (4-d)/2$ are expressed as linear combinations of transcendental constants with rational coefficients. The set of these constants essentially depends on the type of Feynman integrals. Probably, the simplest type of one-scale Feynman integrals are massless propagator integrals depending on one external momentum. Here the world record is set at four loops — see Ref. [1] where all the corresponding master integrals were analytically evaluated in an epsilon expansion up to transcendentality weight seven.

Practical calculations show that only multiple zeta values (MZV) (see, e.g., [2]) appear in results. Brown proved [3] that convergent scalar massless propagator integrals with the degree of divergence $\omega \equiv 4h - 2L = -2$ (where h and L are numbers of loops and edges, correspondingly) up to five loops contain only MZV in their epsilon expansions. (In two loops, a proof was earlier presented in Ref. [4].) He also proved that for the three diagrams depicted in Fig. 1, every coefficient in a Taylor expansion in ϵ is a rational linear combination of MZV and Goncharov's polylogarithms [5] with sixth roots of unity as arguments.

The goal of this brief communication is to study these diagrams experimentally. We present results in an epsilon expansion up to transcendentality weight twelve. To do this we apply the DRA method recently suggested by one of the authors (R.L.), Ref. [6]. The method is based on the use of dimensional recurrence relations (DRR) [7] and analytic properties of Feynman integrals as functions of the parameter of dimensional regularization, d, and was already successfully applied in previous calculations [8–12]. To apply this method it is essential to perform an integration by parts (IBP) [13] reduction of integrals that participate in dimensional recurrence relations to master integrals. To do this, we use the `C++` version of the code `FIRE` [14].

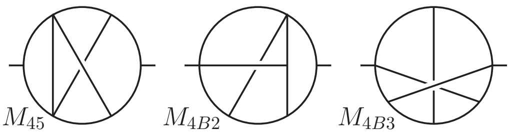

Fig. 1. Diagrams considered in Ref. [3] as possible candidates for the presence of sixth roots of unity in the ϵ-expansion.

To study analytic properties of solutions of dimensional recurrence relations, i.e. to reveal the position and the order of poles in d in a basic stripe, we used a sector decomposition [15–17] implemented in the code `FIESTA` [17,18]. To fix remaining constants in the homogenous solutions of dimensional recurrence relations it was quite sufficient for us to use analytic results for the four-loop massless propagators master integrals [1] (confirmed numerically by `FIESTA` [19]). Finally, after obtaining results for master integrals in terms of multiple series we calculated resulting coefficients at powers of ϵ numerically with a high precision and then applied the PSLQ algorithm [20]. We also applied the code `HPL` [21] for dealing with harmonic polylogarithms.

For all the three diagrams of Fig. 1, we performed evaluation up to transcendentality weight twelve where

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the basis of transcendental numbers we used includes 48 constants. Since coefficients in our results turn out to be cumbersome the accuracy necessary for a successful calculation by PSLQ was rather high so that we were forced to perform numerical calculations with the accuracy of around 800 digits.

The first diagram of Fig. 1 is a master diagram. This is nothing but M_{45} in Fig. 2 of Ref. [1] where all the master integrals for four-loop massless propagators are shown. Following the method of [6] we needed, first to calculate lower master integrals $M_{01}, M_{11}, M_{13}, M_{14}, M_{21}, M_{27}, M_{35}$. Eventually, we arrived at the following result which is made homogeneously transcendental by pulling out an appropriate rational function of ϵ and normalizing it at the fourth power of the one-loop integral (i.e. M_{31} according to the notation of [1]):

$$
\begin{aligned}
\frac{M_{45}(4-2\epsilon)}{\epsilon^4 M_{31}(4-2\epsilon)} = \frac{(1-2\epsilon)^3}{1-6\epsilon}\Bigg\{ & 36\zeta_3^2 - \left(-\frac{6}{5}\pi^4\zeta_3 + 378\zeta_7\right)\epsilon \\
& + \left(-\frac{427\pi^8}{1500} + 2844\zeta_3\zeta_5 + \frac{3024\zeta_{5,3}}{5}\right)\epsilon^2 - \left(-\frac{22}{3}\pi^6\zeta_3 \right. \\
& \left. + 732\zeta_3^3 + 3\pi^4\zeta_5 + \frac{42458\zeta_9}{3}\right)\epsilon^3 + \left(-\frac{60329\pi^{10}}{24948} - \frac{183}{5}\pi^4\zeta_3^2 \right. \\
& \left. + 62403\zeta_5^2 + 149895\zeta_3\zeta_7 - \frac{58563\zeta_{8,2}}{2}\right)\epsilon^4 - \left(-\frac{19817}{500}\pi^8\zeta_3 \right. \\
& + \frac{29578\pi^6\zeta_5}{315} + 101952\zeta_3^2\zeta_5 - \frac{50761\pi^4\zeta_7}{50} - 325152\pi^2\zeta_9 \\
& \left. + \frac{73041423\zeta_{11}}{20} + \frac{175392}{5}\zeta_3\zeta_{5,3} - \frac{216768}{5}\zeta_{5,3,3}\right)\epsilon^5 \\
& + \left(-\frac{98988919597\pi^{12}}{17027010000} - \frac{19242}{35}\pi^6\zeta_3^2 + \frac{37216\zeta_3^4}{3} \right. \\
& - \frac{10280}{9}\pi^4\zeta_3\zeta_5 + \frac{210112}{3}\pi^2\zeta_5^2 + \frac{600320}{3}\pi^2\zeta_3\zeta_7 \\
& + 2100799\zeta_5\zeta_7 + \frac{9455470\zeta_3\zeta_9}{9} + \frac{160424}{75}\pi^4\zeta_{5,3} \\
& \left. + 60032\pi^2\zeta_{7,3} - \frac{1132952\zeta_{9,3}}{3} - 120064\zeta_{6,4,1,1}\right)\epsilon^6 \\
& + O\left(\epsilon^7\right)\Bigg\}. \qquad (1)
\end{aligned}
$$

Here $\zeta_{m_1,\ldots,m_k}$ are MZV given by

$$
\zeta(m_1,\ldots,m_k) = \sum_{i_1=1}^{\infty}\sum_{1}^{i_1-1}\cdots\sum_{1}^{i_{k-1}-1}\prod_{j=1}^{k}\frac{\mathrm{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}. \qquad (2)
$$

The other two diagrams of Fig. 1 *are not* master integrals and, consequently, they do not appear in Fig. 2 of Ref. [1]. We found their reduction to lower master integrals by FIRE and then evaluated resulting master integrals by our technique. Let us stress that the master integrals involved in the IBP reduction of the second and the third non-planar integrals of Fig. 1 are all *planar* diagrams so that they did not have any chance to involve something in addition to MZV, according to [3]. These are our results for them in the same normalization where we again managed to reveal homogenous transcendentality:

$$
\begin{aligned}
\frac{M_{4B2}(4-2\epsilon)}{\epsilon^4 M_{31}(4-2\epsilon)} = \frac{(1-2\epsilon)^3}{1-\epsilon}\Bigg\{ & 36\zeta_3^2 + \left(\frac{6\pi^4\zeta_3}{5} + \frac{189\zeta_7}{2}\right)\epsilon \\
& + \left(\frac{1359\pi^8}{7000} + 144\zeta_3\zeta_5 - \frac{2916\zeta_{5,3}}{5}\right)\epsilon^2 + \left(\frac{4\pi^6\zeta_3}{21} - 1392\zeta_3^3 \right. \\
& \left. + 51\pi^4\zeta_5 + \frac{25549\zeta_9}{6}\right)\epsilon^3 + \left(\frac{146255\pi^{10}}{99792} - \frac{348}{5}\pi^4\zeta_3^2 \right. \\
& \left. - \frac{75843\zeta_5^2}{4} - \frac{206655\zeta_3\zeta_7}{4} + \frac{152163\zeta_{8,2}}{8}\right)\epsilon^4 \\
& + \left(-\frac{313039\pi^8\zeta_3}{31500} + \frac{25982\pi^6\zeta_5}{315} - 39372\zeta_3^2\zeta_5 + \frac{234339\pi^4\zeta_7}{200} \right. \\
& \left. + 29412\pi^2\zeta_9 - \frac{26995129\zeta_{11}}{160} + \frac{71928}{5}\zeta_3\zeta_{5,3} + \frac{19608}{5}\zeta_{5,3,3}\right)\epsilon^5 \\
& + \left(\frac{107930288857\pi^{12}}{68108040000} - \frac{42968}{315}\pi^6\zeta_3^2 + \frac{85696\zeta_3^4}{3} \right. \\
& - \frac{23510}{9}\pi^4\zeta_3\zeta_5 + \frac{31822}{3}\pi^2\zeta_5^2 + \frac{90920}{3}\pi^2\zeta_3\zeta_7 + 165244\zeta_5\zeta_7 \\
& - \frac{6321395\zeta_3\zeta_9}{9} + \frac{45614}{75}\pi^4\zeta_{5,3} + 9092\pi^2\zeta_{7,3} - \frac{803569\zeta_{9,3}}{6} \\
& \left. - 18184\zeta_{6,4,1,1}\right)\epsilon^6 + O\left(\epsilon^7\right)\Bigg\}, \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{M_{4B3}(4-2\epsilon)}{\epsilon^4 M_{31}(4-2\epsilon)} = \frac{(1-2\epsilon)^3}{1+4\epsilon}\Bigg\{ & 36\zeta_3^2 + \left(\frac{6\pi^4\zeta_3}{5} + 567\zeta_7\right)\epsilon \\
& + \left(\frac{211\pi^8}{1750} + 3744\zeta_3\zeta_5 + \frac{1944\zeta_{5,3}}{5}\right)\epsilon^2 + \left(\frac{68\pi^6\zeta_3}{7} + 288\zeta_3^3 \right. \\
& \left. + 30\pi^4\zeta_5 + 22094\zeta_9\right)\epsilon^3 + \left(\frac{1255\pi^{10}}{4158} + \frac{72}{5}\pi^4\zeta_3^2 \right. \\
& \left. + 51318\zeta_5^2 + 95490\zeta_3\zeta_7 - 11799\zeta_{8,2}\right)\epsilon^4 + \left(\frac{3283\pi^8\zeta_3}{125} \right. \\
& + \frac{12484\pi^6\zeta_5}{105} - 65952\zeta_3^2\zeta_5 + \frac{17538\pi^4\zeta_7}{25} + 65232\pi^2\zeta_9 \\
& \left. - \frac{360909\zeta_{11}}{10} - \frac{54432}{5}\zeta_3\zeta_{5,3} + \frac{43488}{5}\zeta_{5,3,3}\right)\epsilon^5 \\
& + \left(\frac{2972813873\pi^{12}}{1064188125} - \frac{15056}{105}\pi^6\zeta_3^2 - 29608\zeta_3^4 - \frac{3640}{3}\pi^4\zeta_3\zeta_5 \right. \\
& - 8176\pi^2\zeta_5^2 - 23360\pi^2\zeta_3\zeta_7 + 1091724\zeta_5\zeta_7 + \frac{4198640\zeta_3\zeta_9}{3} \\
& \left. - \frac{6752}{25}\pi^4\zeta_{5,3} - 7008\pi^2\zeta_{7,3} + 95936\zeta_{9,3} + 14016\zeta_{6,4,1,1}\right)\epsilon^6 \\
& + O\left(\epsilon^7\right)\Bigg\}. \qquad (4)
\end{aligned}
$$

We see that only MZV are present in our results. Although Goncharov's polylogarithms at sixth roots of unity were allowed to appear according to the analysis of Ref. [3]

they have not appeared. In fact, such transcendental numbers do appear in epsilon expansions of some classes of *massive* Feynman integrals [22–25].

Taking our results into account it is plausible to conjecture that there are only MZV in massless propagator diagrams. For the moment, it seems to be unclear how to prove this conjecture. It also seems to be very difficult to find a counterexample.

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