SFB/CPP-10-108
TTP10-46

Towards a Two-Loop Matching of Gauge Couplings in Grand Unified Theories

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Abstract

We calculate the two-loop matching corrections for the gauge couplings at the Grand Unification scale in a general framework that aims at making as few assumptions on the underlying Grand Unified Theory (GUT) as possible. In this paper we present an intermediate result that is general enough to be applied to the Georgi-Glashow $SU(5)$ as a "toy model". The numerical effects in this theory are found to be larger than the current experimental uncertainty on α_s. Furthermore, we give many technical details regarding renormalization procedure, tadpole terms, gauge fixing and the treatment of group theory factors, which is useful preparative work for the extension of the calculation to supersymmetric GUTs.

PACS numbers: 12.10.Kt 12.38.Bx 02.20.Sv 11.25.Db

1 Introduction

Grand Unified Theories (GUTs) provide an appealing framework for physics beyond the Standard Model (SM) of particle physics. Particularly supersymmetric (SUSY) GUTs have gained a lot of interest in the past decades as they seem to be consistent with the measured values of $\alpha_s(M_Z)$, $\alpha_{em}(M_Z)$ and $\sin\theta_W(M_Z)$ [1–3] and offer many other beautiful features. Up to date most Renormalization Group (RG) analyses that study gauge coupling unification use two-loop Renormalization Group Equations (RGEs) and one-loop matching at the SUSY and GUT scales [4–11]. These matching corrections arise from integrating out heavy particles at the SUSY and GUT thresholds. They depend sensitively on the mass splittings between the heavy particles and can be used to constrain the mass spectrum of the theory. As experimental accuracy is increasing, also higher order corrections become more and more important. It has been shown that $\mathcal{O}(\alpha_s^3)$ effects at the SUSY decoupling scale can be as large as the current experimental uncertainties on the gauge couplings [12–14]. Furthermore, in GUT models that contain large representations, as for example the so-called Missing Doublet Model [15, 16], the decoupling scale dependence at the GUT scale can exceed the experimental uncertainty by an order of magnitude [14].

These facts encourage us to try to improve the theoretical accuracy of the unification study and aim at a complete three-loop RGE analysis. This requires three-loop RGEs for the SM, the Minimal Supersymmetric Standard Model (MSSM) and the SUSY GUT model under consideration and two-loop matching corrections at the SUSY scale and the GUT scale. In Quantum Chromodynamics (QCD) the gauge β function in the modified minimal substraction scheme ($\overline{\text{MS}}$) is known to four loops [17, 18], though a complete three-loop SM gauge β function is still missing. For the MSSM full three-loop RGEs are available [19, 20] in the so-called $\overline{\text{DR}}$ scheme. The same is true for the most general single gauge coupling theory in $\overline{\text{MS}}$ and a general supersymmetric GUT in $\overline{\text{DR}}$ [21, 22]. Matching corrections at the SUSY scale are known to one-loop order for the $U(1)$ and $SU(2)$ gauge couplings $\alpha_{1/2}$ [7] and to two-loop for the strong coupling α_s [12, 13]. For the GUT scale thresholds a general formula at the one-loop level is known [23–25], but two-loop corrections have still been missing up to date and thus are not included in present unification analyses [4–6, 10, 11, 14].

The aim of this paper is to provide a first step towards a general formula for the two-loop GUT matching corrections in a similar fashion as the one-loop corrections in refs. [23–25]. Unfortunately it turns out that at the two-loop level it is much harder to carry out the calculation in a general way, making as few assumptions about the underlying GUT model as possible. Therefore, the result given in this paper is not yet applicable to SUSY GUT models, as it is not yet general enough. Nevertheless, by applying it to the Georgi-Glashow $SU(5)$ [26] as a "toy model", we provide an important intermediate step on the way to a full three-loop unification analysis. The generalization for SUSY GUTs actually is in progress.

The remainder of the paper is organized as follows: in the next section we describe the theoretical framework that is used for the calculation. There the Lagrangian is defined and it is shown how delicate issues, as gauge fixing and renormalization are carried out. In section 3 we present an academic study of our general formula when applied to the Georgi-Glashow model, which is the simplest (yet already ruled out) possible GUT model. In particular the issue of reducing the decoupling scale dependence is discussed. Finally, we present our conclusions. In appendix A of this paper we describe the procedure of defining and reducing the group theory factors that appear in our calculation. The rest of the appendix is dedicated to some supplementary material to the main text.

2 Theoretical framework

Since there is a host of well motivated GUT models, we want our final result to be applicable to as many of them as possible. Therefore, it is desirable to carry out the calculation of two-loop matching corrections at the GUT scale in a framework that makes as few assumptions on the underlying GUT model as possible. The idea is to have a general formula that depends on the Casimir invariants and the spectrum of the theory. Choosing a specific model specifies those Casimir invariants and gives an expression that depends only on the masses and couplings of the model. The actual calculation that is presented in this paper is not yet done in full generality, but makes some additional assumptions about the model. These assumptions are described in subsection 2.3. Nevertheless, we present the theoretical framework that is needed for the calculation of the relevant Green's functions (almost) as general as possible below in order to be armed for future improvements of the calculation.

2.1 The Lagrangian

We consider a general renormalizable quantum field theory defined by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_\alpha^{\mu\nu}F^\alpha_{\mu\nu} + \overline{\Psi}iD_\mu\gamma^\mu\Psi + \frac{1}{2}(D^\mu\Phi)^T D_\mu\Phi - V(\Phi) + \mathcal{L}_\mathrm{Y} + \mathcal{L}_\mathrm{gf} + \mathcal{L}_\mathrm{gh}. \quad (1)$$

The chiral Dirac fermion field Ψ and the real scalar field Φ reside in (not necessarily irreducible) representations of the gauge group $\mathbf{G}$. The dynamics of the gauge field that transforms according to the adjoint representation of $\mathbf{G}$, is described by the Yang Mills curl $F_\alpha^{\mu\nu} = \partial^\mu A_\alpha^\nu - \partial^\nu A_\alpha^\mu + g f_{\alpha\beta\gamma}A_\beta^\mu A_\gamma^\nu$. Moreover, $V(\Phi)$, $\mathcal{L}_\mathrm{Y}$, $\mathcal{L}_\mathrm{gf}$ and $\mathcal{L}_\mathrm{gh}$ are the scalar potential, the Yukawa interactions, the gauge fixing and the ghost parts of the Lagrangian, respectively. They will be described in detail later in this section. $V(\Phi)$ is chosen such that the scalar field $\Phi = v + \Phi'$ contains one $\mathbf{G}$-irreducible subspace that develops a vacuum expectation value (vev) v, that breaks $\mathbf{G}$ down to the SM gauge group $\prod_k \mathbf{G}_k = SU(3)\times SU(2)\times U(1)$. Models that have more than one vev of $\mathcal{O}(M_\mathrm{G})$, where

$M_{\rm G}$ is the Grand Unification scale, are not covered by our framework yet. The indices α, β, γ... belong to the adjoint representation and label the generators of $\mathbf{G}$ which fulfill the commutation relations

$$[T^\alpha, T^\beta] = i f^{\alpha\beta\gamma}\, T^\gamma \quad \text{and also:} \quad [\tilde{T}^\alpha, \tilde{T}^\beta] = i f^{\alpha\beta\gamma}\, \tilde{T}^\gamma\,, \tag{2}$$

with the structure constants $f^{\alpha\beta\gamma}$. We use the tilde to denote the generators of the real[1] scalar representation which fulfill $(\tilde{T}^\alpha)^T = -\tilde{T}^\alpha$. The generator that acts on the fermion field satisfies $(T^\alpha)^\dagger = T^\alpha$. Again, T^α and $\tilde{T}^\alpha$ need not necessarily be defined on irreducible representations of $\mathbf{G}$, but can also have block diagonal form. In order to distinguish between broken and unbroken generators, we introduce the notation:

$$\{\alpha\} = \sum_i \{A_i\} + \sum_i \{a_i\} = \{A\} + \{a\}\,, \tag{3}$$

where A_i label the broken generators of $\mathbf{G}$ belonging to the SM-irreducible subspace labeled by i. If there is only one SM-irreducible subspace in the adjoint representation of $\mathbf{G}$, as e.g. is practically the case in $SU(5)$, we can omit the sub-index i. In contrast, a_i label the unbroken generators belonging to the subgroup[2] $\mathbf{G}_i$:

$$\begin{aligned} \tilde{T}^{a_i} v &= 0\,, \\ \tilde{T}^{A_i} v &\neq 0\,. \end{aligned} \tag{4}$$

The Lagrangian in eq. (1) is invariant under local gauge transformations with the real parameter $\theta = \theta(x)$:

$$\begin{aligned} \Psi &\rightarrow \Psi - i\theta^\alpha T^\alpha \Psi\,, \\ \Phi &\rightarrow \Phi - i\theta^\alpha \tilde{T}^\alpha \Phi\,, \\ A^\alpha_\mu &\rightarrow A^\alpha_\mu + f^{\alpha\beta\gamma}\theta^\beta A^\gamma_\mu - \frac{1}{g}\partial_\mu \theta^\alpha\,. \end{aligned} \tag{5}$$

The covariant derivatives are defined as:

$$\begin{aligned} D_\mu \Psi &= (\partial_\mu - i g T^\alpha A^\alpha_\mu)\Psi\,, \\ D_\mu \Phi &= (\partial_\mu - i g \tilde{T}^\alpha A^\alpha_\mu)\Phi\,. \end{aligned} \tag{6}$$

Using eq. (2), eq. (4), and $f^{a_i b_j A_k} = 0$ (cf. appendix A), the gauge-kinetic term for the scalar field $\Phi = v + \Phi'$ can be written as:

$$\begin{aligned} \frac{1}{2}(D^\mu\Phi)^T D_\mu \Phi = {} & \frac{1}{2}(\partial^\mu \Phi')^T \partial_\mu \Phi' + \frac{1}{2} g^2 v \tilde{T}^A \tilde{T}^B v A^\mu_A A_{\mu B} \\ & + i g v \tilde{T}^A \partial_\mu \Phi A^\mu_A + i g^2 f_{ABa} A^\mu_B A_{\mu a} v \tilde{T}^A \Phi' \\ & + i g \Phi' \tilde{T}^\alpha \partial_\mu \Phi' A^\mu_\alpha + \frac{1}{2} g^2 \Phi' \tilde{T}^\alpha \tilde{T}^\beta \Phi' A^\mu_\alpha A_{\mu\alpha} \\ & + g^2 v \tilde{T}^A \tilde{T}^B \Phi' A^\mu_A A_{\mu B}\,, \end{aligned} \tag{7}$$

[1]This is no loss of generality since every complex scalar can be written as two real scalars.

[2]Please note the different meanings of the sub-index i when attached to the capital adjoint index opposed to when attached to a lowercase adjoint index.

where we can identify the (diagonal) gauge boson mass matrix

$$(M_{\rm X})_{A_iB_i} \equiv g^2 v\tilde{T}^{A_i}\tilde{T}^{B_i}v\,, \tag{8}$$

with eigenvalues denoted by $M_{{\rm X}_i}$. Again, the sub-index i labels the SM-irreducible subspace that is meant, because each SM-irreducible subspace can be assigned to a definite gauge boson mass. Note that the position of the adjoint indices α, A_i, a_i... is irrelevant. Furthermore, it is understood that a partial derivative acts only on the single field, which is next to it. The gauge-kinetic term for the scalars contains the undesired quadratic mixing $igv\tilde{T}^A\partial_\mu\Phi A^\mu_A$ between Goldstone bosons and heavy gauge bosons. As we will see in a moment, the gauge fixing Lagrangian $\mathcal{L}_{\rm gf}$ can be chosen in such a way that this term is cancelled, at least at tree-level.

In order to fix the gauge, we choose the R_ξ gauge fixing functional [27]

$$\begin{aligned} f_{A_i} &= \frac{1}{\sqrt{\xi_{1i}}}\partial_\mu A^\mu_{A_i} - ig\sqrt{\xi_{2i}}v\tilde{T}^{A_i}\Phi'\,, \\ f_{a_i} &= \frac{1}{\sqrt{\eta_i}}\partial_\mu A^\mu_{a_i}\,. \end{aligned} \tag{9}$$

Note that we have chosen two distinct gauge parameters ξ_1 and ξ_2 for each SM-irreducible subspace of the heavy gauge bosons. They renormalize differently and thus can only be equated with each other after renormalization. Otherwise not all the Green's functions can be made finite. In the same way each SM group factor receives its own gauge parameter η_i. This subtlety arises first at the two-loop level and is not relevant for computing one-loop matching coefficients. Although there are other ways to treat the gauge fixing [23,28–30], we find this one most convenient for our purposes. Employing the procedure described e.g. in ref. [31], eq. (9) gives us the gauge fixing Lagrangian and the ghost interactions:

$$\begin{aligned} \mathcal{L}^{R_\xi}_{\rm gf} &= -\frac{1}{2}\sum_i f^2_{A_i} - \frac{1}{2}\sum_i f^2_{a_i} \\ &= \sum_i \left[-\frac{1}{2\xi_{1i}}(\partial_\mu A^\mu_{A_i})^2 - \frac{1}{2}g^2\xi_{2i}\,\Phi'\tilde{T}^{A_i}vv\tilde{T}^{A_i}\Phi' + ig\sqrt{\frac{\xi_{2i}}{\xi_{1i}}}v\tilde{T}^{A_i}\Phi'\partial^\mu A^{A_i}_\mu\right] \\ &\quad -\sum_i \frac{1}{2\eta_i}(\partial_\mu A^\mu_{a_i})^2\,, \end{aligned} \tag{10}$$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{gh}}^{R_\xi} \;=\; & \sum_{ij} \Big[\partial^\mu c^\dagger_{A_i} (\delta_{A_i B_j} \partial_\mu - g f_{A_i B_j \alpha} A^\alpha_\mu) c_{B_j} \\
& - g^2 \sqrt{\xi_{1i}\xi_{2i}}\; v \tilde{T}^{A_i} \tilde{T}^{B_j} v \; c^\dagger_{A_i} c_{B_j} - g^2 \sqrt{\xi_{1i}\xi_{2i}}\; v \tilde{T}^{A_i} \tilde{T}^{B_j} \Phi' c^\dagger_{A_i} c_{B_j} \Big] \\
& - \sum_{ij} \Big[g (\eta_j/\xi_{1i})^{1/4} f_{A_i b_j B_i} \partial^\mu c^\dagger_{A_i} A^{B_i}_\mu c_{b_j} + i g^2 (\xi_{1i}^{1/4} \eta_j^{1/4} \xi_{2i}^{1/2}) f_{A_i b_j B_i}\; v \tilde{T}^{B_i} \Phi' c^\dagger_{A_i} c_{b_j} \Big] \\
& - \sum_{ij} g\; (\xi_{1j}/\eta_i)^{1/4} f_{a_i B_j A_j} \partial^\mu c^\dagger_{a_i} A^{A_j}_\mu c_{B_j} \\
& + \sum_{i} \partial^\mu c^\dagger_{a_i} (\delta_{a_i b_i} \partial_\mu - g f_{a_i b_i c_i} A^{c_i}_\mu) c_{b_i} \,. \qquad (11)
\end{aligned}
$$

Here c_{A_i} and c_{a_i} denote the ghost fields belonging to the heavy and light gauge bosons, respectively. Note again that for an $SU(5)$ GUT we could do the replacement $A_i \to A$ for the capital adjoint indices and the notation would become less clumsy. Here, however, we keep the sub-index i in order to stay as general as possible. As mentioned before, after partial integration the term $ig\sqrt{\xi_{2i}/\xi_{1i}}\; v\tilde{T}^{A_i}\Phi'\partial^\mu A^{A_i}_\mu$ in eq. (10) exactly cancels the corresponding term in eq. (7) at tree level, where $\xi_{1i} = \xi_{2_i}$ is a valid choice. However, when considering higher orders in perturbation theory, the bare gauge parameters ξ_{1i} and ξ_{2_i} are not equal to each other and the above term must be kept explicitly as a counterterm in our calculation (cf. also subsection 2.2). The quadratic term in eq. (10) can be identified with the (unphysical) Goldstone boson mass matrix:

$$
M^2_{\mathrm{Gold}} \;\equiv\; \frac{1}{2} g^2 \sum_i \xi_{2i}\, \tilde{T}^{A_i} v v \tilde{T}^{A_i} \qquad (12)
$$

with the property $\mathrm{Tr}(M^2_{\mathrm{Gold}}) = \sum_i \xi_{2i}\, M^2_{\mathrm{X}_i}\, D_i^{\mathcal{A}}$, where $D_i^{\mathcal{A}}$ is the dimension of the i-th SM-irreducible representation of the heavy gauge bosons. From the Goldstone theorem it follows that the structure $v\tilde{T}^{A_i}$ projects on the subspace of Goldstone bosons, i.e. on the subspace that obtains no mass term from $V(\Phi)$. Hence, the matix M^2_{Gold} has non-zero entries only on the subspace of Goldstone bosons.

For $V(\Phi)$ we consider the most general renormalizable scalar potential with the discrete symmetry $\Phi \to -\Phi$:

$$
V(\Phi) = -\frac{1}{2}\mu^2_{ij}\Phi_i\Phi_j + \frac{1}{4!}\, \lambda_{ijkl}\; \Phi_i\Phi_j\Phi_k\Phi_l \,, \qquad (13)
$$

with totally symmetric tensors μ^2_{ij} and λ_{ijkl}. We impose the requirements

$$
\begin{aligned}
0 &= [\mu^2, \tilde{T}^\alpha] \,, \\
0 &= \tilde{T}^\alpha_{im}\, \lambda_{mjkl} + \tilde{T}^\alpha_{jm}\, \lambda_{imkl} + \tilde{T}^\alpha_{km}\, \lambda_{ijml} + \tilde{T}^\alpha_{lm}\, \lambda_{ijkm} \,, \qquad (14)
\end{aligned}
$$

in order to make $V(\Phi)$ gauge invariant under $\mathbf{G}$. The first equation implies that the matrix μ^2 is proportional to the unit matrix on each subspace irreducible under $\mathbf{G}$.

To break the GUT symmetry, there has to be one **G**-irreducible Higgs representation contained in Φ_i that develops a vev. In order to treat the symmetry breaking appropriately, we define the projector $\Pi^{\mathcal{H}}$ on this particular **G**-irreducible subspace (clearly, $[\tilde{T}^\alpha, \Pi^{\mathcal{H}}] = 0$). This subspace is further divided into the subspace of Goldstone bosons and the subspace of physical Higgs bosons with projectors $P^{\mathcal{G}}$ and $P^{\tilde{\mathcal{H}}}$, respectively ($[\tilde{T}^a, P^{\mathcal{G}}] = 0 = [\tilde{T}^a, P^{\tilde{\mathcal{H}}}]$). The physical Higgs bosons receive masses of order M_{G} from $V(\Phi)$, the Goldstone bosons do not. Using the Goldstone boson mass matrix from eq. (12), the projector on the space of Goldstone bosons can be explicitly written down as [32]:

$$(P^{\mathcal{G}})_{ij} = (\Pi^{\mathcal{H}} - P^{\tilde{\mathcal{H}}})_{ij} = g^2 \tilde{T}^A_{ik} v_k \left(\frac{1}{M_X^2}\right)_{AB} v_l \tilde{T}^B_{lj} \,. \tag{15}$$

Now we can parametrize the scalars in the following way:

$$\Phi_i = v_i + \Phi'_i = v_i + H_i + G_i + S_i \tag{16}$$

where v, H and G live only on the subspace defined by $\Pi^{\mathcal{H}}$. S parametrizes all the other scalars[3]:

$$\begin{aligned}
(\mathbb{1} - \Pi^{\mathcal{H}})_{ij}\Phi'_j &= S_i \,, \\
\Pi^{\mathcal{H}}_{ij}\Phi'_j &= H_i + G_i \,, \\
P^{\tilde{\mathcal{H}}}_{ij}\Phi'_j &= H_i \,, \\
P^{\mathcal{G}}_{ij}\Phi'_j &= G_i \,.
\end{aligned} \tag{17}$$

First let us focus on the subspace that $\Pi^{\mathcal{H}}$ projects on. In order to develop a vev on this subspace, the parameter $\mu^2_{\mathcal{H}}$, defined by $\Pi^{\mathcal{H}}\mu^2 \equiv \mu^2_{\mathcal{H}}\mathbb{1}$, has to be positive. If this is the case, it is convenient to parametrize this part of the scalar potential in terms of physical parameters as the Higgs mass, the heavy gauge boson mass, the gauge coupling g and the tadpole instead of the unphysical couplings $\mu^2_{\mathcal{H}}$ and

$$\lambda^{\mathcal{H}}_{ijkl} \equiv \lambda_{i'j'k'l'}\Pi^{\mathcal{H}}_{i'i}\Pi^{\mathcal{H}}_{j'j}\Pi^{\mathcal{H}}_{k'k}\Pi^{\mathcal{H}}_{l'l} \,. \tag{18}$$

In principle, this is analogous to what is usually done for the SM Higgs potential [33]. Here, however, it is more involved due to the appearance of the general invariant tensor $\lambda^{\mathcal{H}}_{ijkl}$. Using essentially eq. (14) and Schur's lemma, it is possible to rewrite the up to quadratic terms of the potential in terms of new parameters M_H^2 (diagonal Higgs mass matrix) and t (tadpole). For the trilinear and quartic terms this does not seem to be possible at the level of the Lagrangian. Thus, for the moment, we leave those terms expressed by the old parameters $\lambda^{\mathcal{H}}_{ijkl}$. They have to be eliminated in favor of M_H^2, g and M_{X}^2 at diagram level to make our choice of parameters consistent.

[3] To see that the number of Goldstone bosons is equal to the number of broken generators it is also possible to define the Goldstone field as $G^A := g\left(\frac{i}{M_{\mathrm{X}}}\right)_{AB} v_i \tilde{T}^B_{ij}\Phi'_j$

Now, including also the scalars on the subspace defined by $\mathbb{1} - \Pi^{\mathcal{H}}$, which is straightforward, the scalar potential can be parametrized as follows[4]:

$$\begin{aligned}
V(\Phi) &= t\ v_i H_i + \frac{1}{2}(M_H^2)_{ij} H_i H_j + \frac{1}{2} t\ H_i H_i + \frac{1}{2} t\ G_i G_i + \frac{1}{2}(M_S^2)_{ij} S_i S_j \\
&+ \frac{1}{2} v_i \lambda^{\mathcal{H}}_{ijkl} H_j G_k G_l + \frac{1}{2} v_i \lambda^{\mathcal{H}}_{ijkl} H_j H_k G_l + \frac{1}{6} v_i \lambda^{\mathcal{H}}_{ijkl} H_j H_k H_l \\
&+ \frac{1}{2} v_i \lambda_{ijkl} H_j H_k S_l + v_i \lambda_{ijkl} H_j G_k S_l + \frac{1}{2} v_i \lambda_{ijkl} G_j G_k S_l \\
&+ \frac{1}{2} v_i \lambda_{ijkl} H_j S_k S_l + \frac{1}{2} v_i \lambda_{ijkl} G_j S_k S_l + \frac{1}{6} v_i \lambda_{ijkl} S_j S_k S_l \\
&+ \frac{1}{24} \lambda^{\mathcal{H}}_{ijkl} G_i G_j G_k G_l + \frac{1}{6} \lambda^{\mathcal{H}}_{ijkl} G_i G_j G_k H_l + \frac{1}{6} \lambda^{\mathcal{H}}_{ijkl} G_i H_j H_k H_l \\
&+ \frac{1}{4} \lambda^{\mathcal{H}}_{ijkl} G_i G_j H_k H_l + \frac{1}{24} \lambda^{\mathcal{H}}_{ijkl} H_i H_j H_k H_l \\
&+ \frac{1}{6} \lambda_{ijkl} H_i H_j H_k S_l + \frac{1}{2} \lambda_{ijkl} H_i H_j G_k S_l + \frac{1}{2} \lambda_{ijkl} H_i G_j G_k S_l \\
&+ \frac{1}{6} \lambda_{ijkl} G_i G_j G_k S_l + \frac{1}{4} \lambda_{ijkl} H_i H_j S_k S_l + \frac{1}{2} \lambda_{ijkl} H_i G_j S_k S_l \\
&+ \frac{1}{4} \lambda_{ijkl} G_i G_j S_k S_l + \frac{1}{6} \lambda_{ijkl} H_i S_j S_k S_l + \frac{1}{6} \lambda_{ijkl} G_i S_j S_k S_l \\
&+ \frac{1}{24} \lambda_{ijkl} S_i S_j S_k S_l
\end{aligned} \tag{19}$$

where

$$t = -\mu_{\mathcal{H}}^2 + \frac{1}{6v^2}\ \lambda^{\mathcal{H}}_{ijkl} v_i v_j v_k v_l\,, \tag{20}$$

$$(M_H^2)_{ij} = \frac{1}{2} \lambda^{\mathcal{H}}_{ijkl} v_k v_l - \frac{1}{6v^2}\ \lambda^{\mathcal{H}}_{klmn} v_k v_l v_m v_n\ \Pi^{\mathcal{H}}_{ij}\,, \tag{21}$$

$$(M_{\mathcal{S}}^2)_{ij} = \frac{1}{2} \lambda^{\mathcal{S}}_{ijkl} v_k v_l - (\mu^2(\mathbb{1} - \Pi^{\mathcal{H}}))_{ij}\,. \tag{22}$$

Due to gauge invariance under the SM group (eqs. (4) and (14)), both mass matrices are diagonal and proportional to the unit matrix on each SM-irreducible subspace. M_H^2 has only non-zero entries on the subspace defined by $P^{\tilde{\mathcal{H}}}$ and M_S^2 only on the subspace defined by $(\mathbb{1} - \Pi^{\mathcal{H}})$ (we have defined $\lambda^{\mathcal{S}}_{ijkl} v_k v_l \equiv (\mathbb{1} - \Pi^{\mathcal{H}})_{ii'} (\mathbb{1} - \Pi^{\mathcal{H}})_{jj'} \lambda^{\mathcal{S}}_{i'j'kl} v_k v_l$).

It is important to see that M_S^2 must have only positive or zero entries. If there are negative entries, some of the S_i would develop a vev and our formalism would not apply. Strictly speaking, we have $(M_S^2)_{ij} < 0$ for the SM Higgs doublet that is contained in S_i, which would exclude it from our treatment. But since in that case the scales involved have the strong hierarchy $\mathcal{O}(M_W) \ll \mathcal{O}(M_{GUT})$, we can safely set the entry to zero here. To do this, some of the $\lambda^{\mathcal{S}}_{ijkl} v_k v_l$ must be fine-tuned against the corresponding $(\mu^2(\mathbb{1} - \Pi^{\mathcal{H}}))_{ij}$ in eq. (22) which is known as the doublet triplet splitting problem, inherent to generic GUTs.

[4]For more details of how this reparametrization is done, please refer to appendix B.

Note that the classical minimum of the GUT-breaking Higgs potential is defined by the equation $t = 0$. However, if we compute higher order corrections, the parameter $t \equiv 0-\delta t$, where $\delta t = \mathcal{O}(\alpha)$ is a counterterm, has to be adjusted in such a way that the renormalized Higgs one point function is zero at all orders of perturbation theory.

The last term in eq. (1) to be specified is $\mathcal{L}_{\mathrm{Y}}$. The most general Yukawa interaction of the chiral Dirac fermion multiplet Ψ with the real scalar multiplet Φ can be written as follows:

$$\mathcal{L}_{\mathrm{Y}} = -\frac{1}{2}\left(Y^k_{ij}\Psi^T_i C\Psi_j\Phi_k + Y^{k\star}_{ij}\Psi^{cT}_i C\Psi^c_j\Phi_k\right) . \tag{23}$$

Here Y^k is a complex, symmetric matrix and $C \equiv i\gamma_2\gamma_0$ denotes the Dirac charge conjugation matrix. $\Psi^c \equiv C\overline{\Psi}^T$ is the charge conjugated Dirac spinor (the ^{T} refers only to Lorentz space). We take $\Psi = \frac{1}{2}(1-\gamma_5)\Psi$ to be left-handed so that Ψ^c will be right-handed. Furthermore, due to gauge invariance Y^k_{ij} satisfies the following relation:

$$0 = Y^k_{mj}T^\alpha_{mi} + Y^k_{im}T^\alpha_{mj} + Y^m_{ij}\tilde{T}^\alpha_{mk} . \tag{24}$$

In the following we will need to distinguish between fields with the mass of $\mathcal{O}(M_{\mathrm{G}})$ and massless fields. We will follow the convention of appendix A and use the projectors P^x_i for heavy fields and p^x_i for the light fields.

2.2 Renormalization

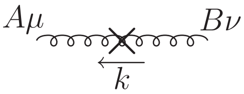
In order to do a two-loop calculation of the matching corrections, a one-loop renormalization program has to be carried out for the theory. The counterterms are adjusted in such a way that all the one-loop Green's functions of the theory are finite. For convenience we use the on-shell scheme for the mass parameters of the theory and $\overline{\mathrm{MS}}$ for the gauge couplings, the gauge parameters and the fields. The renormalized Lagrangian is obtained from eq. (1) by the following replacements:

$$\begin{aligned}
A^{a_i}_\mu &\to \sqrt{Z_{3i}}\ A^{a_i}_\mu , & A^{A_i}_\mu &\to \sqrt{Z^X_{3i}}\ A^{A_i}_\mu , \\
p^{\mathcal{F}}_i\Psi &\to \sqrt{Z_{2i}}\ p^{\mathcal{F}}_i\Psi , & P^{\mathcal{F}}_i\Psi &\to \sqrt{Z^h_{2i}}\ P^{\mathcal{F}}_i\Psi , \\
c^{a_i} &\to \sqrt{\tilde{Z}_{3i}}\ c^{a_i} , & c^{A_i} &\to \sqrt{\tilde{Z}^X_{3i}}\ c^{A_i} , \\
P^{\tilde{\mathcal{H}}}_i H &\to \sqrt{Z_{Hi}}\ P^{\tilde{\mathcal{H}}}_i H , & P^{\mathcal{G}}_i G &\to \sqrt{Z_{Gi}}\ P^{\mathcal{G}}_i G , \\
p^{\mathcal{S}}_i S &\to \sqrt{Z_{Si}}\ p^{\mathcal{S}}_i S , & P^{\mathcal{S}}_i S &\to \sqrt{Z^h_{Si}}\ P^{\mathcal{S}}_i S , \\
M^2_{\mathrm{X}_i} &\to Z^2_{M_{\mathrm{X}i}}\ M^2_{\mathrm{X}_i} , & M^2_{H_i} &\to Z^2_{M_{Hi}}\ M^2_{H_i} , \\
M_{F_i} &\to Z_{M_{Fi}}\ M_{F_i} , & & \\
\xi_{1i} &\to Z_{\xi_{1i}}\ \xi , & \xi_{2i} &\to Z_{\xi_{2i}}\ \xi , \\
\eta_i &\to Z_{3i}\ \eta , & g &\to Z_g\ g .
\end{aligned} \tag{25}$$

Again, we have used the sub-index i to take care of the fact that there might be several SM-irreducible representations for a field that all renormalize differently. No summation is performed over that index. P_i^x and p_i^x are projectors on the various SM-irreducible subspaces of heavy and light fields, respectively. M_F is the fermion mass matrix that can arise from the Yukawa interactions eq. (23). Presently, heavy fermions are not included in the calculation, so the corresponding renormalization constants are only defined for future convenience.

In the following we list the counterterm Feynman rules that are important for our calculation. They are obtained by inserting the renormalization prescriptions from eq. (25) into eq. (1) and considering the up to quadratic terms. For each counterterm we give an expression that is valid to arbitrary loop order in the first line and in the second line a more convenient expression that is valid only for one-loop renormalization. We use the notation $Z_i \equiv 1 - \delta Z_i$ and $t \equiv 0 - \delta t$ where δZ_i and δt are of order α. All the parameters that appear in the equations are renormalized ones.

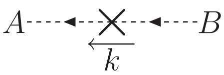
Heavy gauge boson:



$$= i\delta_{AB}\Big[(Z_3^X - \frac{Z_3^X}{\xi Z_{\xi_1}} + \frac{1}{\xi} - 1)k_\mu k_\nu - (Z_3^X - 1)k^2 g_{\mu\nu} + (Z_3^X Z_{M_X}^2 - 1)M_X^2 g_{\mu\nu}\Big]$$

$$= i\delta_{AB}\Big[-\delta Z_{\xi_1} k_\mu k_\nu - (M_X^2 - k^2)\delta Z_3^X g_{\mu\nu} - 2\delta Z_{M_X} M_X^2 g_{\mu\nu}\Big]$$

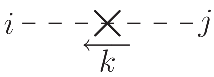
Heavy ghost:



$$= i\delta_{AB}\Big[(\tilde{Z}_3^X - 1)k^2 - (\tilde{Z}_3^X \sqrt{Z_{\xi_1}}\sqrt{Z_{\xi_2}} Z_{M_X}^2 - 1)\xi M_X^2\Big]$$

$$= i\delta_{AB}\Big[\delta\tilde{Z}_3^X(\xi M_X^2 - k^2) + (\frac{1}{2}\delta Z_{\xi_1} + \frac{1}{2}\delta Z_{\xi_2} + 2\delta Z_{M_X})\xi M_X^2\Big]$$

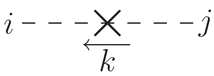
Goldstone boson:



$$= iP_{ij}^{\mathcal{G}}\Big[(Z_G - 1)k^2 - (Z_G Z_{\xi_2} Z_{M_X}^2 - 1)\xi M_X^2 - t\Big]$$

$$= iP_{ij}^{\mathcal{G}}\Big[\delta Z_G(\xi M_X^2 - k^2) + (\delta Z_{\xi_2} + 2\delta Z_{M_X})\xi M_X^2 + \delta t\Big]$$

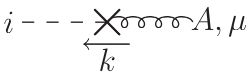
Physical Higgs boson:



$$= i\delta_{ij}\Big[(Z_H - 1)k^2 - (Z_H Z_{M_H}^2 - 1)M_H^2 - t\Big]$$

$$= i\delta_{ij}\Big[\delta Z_H(M_H^2 - k^2) + 2\delta Z_{M_H} M_H^2 + \delta t\Big]$$

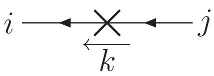
Mixed counterterm:



$$= \quad ig\left[\sqrt{\frac{\xi_2}{\xi_1}}-1\right]v_k\tilde{T}^A_{ki}k_\mu$$

$$= \quad ig\,\frac{1}{2}(\delta Z_{\xi_1}-\delta Z_{\xi_2})\;v_k\tilde{T}^A_{ki}k_\mu$$

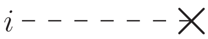
Heavy fermion:



$$= \quad i\delta_{ij}\left[(Z_2^h-1)\not{k}-(Z_2^h Z_{M_F}-1)M_F\right]P_L$$

$$= \quad i\delta_{ij}\left[\delta Z_2^h(M_F-\not{k})+\delta Z_{M_F}M_F\right]P_L$$

Higgs tadpole:



$$= \quad iv_i\delta t$$

In order to avoid clutter with the notation, we have omitted the sub-index i here. From the context it is always unambiguous that the SM-irreducible representation of the field under consideration is meant. For our calculation we need the renormalization constants of all mass and gauge parameters at one-loop. As can be seen from the above Feynman rules, the set of equations that is used to determine δZ_{ξ_1}, δZ_{ξ_2} and δt is overconstrained (cf. also refs. [34, 35]). This provides a useful check for our calculation: we computed $\delta Z_{\xi_1}-\delta Z_{\xi_2}$ from the pole of a combination of the heavy gauge boson and heavy ghost propagator as well as from the mixed Goldstone boson heavy gauge boson propagator. Both calculations lead to the same result. In the same way δt was computed from the pole of the physical Higgs tadpole as well as from the Goldstone propagator yielding the same result. Furthermore, the $\overline{\text{MS}}$ renormalization constant for the gauge coupling Z_g will be needed to two-loop order (cf. subsection 2.3), including corrections from the Yukawa couplings. We did the calculation in our framework and found agreement with the known result in the literature [21, 36, 37].

2.3 Decoupling of heavy particles

When studying gauge coupling unification, it is most convenient to use RGEs defined in mass-independent renormalization schemes, such as $\overline{\text{MS}}$ or $\overline{\text{DR}}$. In these "unphysical" schemes the computation of β functions is simplified significantly, which makes them most suitable for this application. However, these schemes have the well known drawback that the decoupling theorem [38] does not hold in its naive form. As a consequence observables of low energy processes will depend logarithmically[5] on all the heavy particle masses of the GUT. This is unacceptable, since it would spoil perturbation theory by the presence of large logarithms $\ln(M_\text{G}/\mu)$, where μ is the typical energy scale of the process. The way out of this dilemma is to use an effective theory [24, 39], where the heavy particles are

[5] At the two-loop order also more complicated functions of the heavy masses can appear.

integrated out at the GUT scale. This means that the dynamical degrees of freedom of the heavy particles are removed, which manifestly leads to power-suppressed contributions of $\mathcal{O}(1/M_\mathrm{G})$ in the effective Lagrangian. Moreover, the effects of the heavy particles are encoded in a multiplicative redefinition of all the masses and couplings of the theory. For the case of the gauge coupling this so-called decoupling relation reads:

$$\begin{aligned}\alpha_i(\mu_\mathrm{GUT}) &= \zeta_{\alpha_i}(\mu_\mathrm{GUT}, \alpha(\mu_\mathrm{GUT}), M_\mathrm{h})\ \alpha(\mu_\mathrm{GUT})\,,\\ \alpha &\equiv \frac{g^2}{4\pi}\,, \qquad \alpha_i \equiv \frac{g_i^2}{4\pi}\,, \quad i = 1,2,3\,.\end{aligned} \tag{26}$$

Here α_i and α stands for the $\overline{\mathrm{MS}}$ gauge coupling[6] in the effective theory (the SM or the MSSM) and full theory (GUT), respectively. μ_GUT is the unphysical scale, at which the decoupling is performed. At sufficiently high loop order predictions of physical observables must not depend on μ_GUT anymore. The remaining dependence on this scale gives us an estimation of the theory uncertainty of the prediction. ζ_{α_i} is the so-called matching coefficient that depends on all the mass parameters of the particles that have been integrated out. They are abbreviated by M_h in eq. (26). The construction of the effective Lagrangian is described in detail e.g. in refs. [40,41] for the case of QCD. Here we only list the formulas that are relevant for the computation of ζ_{α_i}. For our calculation we find it most convenient to use the three-point Green's function with light ghosts and light gauge bosons as external particles. Applying Slavnov-Taylor identities to this vertex, the formula for the $\overline{\mathrm{MS}}$ matching coefficient reads:

$$\zeta_{\alpha_i} = \left(\frac{Z_g}{Z_{g_i}}\frac{\tilde{\zeta}_{1i}^0}{\tilde{\zeta}_{3i}^0\sqrt{\zeta_{3i}^0}}\right)^2\,, \quad i = 1,2,3\,. \tag{27}$$

The $\overline{\mathrm{MS}}$ renormalization constants for the gauge coupling in the full and effective theory are denoted by Z_g and Z_{g_i}, respectively. The bare matching coefficients for the light ghost-gauge-boson vertex, the light ghost field and the light gauge field respectively are given by:

$$\begin{aligned}\tilde{\zeta}_{1i}^0 &= 1 + \Gamma^{0,h}_{Ac^\dagger c,i}(0,0)\,,\\ \tilde{\zeta}_{3i}^0 &= 1 + \Pi^{0,h}_{c,i}(0)\,,\\ \zeta_{3i}^0 &= 1 + \Pi^{0,h}_{A,i}(0)\end{aligned} \tag{28}$$

i.e. they are computed from the "hard part"[7] of the one particle irreducible Green's functions with zero external momentum. All the masses of SM particles are set to zero and therefore only masses of $\mathcal{O}(M_\mathrm{G})$ appear in the diagrams. The external particles for these Green's functions are: two light ghosts with one light gauge boson, two light ghosts and two light gauge bosons, respectively. In all cases the relevant group structure has to

[6]For simplicity we will only speak about $\overline{\mathrm{MS}}$ parameters from now on. If a SUSY GUT is considered, all $\overline{\mathrm{MS}}$ parameters will be replaced by $\overline{\mathrm{DR}}$ parameters.

[7]This denotes all the diagrams that contain at least one heavy particle.

be projected out and in the case of the light gauge boson propagator only the transverse part is needed. The index i takes care of the fact that the SM gauge group is not simple and labels whether an external gauge bosons and ghosts belonging to $U(1)$, $SU(2)$ or $SU(3)$ have to be taken.

Before we come to the technical details of the calculation, it is in order to describe all the assumptions that have been made about the underlying GUT model:

- There is no trilinear scalar coupling in $V(\Phi)$.
- There is only one vev of $\mathcal{O}(M_{\mathrm{G}})$ in the theory.
- There are no heavy fermions in the theory.
- The heavy gauge bosons decompose in SM-irreducible representations with a common mass.
- The GUT-breaking Higgs decomposes into three SM-irreducible representations (+ Goldstone bosons) at most. They can all have different masses.
- The other scalars in the theory decompose in SM-irreducible representations that have a common mass (+ light scalars).
- The light particles in the theory can decompose in arbitrarily many SM-irreducible representations.

As can be seen, the main limitation comes from the number of heavy degrees of freedom in the theory. The above constraints are designed such that the resulting formula for ζ_{α_i} is applicable to the simplest GUT, the Georgi-Glashow model, yet keeping the calculation as simple as possible. The computational framework for our calculation is set up in such a way that it can be generalized to more heavy degrees of freedom, in order to apply it to SUSY GUTs in the future.

Given the large number of Feynman diagrams, an automated computation is indispensable. The diagrams were generated with `QGRAF` [42] and further processed with `q2e` and `exp` [43,44]. In the next step we used a `FORM` [45] implementation of the two-loop topologies of ref. [46] by the authors of ref. [13] and also the `FORM` packages `MINCER` [47] and `MATAD` [48]. Let us emphasize that no assumptions about the mass hierarchies of the heavy particles have been made. Therefore, the result is valid for arbitrary numerical values of the mass parameters as long as their mass splitting is not to large which would lead to power enhanced contributions and spoil perturbation theory. The reduction of the group theory factors, which was the most time consuming part, has been automated and implemented in `FORM`. The relevant group theory and notational conventions are collected in appendix A of this paper.

Although the present calculation is just general enough to be applied to the simplest GUT, the number of Feynman diagrams for the two-loop Green's functions described above already is considerable. For the light gauge boson two-point function it amounts to 6278, whereas for the ghost-gauge-boson vertex and the light ghost two-point function we have 4109 and 374 diagrams, respectively. Sample diagrams[8] for all three processes are depicted in fig. 1.

In order to subtract the subdivergencies that occur in these two-loop Green's functions, the mass and gauge parameters as well as the gauge coupling that appear in the corresponding one-loop Green's functions have to be renormalized as described in subsection 2.2. We have performed our calculation for arbitrary gauge parameters and verified that they cancel out in the final result, which is a powerful and highly non-trivial check of the calculation. The result for ζ_{α_i} is available in general form, i.e. with group theory factors and couplings not specified to a particular Lie group or model. In the next section we will assign definite values to these quantities in order to show the application of the result exemplarily. Note also that we always need three different sets of group theory factors for $i = 1, 2, 3$, respectively.

3 Numerical study for the Georgi-Glashow model

In the following we perform an academic study of our results applied to the simplest possible GUT, the Georgi-Glashow model [26]. Although this theory is ruled out experimentally [2], we use it as a toy model to demonstrate some simple numerics. It is based on the gauge group $SU(5)$ and the SM fermions sit in the representations

$$\begin{aligned} \overline{\mathbf{5}} \quad &\rightarrow \quad (\overline{\mathbf{3}}, \mathbf{1}, \tfrac{1}{3}) \oplus (\mathbf{1}, \overline{\mathbf{2}}, -\tfrac{1}{2})\,, &(29)\\ \mathbf{10} = [\mathbf{5} \times \mathbf{5}]_a \quad &\rightarrow \quad (\overline{\mathbf{3}}, \mathbf{1}, -\tfrac{2}{3}) \oplus (\mathbf{3}, \mathbf{2}, \tfrac{1}{6})_a \oplus (\mathbf{1}, \mathbf{1}, 1)\,. &(30)\end{aligned}$$

where we have displayed their decomposition into SM multiplets.[9] The gauge bosons as well as the $SU(5)$-breaking Higgs bosons Σ live in the real $\mathbf{24}$ representation:

$$\mathbf{24} \rightarrow (\mathbf{8}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{3}, 0) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{3}, \overline{\mathbf{2}}, -\tfrac{5}{6}) \oplus (\overline{\mathbf{3}}, \mathbf{2}, \tfrac{5}{6})\,. \tag{31}$$

In the case of the gauge bosons the first three parts of the decomposition constitute the light SM gauge bosons and the last two parts the heavy gauge bosons with a common mass M_{X}. For the GUT-breaking Higgs bosons the first three parts represent the three physical Higgs bosons with masses M_Σ, $2\,M_\Sigma$ and M_{24}, respectively. The last two multiplets give the Goldstone bosons with the unphysical mass $\sqrt{\xi_2}\,M_{\mathrm{X}}$ (cf. eq. (12)). Note that we write the field Σ as a 24-dimensional vector multiplet in order to be consistent with our notation in section 2, and not as a hermitian 5×5 matrix as usually done. Therefore, the vev of Σ simply is $\langle \Sigma \rangle = v$, where v is a 24-dimensional vector with a single non-zero entry in

[8]The figure has been created with help of the LaTeX package AXODRAW [49]

[9]The first and second number label the $SU(3)$ and $SU(2)$ representations, respectively and the third number is the hypercharge of the multiplet. The index a indicates that the antisymmetric part of the representation has to be taken.

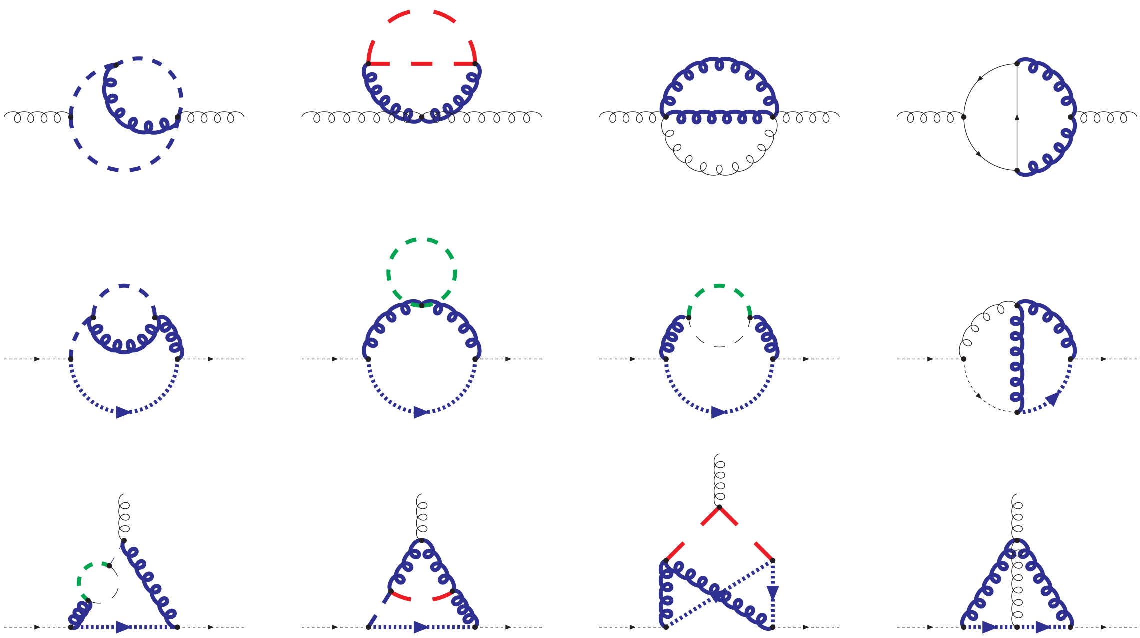


Figure 1: Sample two-loop diagrams that appear in the calculation of ζ_{α_i}. The first line shows the process $A^{a_i}_\mu \to A^{b_i}_\nu$ contributing to $\Pi^{0,h}_{A,i}(0)$. The second and third line depict $c^{a_i} \to c^{b_i}$ and $c^{a_i} \to c^{b_i} + A^{c_i}_\mu$ contributing to $\Pi^{0,h}_{c,i}(0)$ and $\Gamma^{0,h}_{Ac^\dagger c,i}(0,0)$, respectively. Colored (bold) lines represent fields with mass of $\mathcal{O}(M_{\mathrm{G}})$ and black (thin) lines massless fields. Furthermore, curly lines denote gauge bosons, dotted lines ghosts, dashed lines scalar fields and solid lines fermions. Goldstone bosons are marked green (light gray, short-dashed), physical Higgs bosons red (gray, long-dashed) and other heavy scalars blue (dark, short-dashed). Note also that two identical lines in one diagram need not have the same mass because of the non-degenerate mass spectrum.

the 24th component. Additionally we have a complex scalar in the **5** representation that decomposes according to eq. (29) and contains the SM Higgs doublet as well as a heavy colored triplet with the mass $M_{\mathrm{H_c}}$.

For the convenience of the reader it might be helpful to explicitly give the parametrization of the quartic scalar coupling λ_{ijkl} from eq. (13) for the Georgi-Glashow model. It splits up into three parts. The first part is a quartic coupling of the **24** Higgs, the second one a quartic coupling of the **5** Higgs and the last one is a mixed $\mathbf{5}-\mathbf{24}$ coupling [32, 50]:

$$\begin{aligned}
\lambda^{24}_{\alpha\beta\gamma\delta} &= A\ \mathrm{sTr}(T^\alpha T^\beta T^\gamma T^\delta) + \frac{1}{3}\ B\ (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\,, \\
\lambda^{5}_{ijkl} &= \frac{1}{3}\ b\ (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,, \quad i,j,k,l = 1,...,10\,, \\
\lambda^{5-24}_{\alpha\beta ij} &= c\ (\tau^\alpha\tau^\beta + \tau^\beta\tau^\alpha)_{ij}\,, \qquad \alpha,\beta,\gamma,\delta = 1,...,24\,. \qquad (32)
\end{aligned}$$

We have used the symmetrized trace

$$\mathrm{sTr}(T^{\alpha_1}...T^{\alpha_n}) \equiv \frac{1}{n!}\sum_{\pi} \mathrm{Tr}(T^{\alpha_{\pi(1)}}...T^{\alpha_{\pi(n)}}) \tag{33}$$

where the sum is over all the permutations of the indices. Furthermore, it is noticeable that the indices i, j, k, l run from 1 to 10 although they belong to the fundamental **5** representation. This is because the corresponding scalar is complex and we have written it as twice as many real scalars that are transformed by the 10×10 generator matrices τ:

$$\tau^\alpha = \begin{pmatrix} i\ \mathrm{Im}(T^\alpha) & i\ \mathrm{Re}(T^\alpha) \\ -i\ \mathrm{Re}(T^\alpha) & i\ \mathrm{Im}(T^\alpha) \end{pmatrix}, \tag{34}$$

where in this section T^α is the 5×5 generator matrix in the fundamental representation. Inserting eq. (32) into eqs. (21) and (22), we obtain the scalar mass matrices. Additionally we need to impose the tree-level fine-tuning condition $\mu_{\mathbf{5}}^2 = \frac{3}{20}cv^2$, where $\mu_{\mathbf{5}}^2$ is the quardatic term of the **5** Higgs in eq. (13), in order to obtain massless Higgs doublets. Note that in principle one would need to calculate the one-loop fine-tuning condition in order to obtain massless Higgs doublets in a two-loop calculation. However, the light Higgs doublets show up at the first time in the two-loop Green's functions in the matching calculation so that it is sufficient to use the tree-level fine-tuning condition. This gives us the relations between the physical scalar masses and the parameters in eq. (32):

$$M_\Sigma^2 = \tfrac{1}{144}\ A\ v^2\,, \quad M_{24}^2 = \tfrac{1}{3}(\tfrac{7}{120}A + B)\ v^2\,, \quad M_{\mathrm{H_c}}^2 = \tfrac{1}{12}\ c\ v^2\,. \tag{35}$$

The vev is connected to the physical gauge boson mass by

$$M_{\mathrm{X}}^2 = \tfrac{5}{12}\ g^2\ v^2\,. \tag{36}$$

The Yukawa interactions of the Georgi-Glashow model [26, 32] are obtained by inserting the Yukawa matrix

$$Y_{sr}^n = \begin{pmatrix} -Y_{IJ}^U \epsilon_{ijklm}\ T_{ij}^\alpha T_{kl}^\beta S_{mn}^* & 2i\ Y_{IJ}^D\ \mathrm{Im}(T_{kl}^\alpha) S_{ln} \\ 2i\ Y_{IJ}^D\ \mathrm{Im}(T_{kl}^\alpha) S_{ln} & 0 \end{pmatrix}_{sr} \tag{37}$$

into the general Yukawa Lagrangian eq. (23). Here $s = (I, \tilde{s})$ and $r = (J, \tilde{r})$ are multi-indices, where I, J stand for the flavor indices of the $SU(5)$ Yukawa matrices Y^U and Y^D. The indices $\tilde{s}, \tilde{r} = 1, ..., 29$ run over $\{\alpha, j\}$ and $\{\beta, k\}$, respectively. Note that we have written the fermions of the **10** representation as a 24-dimensional vector instead of an antisymmetric 5×5 matrix as usually. The Clebsch-Gordan coefficients for this transformation are given by the following equations:

$$\mathbf{10}^\alpha = \sqrt{2}\ T_{ij}^\alpha\ \mathbf{10}_{ij}\,, \quad \mathbf{10}_{ij} = -\sqrt{2}\ i\ \mathrm{Im}(T_{ij}^\alpha)\ \mathbf{10}^\alpha\,, \tag{38}$$

where $\mathbf{10}_{ij}$ is the usual antisymmetric 5×5 matrix with the normalization as in ref. [26]. Furthermore, $S = \frac{1}{\sqrt{2}}(\mathbb{1},\ i\mathbb{1})$ is a 5×10 matrix and ϵ_{ijklm} is the totally antisymmetric tensor with $\epsilon_{12345} = 1$. As can be seen from eq. (37), the chiral fermion multiplet Ψ from subsection 2.1 is written as a $3\,(24+5) = 87$-dimensional vector for the case of Georgi-Glashow $SU(5)$ model.

Using the definitions from this section, we computed the numerical values of all the group theory factors that appear in our general result. Furthermore, we set $V_{\rm CKM} = 1\!\!1$ and kept only the third generation Yukawa couplings y_t and y_b. We obtained three two-loop formulas for ζ_{α_i} $(i = 1, 2, 3)$ that depend on the parameters

$$\alpha(\mu_{\rm GUT}),\ y_t(\mu_{\rm GUT}),\ y_b(\mu_{\rm GUT}),\ M_{\rm X},\ M_{\rm H_c},\ M_\Sigma,\ M_{24},\ \mu_{\rm GUT}\,. \tag{39}$$

The `Mathematica` package that contains the expressions can be downloaded from

`http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp10/ttp10-46/`

In order to examine the numerical impact of the two-loop matching corrections in this model, we have implemented a RGE analysis in `Mathematica`. Since in this model the gauge couplings do not unify, we just focus on examining the reduction of the decoupling scale dependence, as an illustration of our results. We start with the precise values of the three gauge couplings at the electroweak scale. They are obtained from the effective weak mixing angle in the $\overline{\rm MS}$ scheme [51], the QED coupling constant at zero momentum transfer and its hadronic [52] contribution in order to obtain its counterpart at the Z-boson scale, and the strong coupling constant [53].[10] These quantities need to be transformed to a six-flavor theory, which is described in detail in ref. [14]. Our starting values are then:

$$\begin{aligned}
\alpha_{em}^{(6),\overline{\rm MS}}(M_Z) &= 1/(128.129 \pm 0.021)\,, \\
\sin^2\Theta^{(6),\overline{\rm MS}}(M_Z) &= 0.23138 \pm 0.00014\,, \\
\alpha_s^{(6)}(M_Z) &= 0.1173 \pm 0.0020\,.
\end{aligned} \tag{40}$$

These quantities are related to the three gauge couplings via

$$\begin{aligned}
\alpha_1 &= \frac{5}{3}\frac{\alpha^{(6),\overline{\rm MS}}}{\cos^2\Theta^{(6),\overline{\rm MS}}}\,, \\
\alpha_2 &= \frac{\alpha^{(6),\overline{\rm MS}}}{\sin^2\Theta^{(6),\overline{\rm MS}}}\,, \\
\alpha_3 &= \alpha_s^{(6)}\,.
\end{aligned} \tag{41}$$

which holds for any renormalization scale μ. We also need the W and Z boson pole masses M_W and M_Z, the top quark and tau lepton pole masses M_t and M_τ and the

[10] We adopt the central value from ref. [53], however, use as our default choice for the uncertainty 0.0020 instead of 0.0007.

running bottom quark mass $m_b^{\overline{\rm MS}}$. For the convenience of the reader we also specify their numerical values [51, 54, 55]:

$$\begin{aligned}
M_W &= 80.398 \text{ GeV}\,, \\
M_Z &= 91.1876 \text{ GeV}\,, \\
M_t &= 173.3 \text{ GeV}\,, \\
M_\tau &= 1.77684 \text{ GeV}\,, \\
m_b^{\overline{\rm MS}}(m_b^{\overline{\rm MS}}) &= 4.163 \text{ GeV}\,.
\end{aligned} \tag{42}$$

The corresponding uncertainties are not important for our analysis. These parameters are converted to six-flavor theory using `RunDec` [56] and then used to compute the starting values for y_t, y_b and y_τ at the electroweak scale. The RGE running in the SM was implemented at two loops [36,37,57,58] for the electroweak sector and at three loops [59,60] for QCD. We take into account the tau, bottom and top Yukawa couplings and thus solve the coupled system of six differential equations. Since the quartic SM Higgs coupling b enters the equations of the Yukawa couplings starting from two-loop order only, we neglect its contribution. After taking into account the two-loop decoupling relations, we compute the running from $\mu_{\rm GUT}$ to the Planck scale using three-loop RGEs for the gauge coupling and one-loop RGEs for the Yukawa couplings. The RGEs are obtained by inserting the general expressions for the Yukawa and scalar couplings (eqs. (37) and (32)) as well as the numerical values for the group theory factors into the general formulas of refs. [21,37] (see appendix C for the details).

In figure 2 the dependence on the decoupling scale of $\alpha(10^{18}\text{GeV})$ is shown. Since only for QCD the full three-loop β function could be implemented and there is no unification of gauge couplings anyway, we took $\alpha(\mu_{\rm GUT}) = \zeta_{\alpha_3}^{-1}(\mu_{\rm GUT})\ \alpha_3(\mu_{\rm GUT})$ as a starting value for the gauge coupling above the GUT scale. For illustration we use the following set of mass parameters:

$$\begin{aligned}
M_{\rm X} &= 10^{15} \text{ GeV}\,, \\
M_{\rm H_c} &= 4 \cdot 10^{13} \text{ GeV}\,, \\
M_\Sigma &= 10^{14} \text{ GeV}\,, \\
M_{24} &= 6 \cdot 10^{13} \text{ GeV}
\end{aligned} \tag{43}$$

which are chosen to obey the restriction $M_{\rm X} \gtrsim M_i$ for $i = {\rm H_c}, \Sigma, 24$. Otherwise the scalar self-couplings easily become non-perturbative and blow up the gauge coupling above the GUT scale. The scale dependence is shown for n-loop running and $(n-1)$-loop decoupling with $n = 1, 2, 3$. We observe a dramatic improvement when going from $n = 2$ to $n = 3$. In particular the three-loop corrections can be larger than the experimental error band depending on $\mu_{\rm GUT}$. Note also that for $n = 2$ choosing $\mu_{\rm GUT}$ naively as a mean value of the GUT masses which would be of $\mathcal{O}(10^{14}$ GeV) in our case is not a good choice.

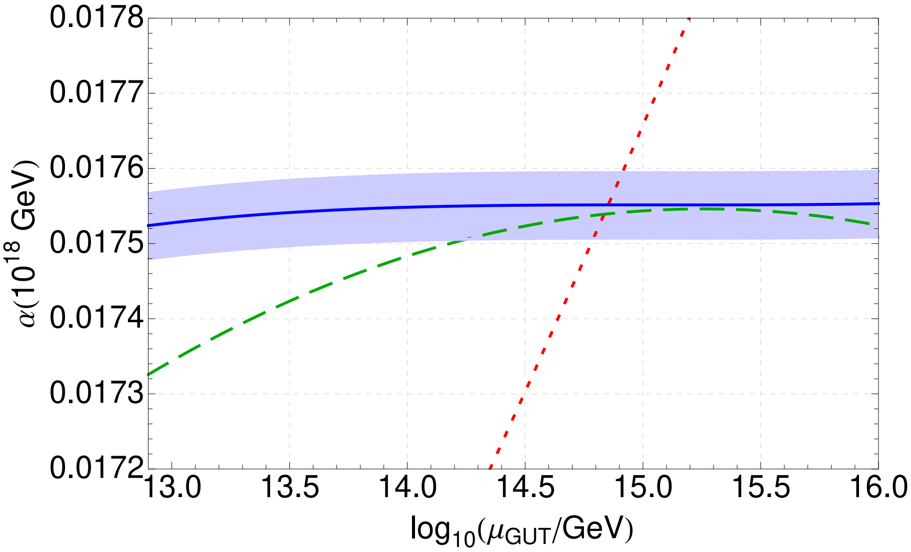


Figure 2: Dependence of $\alpha(10^{18}$ GeV) on the decoupling scale $\mu_{\rm GUT}$. The red (dotted), green (dashed) and blue (solid) lines correspond to the one-, two- and three-loop analysis, respectively. For the three-loop curve also the experimental error band with $\delta\alpha_s = 0.0020$ has been indicated.

The described qualitative behavior does not depend much on our choice of the GUT masses. Though the numerical effect of the two-loop matching is already significant in the Georgi-Glashow model, we emphasize that in certain models that contain large representations, as e.g. the Missing Doublet Model [15, 16], we expect these corrections to be even larger [14]. Our goal for the future, of course, is to generalize the formula for ζ_{α_i} to make it applicable to these models.

To provide a check of the result for the Georgi-Glashow model, we have verified analytically that the matching coefficients $\zeta_{\alpha_i}(\mu_{\rm GUT})$ exhibit the correct $\mu_{\rm GUT}$ dependence. This can be derived from the knowledge of the two-loop beta functions of the SM and the $SU(5)$ model by computing the derivative w.r.t. $t_{\rm GUT} \equiv \ln(\mu_{\rm GUT})$ of eq. (26). Solving the resulting differential equation order by order, we arrive at a general formula[11] for the $\mu_{\rm GUT}$-dependent terms in $\zeta_{\alpha_i}(\mu_{\rm GUT})$:

$$\begin{aligned}\zeta_{\alpha_i}(\mu_{\rm GUT}) &= 1 + \frac{\alpha(\mu_{\rm GUT})}{\pi}\left[\tfrac{1}{2}(\beta_0^i - \beta_0)\, t_{\rm GUT} - C_0(M_{\rm h})\right] \\ &+ \left(\frac{\alpha(\mu_{\rm GUT})}{\pi}\right)^2 \left[\tfrac{1}{4}(\beta_0^i - \beta_0)^2\, t_{\rm GUT}^2 + \left[\tfrac{1}{8}(\beta_1^i - \beta_1) - C_0(M_{\rm h})\,(\beta_0^i - \beta_0)\right] t_{\rm GUT} + C_1(M_{\rm h})\right].\end{aligned} \tag{44}$$

C_0 and C_1 are $\mu_{\rm GUT}$-independent terms that depend only on the heavy GUT masses.

[11] For simplicity we neglect the Yukawa corrections in this formula. However, the generalization is straightforward. Of course, in our analytical check we took care of them too.

The β function coefficients are defined by:

$$\frac{1}{2}\frac{d}{dt}\frac{\alpha}{4\pi} = \sum_{k=0}^{N-1}\left(\frac{\alpha}{4\pi}\right)^{k+2}\beta_k\,, \qquad \alpha = \frac{g^2}{4\pi} \tag{45}$$

and similarly for α_i. We find agreement in the μ_{GUT} dependence of our explicit calculation with the form of eq. (44).

4 Conclusions and Outlook

As experimental accuracy for α_s, α_{em} and $\sin\Theta$ is increasing, also theoretical unification analyses must improve their precision in order to find better exclusion limits for GUTs. Therefore, we have performed a first step towards the calculation of the two-loop matching corrections for the gauge couplings at the GUT scale in a framework that aims at making as few assumptions on the underlying GUT as possible. The assumptions that were made can be found in subsection 2.3. The result is general enough to be applied to the simplest GUT, the Georgi-Glashow $SU(5)$. The numerical impact in this model was found to be larger than the current experimental uncertainty on α_s depending on the choice of the decoupling scale μ_{GUT}. We expect even larger effects in GUT models that contain large representations, as the so-called Missing Doublet Model. Moreover, the two-loop matching coefficients provide a significant stabilization of our predictions w.r.t the variation of the decoupling scale.

Furthermore, we have described in detail the proper treatment of the gauge fixing and the renormalization procedure for this calculation. In this context also the issue of Higgs tadpoles in theories with spontaneous symmetry breaking was discussed in a general manner. The appendix contains an in-depth introduction of the group theoretical framework that we used. There we also give many useful reduction identities that are essential for performing multi-loop calculations in GUTs and can also be applied to SUSY GUTs.

We consider the result computed in this paper only as an intermediate step towards a more general calculation of ζ_{α_i}. Particularly, we are interested in performing a consistent three-loop RGE analysis for SUSY GUTs, where two-loop matching at the GUT scale is needed. In order to apply our result to the simplest SUSY GUT, the minimal SUSY $SU(5)$, several generalizations are needed:

- Add a trilinear term to the scalar potential eq. (13).
- Add two massive Dirac and three massive Majorana fermion SM-irreducible representations as well as all possible kinds of Yukawa interactions for them.
- Increase the number of SM-irreducible representations within $\mathcal{R}^{\mathcal{H}}$ by one in order to cover the case of the non-renormalizable version of minimal SUSY $SU(5)$

- Allow for unitary mixing matrices in the interactions of heavy Dirac fermions with gauge bosons.
- Convert the result to the $\overline{\mathrm{DR}}$ renormalization scheme.

Though there is quite some work to be done, to achieve this, we have already built a solid basis to start with and have overcome many of the main difficulties that were expected to occur.

Acknowledgments

The author is greatly indebted to Luminita Mihaila and Matthias Steinhauser for countless fruitful discussions and many useful comments to the manuscript. Thanks also to Jennifer Girrbach for carefully reading the manuscript and providing useful remarks. This work was supported by the "Studienstiftung des deutschen Volkes", the DFG through SFB/TR 9 "Computational Particle Physics" and the Graduiertenkolleg "Hochenergiephysik und Teilchenastrophysik".

Appendix

A Group theoretical framework

When performing multi-loop calculations for GUTs, one necessarily has to deal with the reduction of the color structure[12] [21, 23, 36, 37, 61, 62]. The aim is to reduce all the color factors of the diagrams to a set of basic invariants. In this way nontrivial cancellations among different diagrams are possible without inserting the actual numerical values for those invariants. In this appendix we develop a notational framework that is appropriate for this task and also give some useful reduction identities that will hopefully prove helpful also for future calculations. For similar reduction algorithms and identities applied to unbroken gauge theories see e.g. refs. [63, 64] and references therein. Here we focus on spontaneously broken gauge theories.

We start with a simple GUT group $\mathbf{G}$ that is broken to a (in general not simple) gauge group $\prod_k \mathbf{G}_k$. In the following the $\mathbf{G}_k$ are called group factors. The generators of $\mathbf{G}$ in a

[12]For convenience we will use the terms "color structure", "color factor" etc. in this appendix following QCD terminology. However, obviously our formalism is not restricted to $SU(3)$, but is meant to be applied to GUT groups.

general reducible representation $\mathcal{R}$ are denoted by T^α in this appendix[13]. They fulfill the commutation relation

$$[T^\alpha, T^\beta] = i f^{\alpha\beta\gamma} T^\gamma \,, \tag{46}$$

where $f^{\alpha\beta\gamma}$ are the structure constants of $\mathbf{G}$. In order to distinguish between broken and unbroken generators, we use the notation of subsection 2.1:

$$\{\alpha\} = \sum_i \{A_i\} + \sum_i \{a_i\} = \{A\} + \{a\} \tag{47}$$

where A_i label the broken generators of $\mathbf{G}$ belonging to the $\prod_k \mathbf{G}_k$-irreducible subspace labeled by i. If there is only one $\prod_k \mathbf{G}_k$-irreducible subspace in the adjoint representation of $\mathbf{G}$, we can omit the sub-index i in A_i. In contrast, a_i label the unbroken generators belonging to the subgroup $\mathbf{G}_i$. As the $\mathbf{G}_i$ are regular subgroups of $\mathbf{G}$, also the following cummutation relations hold:

$$[T^{a_i}, T^{b_j}] = i f^{a_i b_j c_k} T^{c_k} \tag{48}$$

where $f^{a_i b_j c_k} = 0$ unless $i = j = k$. Furthermore, because the subgroup $\mathbf{G}_i$ is closed, we have $f^{a_i b_j A_k} = 0$ for all i, j, k. Otherwise the commutator $[T^{a_i}, T^{b_j}]$ would contain terms proportional to the broken generators T^{A_k}. Note that if not explicitly stated, the sub-indices $i, j...$ of the indices $a_i, b_j..., A_i, B_j...$ are not summed. A repeated index $a, b...$ or $A, B...$ without sub-index means that the sub-index has been summed over. The representation $\mathcal{R}$ that T^α is defined on generally is reducible under $\mathbf{G}$. It decomposes into $\mathbf{G}$-irreducible representations where the gauge bosons, fermions and scalars of the theory live in:

$$\mathcal{R} \to \bigoplus_x \mathcal{R}^x = \mathcal{R}^{\mathcal{A}} \oplus \mathcal{R}^{\mathcal{H}} \oplus \mathcal{R}^{\mathcal{S}_\mathrm{I}} \oplus \mathcal{R}^{\mathcal{S}_\mathrm{II}} \oplus ... \oplus \mathcal{R}^{\mathcal{F}_\mathrm{I}} \oplus \mathcal{R}^{\mathcal{F}_\mathrm{II}} \oplus ... \,. \tag{49}$$

$\mathcal{R}^{\mathcal{A}}$ stands for the adjoint representation and $\mathcal{R}^{\mathcal{H}}$ for the representation of the GUT-breaking scalar. The other symbols represent the irreducible representations for the scalars and fermions, respectively, numbered by roman numerals for convenience. We define projectors on these subspaces denoted by Π^x with $x = \mathcal{A}, \mathcal{H}, \mathcal{S}_\mathrm{I}, \mathcal{S}_\mathrm{II}..., \mathcal{F}_\mathrm{I}, \mathcal{F}_\mathrm{II}....$ Clearly, $[\Pi^x, T^\alpha] = 0$ holds for all x and α. Each of those representations decomposes further under $\prod_k \mathbf{G}_k$:

$$\mathcal{R}^x \to \bigoplus_n \mathcal{R}^x_n = \mathcal{R}^x_1 \oplus \mathcal{R}^x_2 \oplus \mathcal{R}^x_3 \oplus ... \tag{50}$$

with projectors P^x_n and p^x_n. We use a capital P to denote that the respective $\prod_k \mathbf{G}_k$-irreducible representation contains fields with a mass of $\mathcal{O}(M_\mathrm{G})$ and a lowercase p for projectors on a subspace with massless fields. Because the Lagrangian is still invariant under $\prod_k \mathbf{G}_k$ after symmetry breaking, each of those $\prod_k \mathbf{G}_k$-irreducible subspaces can be assigned to a definite mass of the respective field. The indices $n = 1, 2, 3...$ label the

[13] In the main text we employed the symbol T^α only for the fermion representation. In this appendix, however, we will use the symbol for a generic generator of the gauge group.

$\prod_k \mathbf{G}_k$-irreducible representation in $\mathcal{R}^x$. The projectors fulfill $[P_n^x, T^{a_i}] = 0 = [p_n^x, T^{a_i}]$ for all x, a, n and i. Furthermore, we have

$$\sum_n \varrho_n^x \equiv \sum_n p_n^x + \sum_n P_n^x \equiv p^x + P^x = \Pi^x \tag{51}$$

where ϱ can be P or p depending on n. Armed with these definitions, we can define some basic Dynkin indices $I_2(...)$ and Casimir invariants $C_2(...)$ which have real numerical values for a given group.

$$\begin{aligned}
\mathrm{Tr}(\Pi^x T^\alpha T^\beta) &= I_2(\Pi^x)\ \delta^{\alpha\beta}\,, \\
\mathrm{Tr}(\varrho_n^x T^{a_i} T^{b_i}) &= I_2(\varrho_n^x)^i\ \delta^{a_i b_i}\,, \\
\mathrm{Tr}(\varrho_n^x T^{A_i} \varrho_m^x T^{B_i}) &= I_2(\varrho_n^x, \varrho_m^x)^i\ \delta^{A_i B_i}\,, \\
\Pi^x T^\alpha T^\alpha &= C_2(\Pi^x)\ \Pi^x\,, \\
\varrho_n^x T^{a_i} T^{a_i} &= C_2(\varrho_n^x)^i\ \varrho_n^x\,, \\
\varrho_n^x T^{A_i} \varrho_m^x T^{A_i} &= C_2(\varrho_n^x, \varrho_m^x)^i\ \varrho_n^x\,.
\end{aligned} \tag{52}$$

Again ϱ can stand for either P or p. Furthermore, we can define the dimensions of the various irreducible representations by

$$\begin{aligned}
\Delta^x &\equiv \mathrm{Tr}(\Pi^x)\,, \\
D_n^x &\equiv \mathrm{Tr}(P_n^x)\,, \\
d_n^x &\equiv \mathrm{Tr}(p_n^x)\,,
\end{aligned} \tag{53}$$

and specifically for the adjoint representation:

$$\begin{aligned}
\Delta^{\mathcal{A}} &\equiv \delta^{\alpha\alpha}\,, \\
D_n^{\mathcal{A}} &\equiv \delta^{A_n A_n}\,, \\
d_n^{\mathcal{A}} &\equiv \delta^{a_n a_n}\,.
\end{aligned} \tag{54}$$

which gives us the relations

$$\begin{aligned}
I_2(\Pi^x)\ \Delta^{\mathcal{A}} &= C_2(\Pi^x)\ \Delta^x\,, \\
I_2(P_n^x)^i\ d_i^{\mathcal{A}} &= C_2(P_n^x)^i\ D_n^x\,, \\
I_2(p_n^x)^i\ d_i^{\mathcal{A}} &= C_2(p_n^x)^i\ d_n^x\,, \\
I_2(P_n^x, \varrho_m^x)^i\ D_i^{\mathcal{A}} &= C_2(P_n^x, \varrho_m^x)^i\ D_n^x\,, \\
I_2(p_n^x, \varrho_m^x)^i\ D_i^{\mathcal{A}} &= C_2(p_n^x, \varrho_m^x)^i\ d_n^x\,.
\end{aligned} \tag{55}$$

Let us emphasize again that no summation over the sub-indices i that label the $\prod_k \mathbf{G}_k$-irreducible representation is implied here.

However, we introduce the convention that omitting this sub-index implies summation:

$$\begin{aligned}
C_2(...) &\equiv \sum_i C_2(...)^i \,, \\
C_2(\varrho^x_n, \varrho^x)^i &\equiv \sum_m C_2(\varrho^x_n, \varrho^x_m)^i \,, \\
I_2(...\varrho^x...)^i &\equiv \sum_m I_2(...\varrho^x_m...)^i \,. \qquad (56)
\end{aligned}$$

Note also that the invariants $I_2(\Pi^x)$ and $C_2(\Pi^x)$ have only been defined for convenience. Actually they can be decomposed into other "more elementary" invariants:

$$\begin{aligned}
I_2(\Pi^x) &= I_2(p^x)^i + I_2(P^x)^i = I_2(p^x, p^x)^i + I_2(P^x, P^x)^i + 2\, I_2(P^x, p^x)^i \,, \qquad (57) \\
C_2(\Pi^x) &= C_2(p^x_i) + C_2(p^x_i, p^x) + C_2(p^x_i, P^x) = C_2(P^x_i) + C_2(P^x_i, p^x) + C_2(P^x_i, P^x) \,,
\end{aligned}$$

where the summation convention introduced before has been used. The right hand side of these equations does not depend on i anymore due to the summation of the projectors over the full representation space $\mathcal{R}^x$. Since in the calculation of matching coefficients one has to distinguish between heavy and light particles, it is more convenient to use the "more elementary" invariants on the right hand side.

The definitions that have been introduced are general enough to be applied to all color factors that appear in our calculations. However, some of the fields live in representations that deserve special attention. First let us focus on the adjoint representation. The generators here are defined by

$$(T^\alpha_{\mathcal{A}})_{\beta\gamma} \equiv (\Pi^{\mathcal{A}}\, T^\alpha)_{\beta\gamma} = -i f^{\alpha\beta\gamma} \,. \qquad (58)$$

Clearly, the operators $P^{\mathcal{A}}_i$ and $p^{\mathcal{A}}_i$ project on the subspaces with indices A_i and a_i, respectively. Because now $x = \mathcal{A}$ in eq. (55) and $f^{a_i b_j A_k} = 0$, things simplify for the adjoint representation. In fact it is sufficient to define four invariants:

$$\begin{aligned}
f^{\alpha\gamma\delta} f^{\beta\gamma\delta} &= I_2(\Pi^{\mathcal{A}})\ \delta^{\alpha\beta} \,, \\
f^{a_i c_i d_i} f^{b_i c_i d_i} &= I_2(p^{\mathcal{A}}_i)^i\ \delta^{a_i b_i} \,, \\
f^{a_i C_j D_j} f^{b_i C_j D_j} &= I_2(P^{\mathcal{A}}_j)^i\ \delta^{a_i b_i} \,, \\
f^{A_j c_i D_j} f^{B_j c_i D_j} &= I_2(P^{\mathcal{A}}_j, p^{\mathcal{A}}_i)^j\ \delta^{A_j B_j} \qquad (59)
\end{aligned}$$

where we followed the notation of eq. (52). We keep the redundant sub-indices i and j in lines 2 and 4 of this equation in order to be consistent with the notation of eq. (52). Note that $f^{a_i B_j C_k} = 0$ for $j \neq k$ because the indices j and k are used here to label the $\prod_k \mathbf{G}_k$-irreducible representation of the generator $T^{a_i}_{\mathcal{A}}$. There is another quadratic Casimir invariant for the adjoint representation that can be expressed through these and one that vanishes:

$$\begin{aligned}
f^{A_j CD} f^{B_j CD} &= \Big[I_2(\Pi^{\mathcal{A}}) \ - 2\, I_2(P^{\mathcal{A}}_j, p^{\mathcal{A}})^j \Big]\ \delta^{A_j B_j} \,, \\
f^{A_i CD} f^{b_j CD} &= 0 \,. \qquad (60)
\end{aligned}$$

Furthermore, there are relations between these invariants:

$$\begin{aligned} I_2(\Pi^{\mathcal{A}}) &= I_2(P^{\mathcal{A}})^i + I_2(p_i^{\mathcal{A}})^i \,, \\ I_2(P_j^{\mathcal{A}}, p_i^{\mathcal{A}})^j \, D_j^{\mathcal{A}} &= I_2(P_j^{\mathcal{A}})^i \, d_i^{\mathcal{A}} \,. \end{aligned} \tag{61}$$

At the two-loop level these relations are not sufficient to reduce all adjoint color factors because products of up to six structure constants with various contractions can appear in the diagrams. Using the Jacobi identity for $f^{\alpha\beta\gamma}$, we can derive relations for products of three contracted structure constants that have three open indices:

$$\begin{aligned} f^{\alpha\delta\epsilon} f^{\beta\epsilon\phi} f^{\gamma\phi\delta} &= \frac{1}{2} I_2(\Pi^{\mathcal{A}}) \, f^{\alpha\beta\gamma} \,, \\ f^{a_i d_i e_i} f^{b_i e_i f_i} f^{c_i f_i d_i} &= \frac{1}{2} I_2(p_i^{\mathcal{A}})^i \, f^{a_i b_i c_i} \,, \\ f^{a_i DE} f^{b_i EF} f^{c_i FD} &= \frac{1}{2} I_2(P^{\mathcal{A}})^i \, f^{a_i b_i c_i} \,, \\ f^{a_i DE} f^{b_i EF} f^{CFD} &= 0 \,, \\ f^{a_i DE} f^{B_j EF} f^{C_j FD} &= \frac{1}{2} \Big[I_2(\Pi^{\mathcal{A}}) - 2 \, I_2(P_j^{\mathcal{A}}, p^{\mathcal{A}})^j \Big] \, f^{a_i B_j C_j} \,, \\ f^{a_i d_i e_i} f^{B_j e_i F} f^{C_j F d_i} &= \frac{1}{2} I_2(p_i^{\mathcal{A}})^i \, f^{a_i B_j C_j} \,, \\ f^{a_i DE} f^{B_k E f_j} f^{C_k f_j D} &= \frac{1}{2} \Big[2 \, I_2(P_k^{\mathcal{A}}, p_j^{\mathcal{A}})^k - \delta_{ij} \, I_2(p_i^{\mathcal{A}})^i \Big] f^{a_i B_k C_k} \,, \\ f^{A_j De} f^{B_j eF} f^{C_j FD} &= \frac{1}{2} \, I_2(P_j^{\mathcal{A}}, p^{\mathcal{A}})^j f^{A_j B_j C_j} \,, \\ f^{A_j DE} f^{B_j EF} f^{C_j FD} &= \frac{1}{2} \Big[I_2(\Pi^{\mathcal{A}}) - 3 \, I_2(P_j^{\mathcal{A}}, p^{\mathcal{A}})^j \Big] f^{A_j B_j C_j} \,. \end{aligned} \tag{62}$$

These relations are sufficient to do all the reduction for two-point and three-point Green's functions for the adjoint representation at the two-loop level.

Next, let's turn our attention to the representation $\mathcal{R}^{\mathcal{H}}$ where the GUT-breaking scalar field H and the Goldstone field G live in. Some peculiarities occur here due to the appearance of the vev v. $\mathcal{R}^{\mathcal{H}}$ decomposes under $\prod_k \mathbf{G}_k$ into the part of physical Higgs fields and a part of Goldstone bosons:

$$\mathcal{R}^{\mathcal{H}} \to \mathcal{R}^{\tilde{\mathcal{H}}} \oplus \mathcal{R}^{\mathcal{G}} = \mathcal{R}_1^{\tilde{\mathcal{H}}} \oplus \mathcal{R}_2^{\tilde{\mathcal{H}}} \oplus ... \oplus \mathcal{R}_1^{\mathcal{G}} \oplus \mathcal{R}_2^{\mathcal{G}} \oplus ... \tag{63}$$

As already explained in subsection 2.1, the explicit form of the projector on the subspace $\mathcal{R}^{\mathcal{G}}$ is given by [32]:

$$P_i^{\mathcal{G}} = g^2 \tilde{T}^{A_i} v \left(\frac{1}{M_X^2} \right)_{A_i B_i} v \tilde{T}^{B_i} \,, \qquad P^{\mathcal{G}} = \sum_i P_i^{\mathcal{G}} \tag{64}$$

where the (diagonal) gauge boson mass matrix

$$(M_{\mathrm{X}}^2)_{A_i B_i} \equiv g^2 v \tilde{T}^{A_i} \tilde{T}^{B_i} v \equiv M_{\mathrm{X}_i}^2 \delta^{A_i B_i} \tag{65}$$

has been used. The antisymmetric generators in the real representation $\mathcal{R}^{\mathcal{H}}$ have been denoted by $\tilde{T}^\alpha$. Accordingly, the projectors on the subspace of physical Higgs bosons can be written as

$$\sum_i P_i^{\tilde{\mathcal{H}}} = P^{\tilde{\mathcal{H}}} \equiv \Pi^{\mathcal{H}} - P^{\mathcal{G}} = \Pi^{\mathcal{H}} - g^2 \tilde{T}^A v \left(\frac{1}{M_X^2}\right)_{AB} v\tilde{T}^B \,. \tag{66}$$

where $P_i^{\tilde{\mathcal{H}}}$ projects on $\mathcal{R}_i^{\tilde{\mathcal{H}}}$. With these definitions and using eq. (48) as well as the antisymmetry of $\tilde{T}^\alpha$, we already can derive a useful reduction identity for an invariant tensor that appears frequently:

$$v\tilde{T}^{A_i}\tilde{T}^{B_j}\tilde{T}^{C_k}v = \frac{i}{2g^2}(M_{\mathrm{X}_i}^2 - M_{\mathrm{X}_j}^2 + M_{\mathrm{X}_k}^2) f^{A_i B_j C_k} \,. \tag{67}$$

One important property follows from $\tilde{T}^{a_i} v = 0$:

$$\tilde{T}^{a_i}\tilde{T}^{A_j} v = [\tilde{T}^{a_i}, \tilde{T}^{A_j}]\, v = -(T_{\mathcal{A}}^{a_i})_{A_j B_j} \tilde{T}^{B_j} v \,. \tag{68}$$

i.e. $\tilde{T}^{a_i}$ acts like the adjoint generator on the subspace of Goldstone bosons. This leads to various relations between invariants in $\mathcal{R}^{\mathcal{H}}$ and in $\mathcal{R}^{\mathcal{A}}$:

$$\begin{aligned}
D_j^{\mathcal{G}} &= D_j^{\mathcal{A}} \,, \\
I_2(P_j^{\mathcal{G}})^i &= I_2(P_j^{\mathcal{A}})^i \,, \\
C_2(P_j^{\mathcal{G}})^i &= I_2(P_j^{\mathcal{A}}, p_i^{\mathcal{A}})^j \,, \\
I_2(P^{\tilde{\mathcal{H}}})^i &= I_2(\Pi^{\mathcal{H}}) - I_2(P^{\mathcal{A}})^i \,, \\
I_2(P^{\tilde{\mathcal{H}}}, P^{\tilde{\mathcal{H}}})^i &= I_2(\Pi^{\mathcal{H}}) - 2\, C_2(\Pi^{\mathcal{H}}) + \frac{3}{2}\, I_2(P_i^{\mathcal{A}}, p^{\mathcal{A}})^i + \frac{1}{4}\, I_2(\Pi^{\mathcal{A}}) \,.
\end{aligned} \tag{69}$$

There are also two nontrivial important reduction identities that involve both types of invariants:

$$\begin{aligned}
&f^{a_i B_j C_j}\ \mathrm{Tr}(P_n^x \tilde{T}^{B_j} P_m^x \tilde{T}^{C_j} \tilde{T}^{b_i}) = \\
&\frac{i}{2}\delta^{a_i b_i}\Big[I_2(P_j^{\mathcal{A}})^i I_2(P_n^x, P_m^x)^j + C_2(P_n^x, P_m^x)^j I_2(P_n^x)^i - C_2(P_m^x, P_n^x)^j I_2(P_m^x)^i\Big] \,,
\end{aligned} \tag{70}$$

$$\begin{aligned}
&\sum_{jk} f^{a_i B_j C_j} f^{b_i D_k C_k}\ v\tilde{T}^{B_j}\tilde{T}^{D_k} P_n^{\tilde{\mathcal{H}}} \tilde{T}^{C_j}\tilde{T}^{E_k} v = \\
&\delta^{a_i b_i} \sum_j \frac{M_{\mathrm{X}j}^2}{g^2}\Big[I_2(P_j^{\mathcal{A}})^i\, I_2(P_n^{\tilde{\mathcal{H}}}, P^{\mathcal{G}})^j - \frac{1}{2} I_2(P_n^{\tilde{\mathcal{H}}})^i\, C_2(P_n^{\tilde{\mathcal{H}}}, P^{\mathcal{G}})^j\Big]
\end{aligned} \tag{71}$$

where $x \in \{\tilde{\mathcal{H}}, \mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}}, ..., \mathcal{F}_{\mathrm{I}}, \mathcal{F}_{\mathrm{II}}...\}$. Additionally, w.l.o.g. we now define $P_1^{\tilde{\mathcal{H}}}$ to be the operator that projects on the subspace where $v \neq 0$ (i.e. $(P_1^{\tilde{\mathcal{H}}})_{ij} = \frac{v_i v_j}{v^2}$ is a matrix with a single non-zero entry in the component (k,k) where $v_k \neq 0$). Then because of $\tilde{T}^{a_i} v = 0$ and $v\tilde{T}^{A_i} P_n^{\tilde{\mathcal{H}}} = v\tilde{T}^{A_i} P^{\mathcal{G}} P_n^{\tilde{\mathcal{H}}} = 0$, any invariant that contains $P_1^{\tilde{\mathcal{H}}}$ vanishes:

$$C_2(...P_1^{\tilde{\mathcal{H}}}...) = I_2(...P_1^{\tilde{\mathcal{H}}}...) = 0 \,. \tag{72}$$

Since the Higgs mass matrix M_H^2 (cf. eq. (21)) commutes with all $\tilde{T}^{a_i}$, it is diagonal and proportional to the unit matrix on each $\prod_k \mathbf{G}_k$-irreducible subspace. Therefore, it can be written as:

$$M_H^2 = \sum_i P_i^{\tilde{\mathcal{H}}}\, M_{H_i}^2 \,, \tag{73}$$

where $M_{H_i}^2$ are masses of the physical Higgs bosons and particularly $P^{\mathcal{G}}\, M_H^2 = 0$ due to the Goldstone theorem.

Next we will give some useful reduction identities that involve the quartic scalar coupling $\lambda^{\mathcal{H}}_{ijkl} \equiv \lambda_{i'j'k'l'}\, P^{\mathcal{H}}_{i'i}\, P^{\mathcal{H}}_{j'j}\, P^{\mathcal{H}}_{k'k}\, P^{\mathcal{H}}_{l'l}$, which is a totally symmetric invariant tensor under $\mathbf{G}$. These identities can be derived by using eq. (14) and Schur's Lemma as well as the definition of the Higgs mass matrix in eq. (21). Some of these identities can be obtained by multiplying eq. (14) by $\Phi_i\Phi_j\Phi_k\Phi_l$ and performing derivatives w.r.t Φ_m. They are used to eliminate the coupling $\lambda^{\mathcal{H}}_{ijkl}$ from the result by expressing it through the physical Higgs masses.

$$\begin{aligned}
\lambda^{\mathcal{H}}_{ijkl}\; v_k v_m \tilde{T}^A_{ml} &= (M_H^2 \tilde{T}^A)_{ij} - (\tilde{T}^A M_H^2)_{ij} \,, \\
\Rightarrow \lambda^{\mathcal{H}}_{ijkl}\; v_j v_m \tilde{T}^A_{mk} v_n \tilde{T}^B_{nl} &= -(v \tilde{T}^B \tilde{T}^A M_H^2)_i \,, \\
\Rightarrow \lambda^{\mathcal{H}}_{ijkl}\; v_i v_m \tilde{T}^A_{mj} v_n \tilde{T}^B_{nk} v_r \tilde{T}^C_{rl} &= 0 \,, \\
\\
\lambda^{\mathcal{H}}_{ijkl} v_m \tilde{T}^A_{mi} v_n \tilde{T}^B_{nj} v_r \tilde{T}^C_{rk} v_s \tilde{T}^D_{sl} &= v \tilde{T}^D \tilde{T}^C M_H^2 \tilde{T}^A \tilde{T}^B v \,, \\
&\quad + v \tilde{T}^D \tilde{T}^B M_H^2 \tilde{T}^A \tilde{T}^C v \,, \\
&\quad + v \tilde{T}^C \tilde{T}^B M_H^2 \tilde{T}^A \tilde{T}^D v \,, \\
\lambda^{\mathcal{H}}_{ijkl} \lambda^{\mathcal{H}}_{mjkl} v_i v_m &= \frac{v^2}{\Delta^{\mathcal{H}}} \lambda^{\mathcal{H}}_{ijkl} \lambda^{\mathcal{H}}_{ijkl} \,, \\
\lambda^{\mathcal{H}}_{klmn} (v \tilde{T}^{A_i})_k (v \tilde{T}^{B_j})_l (P^{\tilde{\mathcal{H}}}_s)_{mn} &= -\lambda^{\mathcal{H}}_{klmn} (v \tilde{T}^{A_i} \tilde{T}^{B_j})_k v_l (P^{\tilde{\mathcal{H}}}_s)_{mn} \,, \\
&\quad + 2\,\mathrm{Tr}(P^{\tilde{\mathcal{H}}}_s M_H^2 \tilde{T}^{B_j} \tilde{T}^{A_i}) - 2\,\mathrm{Tr}(M_H^2 \tilde{T}^{B_j} P^{\tilde{\mathcal{H}}}_s \tilde{T}^{A_i}) \,, \\
\lambda^{\mathcal{H}}_{ijkl} v_j v_k v_l &= \frac{3 v_i}{v^2} v M_H^2 v \,, \\
\lambda^{\mathcal{H}}_{ijkk} &= \delta_{ij} \left[\frac{2\,\mathrm{Tr}(M_H^2)}{v^2} + \frac{v M_H^2 v}{v^4} D^{\mathcal{H}} \right] , \\
\lambda^{\mathcal{H}}_{ijkl} v_i v_j (P^{\mathcal{H}}_s)_{kl} &= 2\,\mathrm{Tr}(M^2_{H_s} P^{\mathcal{H}}_s) + \frac{v M_H^2 v}{v^2} D^{\mathcal{H}}_s \,.
\end{aligned} \tag{74}$$

In the same way we obtain some important relations for the coupling λ that is not restricted to the space $\mathcal{R}^{\mathcal{H}}$:

$$\begin{aligned}
\lambda_{ijk'l'} (v \tilde{T}^A)_i v_j (\varrho^{\mathcal{S}})_{k'k}\; (\varrho^{\mathcal{S}})_{l'l} &= (M_S^2 \tilde{T}^A)_{kl} - (\tilde{T}^A M_S^2)_{kl} \,, \\
\lambda_{ijkl} (v \tilde{T}^A)_i (v \tilde{T}^A)_j\; (\tilde{T}^{a_n} \tilde{T}^{a_n} \varrho^{\mathcal{S}}_m)_{kl} &= 2\,\mathrm{Tr}(M_S^2 \tilde{T}^A \tilde{T}^A \tilde{T}^{a_n} \tilde{T}^{a_n} \varrho^{\mathcal{S}}_m) \,, \\
&\quad -\, 2\,\mathrm{Tr}(\tilde{T}^A M_S^2 \tilde{T}^A \tilde{T}^{a_n} \tilde{T}^{a_n} \varrho^{\mathcal{S}}_m) \\
&\quad - C_2(P^{\mathcal{H}})\, \lambda_{ijkl} v_i v_j (\tilde{T}^{a_n} \tilde{T}^{a_n} \varrho^{\mathcal{S}}_m)_{kl} \,,
\end{aligned}$$

$$
\begin{aligned}
\lambda_{i_1i_2i_3i_4}\lambda_{j_1j_2j_3j_4} \quad v_{i_1}v_{j_1} \quad & (\varrho^x_{n_1})_{i_2j_2}(\varrho^y_{n_2}T^{a_k})_{i_3j_3}(\varrho^z_{n_3}T^{a_k})_{i_4j_4} = \\
& \tfrac{1}{2}\lambda_{i_1i_2i_3i_4}\lambda_{j_1j_2j_3j_4}v_{i_1}v_{j_1}(\varrho^x_{n_1})_{i_2j_2}(\varrho^y_{n_2})_{i_3j_3}(\varrho^z_{n_3}T^{a_k}T^{a_k})_{i_4j_4} \\
& -\tfrac{1}{2}\lambda_{i_1i_2i_3i_4}\lambda_{j_1j_2j_3j_4}v_{i_1}v_{j_1}(\varrho^y_{n_2})_{i_2j_2}(\varrho^z_{n_3})_{i_3j_3}(\varrho^x_{n_1}T^{a_k}T^{a_k})_{i_4j_4} \\
& -\tfrac{1}{2}\lambda_{i_1i_2i_3i_4}\lambda_{j_1j_2j_3j_4}v_{i_1}v_{j_1}(\varrho^x_{n_1})_{i_2j_2}(\varrho^z_{n_3})_{i_3j_3}(\varrho^y_{n_2}T^{a_k}T^{a_k})_{i_4j_4} \qquad (75)
\end{aligned}
$$

with $x, y, z \in \{\mathcal{H}, \mathcal{S}_{\rm I}, \mathcal{S}_{\rm II}, ...\}$. In the same way we can also derive reduction identities for the Yukawa coupling Y^n using eq. (24)

$$
\begin{aligned}
\mathrm{Tr}(\varrho^x_j Y^n \varrho^y_k Y^{m\star})(\varrho^z_l T^{a_i}T^{a_i})_{nm} &= \mathrm{Tr}(\varrho^x_j Y^{n\star} \varrho^y_k Y^m T^{a_i}T^{a_i})(\varrho^z_l)_{nm} \qquad (76) \\
&\quad +\mathrm{Tr}(\varrho^y_k Y^{n\star} \varrho^x_j Y^m T^{a_i}T^{a_i})(\varrho^z_l)_{nm} \\
&\quad +\mathrm{Tr}(\varrho^x_j Y^{n\star} \varrho^y_k T^{a_i\star} Y^m T^{a_i})(\varrho^z_l)_{nm} \\
&\quad +\mathrm{Tr}(\varrho^y_k Y^{n\star} \varrho^x_j T^{a_i\star} Y^m T^{a_i})(\varrho^z_l)_{nm}\,, \\
\mathrm{Tr}(\varrho^x_j T^{a_i} Y^{n\star} \varrho^y_k Y^m)(\varrho^z_l T^{a_i})_{nm} &= \mathrm{Tr}(\varrho^x_j Y^{n\star} \varrho^y_k Y^m T^{a_i}T^{a_i})(\varrho^z_l)_{nm} \\
&\quad +\mathrm{Tr}(\varrho^y_k Y^n \varrho^x_j T^{a_i} Y^{m\star} T^{a_i\star})(\varrho^z_l)_{nm}\,, \\
\mathrm{Tr}(\varrho^x_j T^{a_i} Y^n \varrho^y_k Y^{m\star})(\varrho^z_l T^{a_i})_{nm} &= -\mathrm{Tr}(\varrho^x_j Y^n \varrho^y_k Y^{m\star} T^{a_i} T^{a_i\star})(\varrho^z_l)_{nm} \\
&\quad -\mathrm{Tr}(\varrho^y_k Y^{n\star} \varrho^x_j T^{a_i} Y^m T^{a_i})(\varrho^z_l)_{nm}\,, \\
\mathrm{Tr}(\varrho^x_j T^{a_i}T^{a_i} Y^{n\star} \varrho^y_k Y^m)(v\tilde{T}^{A_l})_n(v\tilde{T}^{B_l})_m &= -\mathrm{Tr}(\varrho^x_j T^{a_i}T^{a_i}T^{A_l} Y^{n\star} \varrho^y_k Y^m T^{B_l})v_n v_m \\
&\quad -\mathrm{Tr}(\varrho^x_j T^{a_i}T^{a_i} Y^n T^{B_l} \varrho^y_k T^{A_l} Y^{m\star})v_n v_m \\
&\quad -\mathrm{Tr}(\varrho^x_j T^{a_i}T^{a_i}T^{A_l} Y^{n\star} \varrho^y_k T^{B_l\star} Y^m)v_n v_m \\
&\quad -\mathrm{Tr}(\varrho^x_j T^{a_i}T^{a_i}T^{B_l\star} Y^n \varrho^y_k T^{A_l} Y^{m\star})v_n v_m\,.
\end{aligned}
$$

Here $x, y \in \{\mathcal{F}_{\rm I}, \mathcal{F}_{\rm II}, ...\}$ and $z \in \{\mathcal{H}, \mathcal{S}_{\rm I}, \mathcal{S}_{\rm II}, ...\}$. The list of reduction identities may not be exhaustive, but it contains the most important relations. In a similar fashion also other identities can be derived using the invariance relations eqs. (14) and (24) for the invariant tensors.

In the following we briefly sketch the algorithm that is used for the reduction of all the color factors in the diagrams: we have written a `FORM` program that treats the color factors for each individual diagram and reduces them to a basic set of invariants. In a first step all reduction identities that involve the quartic scalar coupling λ_{ijkl} are applied to a given expression. After that any expression will contain traces of strings of generators T^{a_i}, T^{A_i}, projectors ϱ^x_n and Yukawa matrices Y^n. The adjoint indices a_i and A_i are either contracted with each other or with some structure constants. First all the contracted adjoint indices are removed, i.e. traces of the form

$$
\mathrm{Tr}(...T^{a_i}...T^{a_i}...), \qquad \mathrm{Tr}(...T^{A_i}...T^{A_i}...), \qquad (77)
$$

by applying the definitions of the quadratic Casimir invariants eq. (52). If the generators are not next to each other, we commute them until they are and eventually arrive at traces that contain no more contracted indices.

Next, expressions of the form

$$f^{c_i a_i b_i}\,\mathrm{Tr}(...T^{a_i}...T^{b_i}...), \qquad f^{\alpha A_i B_i}\,\mathrm{Tr}(...T^{A_i}...T^{B_i}...), \tag{78}$$

are reduced by using

$$f^{c_i a_i b_i} T^{a_i} T^{b_i} = \tfrac{1}{2} f^{c_i a_i b_i}\,[T^{a_i} T^{b_i}] = \tfrac{i}{2} I_2(p_i^{\mathcal{A}})^i\, T^{c_i} \tag{79}$$

and eq. (70). Again generators that are not next to each other are commuted. Expression that contain the projector $P_i^{\mathcal{G}}$ are treated separately. We insert the explicit form of $P_i^{\mathcal{G}}$ (eq. (64)) and write the traces in the form $v...v$, where "..." stands for a string of generators T^{a_i}, T^{A_i}, and projectors ϱ_n^x. Here we additionally make use of the relations $T^{a_i}v = 0$ and $P_i^{\tilde{\mathcal{H}}}v = 0$ (for $i \neq 1$) in order to eliminate all generators inside the string that have a lowercase adjoint index. After that all color factors involving the Yukawa matrix Y^n are reduced to a basic set of invariants using eq. (76). Finally, we are left with various contractions of structure constants which are expressed by the respective invariants using eqs. (59) and (62). The actual program is slightly more complicated than described above and one needs to introduce repetitive control structures because not all the reduction can be done by a single run. However, the basic procedure is as described.

B Reparametrization of the scalar potential

In this appendix give some details on how the parametrization of the up to quadratic terms in the scalar potential (eq. (19)) arises. We start with eq. (13) and insert the decomposition of the scalar field Φ:

$$\Phi_i = \underbrace{\underbrace{v_i + H_i}_{\mathcal{R}^{\tilde{\mathcal{H}}}} + \underbrace{G_i}_{\mathcal{R}^{\mathcal{G}}}}_{\mathcal{R}^{\mathcal{H}}} + \underbrace{S_i}_{\mathcal{R}^{\mathcal{S}}} \tag{80}$$

where v is the vev with a single non-zero component, H the physical Higgs field, G the Goldstone field and S a field, representing all the other scalars, present in the theory. For clarity we have also indicated the representations where the individual fields live in. Considering only the up to quadratic terms, we arrive at the following expression:

$$V(\Phi) = H_i \tilde{t}_i + \frac{1}{2}\tilde{M}_{ij}^2 H_i H_j + \frac{1}{2}\tilde{M}_{ij}^2 G_i G_j + \frac{1}{2}\tilde{M}_{ij}^2 S_i S_j + \mathcal{O}(\Phi^3) \tag{81}$$

where we have defined the quantities

$$\begin{aligned} \tilde{t}_i &= -\mu_{ij}^2 v_j + \frac{1}{6}\lambda_{ijkl} v_j v_k v_l\,, \\ \tilde{M}_{ij}^2 &= -\mu_{ij}^2 + \frac{1}{2}\lambda_{ijkl} v_k v_l\,. \end{aligned} \tag{82}$$

Here we already have used the fact that, due to gauge invariance (eq. (14)), the matrix μ^2 is proportional to the unit matrix on each **G**-irreducible subspace and therefore

$$\begin{aligned} \mu^2_{ij} v_i G_j &= 0\,, \qquad \mu^2_{ij} v_i S_j = 0\,, \qquad \mu^2_{ij} H_i G_j = 0\,, \\ \mu^2_{ij} H_i S_j &= 0\,, \qquad \mu^2_{ij} G_i S_j = 0\,. \end{aligned} \tag{83}$$

Also due to gauge invariance (eq. (14)) we have $\lambda_{ijkl} v_j v_k v_l \sim v_i$, because the matrix K defined by $K_{ij} = \lambda_{ijkl} v_k v_l$ is diagonal and proportional to the unit matrix on each SM-irreducible subspace. Hence

$$\lambda_{ijkl} G_i v_j v_k v_l = 0 = \lambda_{ijkl} S_i v_j v_k v_l \tag{84}$$

since $(\mathcal{P}^{\mathcal{G}})_{ij} v_j = 0 = (\mathcal{P}^{\mathcal{S}})_{ij} v_j$. The parametrization in eq. (81) has the disadvantage that the masslesness of Goldstone bosons is not manifest there due to the appearance of an explicit mass matrix for the Goldstone field. Note furthermore that on the subspace $\mathcal{R}^{\mathcal{H}}$ the parameter μ^2 is redundant, because it depends on the tadpole term $\tilde{t}$, which has been chosen as a physical parameter in the Lagrangian. Therefore we want μ^2 also to disappear from the mass matrix of the physical Higgs bosons H by trading it for $\tilde{t}$. To solve these issues, we first rewrite the tadpole term by observing that $\tilde{t} \sim v_i$ due to arguments already given above. Therefore we define

$$\begin{aligned} \tilde{t}_i \equiv t\, v_i\,, \qquad\qquad t &\equiv \frac{1}{v^2}\left(-\mu^2_{ij} v_i v_j + \tfrac{1}{6}\lambda_{ijkl} v_i v_j v_k v_l\right) \\ &\equiv -\mu^2_{\mathcal{H}} + \frac{1}{6v^2}\,\lambda^{\mathcal{H}}_{ijkl} v_i v_j v_k v_l\,. \end{aligned} \tag{85}$$

The classical minimum of the potential can be found by setting $t = 0$. In a quantum theory, however, we must keep t as a counterterm, as explained in subsections 2.1 and 2.2. Next we turn to the term $\frac{1}{2}\tilde{M}^2_{ij} G_i G_j$ in eq. (81). From the Goldstone theorem it follows that this mass term must vanish at the classical level. But since we are also interested in quantum corrections, care is needed here. We calculate the term by applying the explicit form of the Goldstone projector $\mathcal{P}^{\mathcal{G}}$ which is given in eq. (15) to $\tilde{M}^2$. Using also the gauge invariance relations in eq. (14), we can show the identities

$$\begin{aligned} (\mathcal{P}^{\mathcal{G}})_{ij} \lambda_{jklm} v_l v_m &= \frac{1}{3v^2} (\mathcal{P}^{\mathcal{G}})_{ik} \lambda_{jlmn} v_j v_l v_m v_n\,, \\ (\mathcal{P}^{\mathcal{G}})_{ij} \mu^2_{jk} &= \frac{1}{v^2} (\mathcal{P}^{\mathcal{G}})_{ik} \mu^2_{jl} v_j v_l = (\mathcal{P}^{\mathcal{G}})_{ik}\, \mu^2_{\mathcal{H}}\,. \end{aligned} \tag{86}$$

Therefore the following identity holds:

$$(\mathcal{P}^{\mathcal{G}})_{ij} \tilde{M}^2_{jk} = t\, (\mathcal{P}^{\mathcal{G}})_{ik}\,. \tag{87}$$

As expected from the Goldstone theorem, the term vanishes at the classical level, but contributes as a counterterm to the two-point function of the Goldstone field via t. Similarly, we calculate $\frac{1}{2}\tilde{M}^2_{ij}H_iH_j$ by applying the projector $\mathcal{P}^{\tilde{\mathcal{H}}} = \Pi^{\mathcal{H}} - \mathcal{P}^{\mathcal{G}}$ to $\tilde{M}^2$:

$$\begin{aligned}
(\mathcal{P}^{\tilde{\mathcal{H}}})_{ij}\tilde{M}^2_{jk} &= (\Pi^{\mathcal{H}})_{ij}\tilde{M}^2_{jk} - (\mathcal{P}^{\mathcal{G}})_{ij}\tilde{M}^2_{jk} \\
&= t\,(\mathcal{P}^{\tilde{\mathcal{H}}})_{ik} + \tilde{M}^2_{jk} - t\,(\Pi^{\mathcal{H}})_{ik} \\
&\equiv t\,(\mathcal{P}^{\tilde{\mathcal{H}}})_{ik} + (M^2_H)_{jk} \qquad (88)
\end{aligned}$$

where we have used eq. (87) and defined the physical Higgs mass matrix

$$(M^2_H)_{ij} = \frac{1}{2}\lambda^{\mathcal{H}}_{ijkl}v_kv_l - \frac{1}{6v^2}\,\lambda^{\mathcal{H}}_{klmn}v_kv_lv_mv_n\,\Pi^{\mathcal{H}}_{ij}\,. \qquad (89)$$

Note that $\mathcal{P}^{\mathcal{G}}M^2_H = 0$ and therefore the Higgs mass matrix, as we have defined it, can have non-zero entries only on the subspace $\mathcal{R}^{\tilde{\mathcal{H}}}$. Also the parameter $\mu^2_{\mathcal{H}}$ does not appear anymore in the definition of M^2_H, as desired. For the fields S_i no peculiarities occur, since they have no vev. Their mass matrix is essentially given by $\tilde{M}^2$:

$$(M^2_{\mathcal{S}})_{ij} = \frac{1}{2}\lambda^{\mathcal{S}}_{ijkl}v_kv_l - (\mu^2(\mathbb{1} - \Pi^{\mathcal{H}}))_{ij}\,. \qquad (90)$$

The up to quadratic terms of the scalar potential can now be written as

$$V(\Phi) = t\;v_iH_i + \frac{1}{2}(M^2_H)_{ij}H_iH_j + \frac{1}{2}t\;H_iH_i + \frac{1}{2}t\;G_iG_i + \frac{1}{2}(M^2_S)_{ij}S_iS_j + \mathcal{O}(\Phi^3) \qquad (91)$$

which is the appropriate parametrization for renormalizing the theory.

C Three-loop gauge β function for the Georgi-Glashow $SU(5)$ model

For performing a consistent three-loop RGE analysis, apart from the two-loop GUT matching corrections also the three-loop gauge β function for the Georgi-Glashow model is needed. The authors of ref. [21] give a general formula for the gauge β function of a general single gauge coupling theory. Specifiying the group theory factors that appear there to the Georgi-Glashow model and insering the scalar self-couplings from eq. (32), as well as the Yukawa coupling from eq. (37) into their general result gives us the desired β function including scalar self-couplings and Yukawa corrections:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\frac{\alpha}{4\pi} \;=\;& -\frac{40}{3}\left(\frac{\alpha}{4\pi}\right)^2 \\
& -\frac{1184}{15}\left(\frac{\alpha}{4\pi}\right)^3 + \left[-\frac{9}{2}\left(\frac{y_t}{4\pi}\right)^2 - 5\left(\frac{y_b}{4\pi}\right)^2\right]\left(\frac{\alpha}{4\pi}\right)^2 \\
& -\frac{1007357}{1080}\left(\frac{\alpha}{4\pi}\right)^4 \\
& +\Bigg[-\frac{1323}{4}\left(\frac{y_t}{4\pi}\right)^2 - \frac{3617}{10}\left(\frac{y_b}{4\pi}\right)^2 \\
& \quad +\frac{155}{96}\frac{A}{(4\pi)^2} + \frac{11}{20}\frac{b}{(4\pi)^2} + \frac{125}{12}\frac{B}{(4\pi)^2} + \frac{25}{4}\frac{c}{(4\pi)^2}\Bigg]\left(\frac{\alpha}{4\pi}\right)^3 \\
& +\Bigg[\frac{51}{4}\left(\frac{y_t}{4\pi}\right)^4 + \frac{47}{4}\left(\frac{y_b}{4\pi}\right)^4 + \frac{839}{8}\frac{y_t^2 y_b^2}{(4\pi)^4} \\
& \quad -\frac{493}{11520}\frac{A^2}{(4\pi)^4} - \frac{47}{144}\frac{AB}{(4\pi)^4} - \frac{1}{12}\frac{b^2}{(4\pi)^4} - \frac{65}{36}\frac{B^2}{(4\pi)^4} - \frac{851}{200}\frac{c^2}{(4\pi)^4}\Bigg]\left(\frac{\alpha}{4\pi}\right)^2 .
\end{aligned} \tag{92}
$$

The first line of this equation represents the one-loop result, the second line the two-loop result and the rest corresponds to the three-loop corrections. Since the Yukawa couplings enter the gauge β function starting from two-loop level only, it is enough to employ the one-loop RGEs for the Yukawa couplings for the precision we are aiming at. These can be derived in a similar manner from the general formula in ref. [37]:

$$
\begin{aligned}
\frac{dy_t}{dt} &= y_t\left[-\frac{108}{5}\left(\frac{\alpha}{4\pi}\right) - 6\left(\frac{y_b}{4\pi}\right)^2 + 9\left(\frac{y_t}{4\pi}\right)^2\right], \\
\frac{dy_b}{dt} &= y_b\left[-18\left(\frac{\alpha}{4\pi}\right) + 11\left(\frac{y_b}{4\pi}\right)^2 - \frac{9}{2}\left(\frac{y_t}{4\pi}\right)^2\right].
\end{aligned} \tag{93}
$$

The scalar self-couplings A, B and c that appear in eq. (92) only at the three-loop level are approximated as constants in our analysis by replacing them by their relations to the physical mass parameters $M_\Sigma, M_{24}, M_{\mathrm{H_c}}, M_{\mathrm{X}}$, and the gauge coupling α by using eqs. (35) and (36). The scalar self-coupling b that appears here can be approximated similarly by a constant using the SM Higgs mass $M^2_{\mathrm{H,SM}}$ and the mass of the W boson M_W:

$$
b = \frac{3}{4}g^2\frac{M^2_{\mathrm{H,SM}}}{M_W^2}. \tag{94}
$$

References

[1] J. R. Ellis, S. Kelley and D. V. Nanopoulos, "Probing the desert using gauge coupling unification," Phys. Lett. B **260** (1991) 131.

[2] U. Amaldi, W. de Boer and H. Furstenau, "Comparison of grand unified theories with electroweak and strong coupling constants measured at LEP," Phys. Lett. B **260**, 447 (1991).

[3] P. Langacker and M. x. Luo, "Implications of precision electroweak experiments for M_t, ρ_0, $\sin^2\theta_W$ and grand unification," Phys. Rev. D **44** (1991) 817.

[4] K. Hagiwara and Y. Yamada, "Grand Unification Threshold Effects In Supersymmetric SU(5) Models," Phys. Rev. Lett. **70** (1993) 709.

[5] J. Hisano, H. Murayama and T. Yanagida, "Nucleon decay in the minimal supersymmetric SU(5) grand unification," Nucl. Phys. B **402** (1993) 46 [arXiv:hep-ph/9207279].

[6] J. Hisano, H. Murayama and T. Yanagida, "Probing GUT scale mass spectrum through precision measurements on the weak scale parameters," Phys. Rev. Lett. **69** (1992) 1014.

[7] Y. Yamada, "SUSY and GUT threshold effects in SUSY SU(5) models," Z. Phys. C **60** (1993) 83.

[8] J. Hisano, T. Moroi, K. Tobe and T. Yanagida, "Limit on the color triplet Higgs mass in the minimum supersymmetric SU (5) model," Mod. Phys. Lett. A **10** (1995) 2267 [arXiv:hep-ph/9411298].

[9] A. Dedes, A. B. Lahanas, J. Rizos and K. Tamvakis, "Threshold effects and radiative electroweak symmetry breaking in SU(5) extensions of the MSSM," Phys. Rev. D **55** (1997) 2955 [arXiv:hep-ph/9610271].

[10] H. Murayama and A. Pierce, "Not even decoupling can save minimal supersymmetric SU(5)," Phys. Rev. D **65** (2002) 055009 [arXiv:hep-ph/0108104].

[11] I. Dorsner, P. Fileviez Perez and G. Rodrigo, "On Unification and Nucleon Decay in Supersymmetric Grand Unified Theories Based on SU(5)," Phys. Lett. B **649** (2007) 197 [arXiv:hep-ph/0610034].

[12] R. Harlander, L. Mihaila and M. Steinhauser, "Two-loop matching coefficients for the strong coupling in the MSSM," Phys. Rev. D **72** (2005) 095009 [arXiv:hep-ph/0509048].

[13] A. Bauer, L. Mihaila and J. Salomon, "Matching coefficients for α_s and m_b to $\mathcal{O}(\alpha_s^2)$ in the MSSM," JHEP **0902** (2009) 037 [arXiv:0810.5101 [hep-ph]].

[14] W. Martens, L. Mihaila, J. Salomon and M. Steinhauser, "Minimal Supersymmetric SU(5) and Gauge Coupling Unification at Three Loops," arXiv:1008.3070 [hep-ph].

[15] A. Masiero, D. V. Nanopoulos, K. Tamvakis and T. Yanagida, "Naturally Massless Higgs Doublets In Supersymmetric SU(5)," Phys. Lett. B **115** (1982) 380.

[16] B. Grinstein, "A Supersymmetric SU(5) Gauge Theory With No Gauge Hierarchy Problem," Nucl. Phys. B **206** (1982) 387.

[17] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, "The four-loop beta function in quantum chromodynamics," Phys. Lett. B **400** (1997) 379 [arXiv:hep-ph/9701390].

[18] M. Czakon, "The four-loop QCD beta-function and anomalous dimensions," Nucl. Phys. B **710** (2005) 485 [arXiv:hep-ph/0411261]. [20]

[19] R. V. Harlander, L. Mihaila and M. Steinhauser, "The SUSY-QCD beta function to three loops," Eur. Phys. J. C **63** (2009) 383 [arXiv:0905.4807 [hep-ph]].

[20] P. M. Ferreira, I. Jack and D. R. T. Jones, "The three-loop SSM beta-functions," Phys. Lett. B **387** (1996) 80 [arXiv:hep-ph/9605440].

[21] A. G. M. Pickering, J. A. Gracey and D. R. T. Jones, "Three loop gauge beta-function for the most general single gauge-coupling theory," Phys. Lett. B **510** (2001) 347 [Phys. Lett. B **512** (2001) 230] [Erratum-ibid. B **535** (2002) 377] [arXiv:hep-ph/0104247].

[22] I. Jack, D. R. T. Jones and C. G. North, "N = 1 supersymmetry and the three loop gauge beta function," Phys. Lett. B **386** (1996) 138 [arXiv:hep-ph/9606323].

[23] L. J. Hall, "Grand Unification Of Effective Gauge Theories," Nucl. Phys. B **178** (1981) 75.

[24] S. Weinberg, "Effective Gauge Theories," Phys. Lett. B **91** (1980) 51.

[25] M. B. Einhorn and D. R. T. Jones, "The Weak Mixing Angle And Unification Mass In Supersymmetric SU(5)," Nucl. Phys. B **196** (1982) 475.

[26] H. Georgi and S. L. Glashow, "Unity Of All Elementary Particle Forces," Phys. Rev. Lett. **32** (1974) 438.

[27] K. Fujikawa, B. W. Lee and A. I. Sanda, "Generalized Renormalizable Gauge Formulation Of Spontaneously Broken Gauge Theories," Phys. Rev. D **6** (1972) 2923.

[28] R. Santos and A. Barroso, "On the renormalization of two Higgs doublet models," Phys. Rev. D **56** (1997) 5366 [arXiv:hep-ph/9701257].

[29] G. Girardi, C. Malleville and P. Sorba, "General Treatment Of The Nonlinear R (Epsilon) Gauge Condition," Phys. Lett. B **117** (1982) 64.

[30] M. B. Gavela, G. Girardi, C. Malleville and P. Sorba, "A Nonlinear R(Xi) Gauge Condition For The Electroweak SU(2) X U(1) Model," Nucl. Phys. B **193** (1981) 257.

[31] T. Muta, "Foundations of quantum chromodynamics. Second edition," World Sci. Lect. Notes Phys. **57** (1998) 1.

[32] P. Langacker, "Grand Unified Theories And Proton Decay," Phys. Rept. **72** (1981) 185.

[33] D. Bardin, G. Passarino, "The Standard Model in the Making: Precision Study of the Electroweak Interactions" Oxford University Press, USA, 1999

[34] M. Bohm, H. Spiesberger and W. Hollik, "On the One Loop Renormalization of the Electroweak Standard Model and Its Application to Leptonic Processes," Fortsch. Phys. **34** (1986) 687.

[35] P. H. Chankowski, S. Pokorski and J. Rosiek, "Complete on-shell renormalization scheme for the minimal supersymmetric Higgs sector," Nucl. Phys. B **423** (1994) 437 [arXiv:hep-ph/9303309].

[36] M. E. Machacek and M. T. Vaughn, "Two Loop Renormalization Group Equations In A General Quantum Field Theory. 1. Wave Function Renormalization," Nucl. Phys. B **222** (1983) 83.

[37] M. E. Machacek and M. T. Vaughn, "Two Loop Renormalization Group Equations In A General Quantum Field Theory. 2. Yukawa Couplings," Nucl. Phys. B **236** (1984) 221.

[38] T. Appelquist and J. Carazzone, "Infrared Singularities And Massive Fields," Phys. Rev. D **11** (1975) 2856.

[39] H. Georgi, "Effective field theory," Ann. Rev. Nucl. Part. Sci. **43** (1993) 209.

[40] M. Steinhauser, "Results and techniques of multi-loop calculations," Phys. Rept. **364** (2002) 247 [arXiv:hep-ph/0201075].

[41] K. G. Chetyrkin, B. A. Kniehl and M. Steinhauser, "Decoupling relations to O(alpha(s)**3) and their connection to low-energy theorems," Nucl. Phys. B **510** (1998) 61 [arXiv:hep-ph/9708255].

[42] P. Nogueira, "Automatic Feynman graph generation," J. Comput. Phys. **105** (1993) 279.

[43] T. Seidensticker, "Automatic application of successive asymptotic expansions of Feynman diagrams," arXiv:hep-ph/9905298.

[44] R. Harlander, T. Seidensticker and M. Steinhauser, "Complete corrections of O(alpha alpha(s)) to the decay of the Z boson into bottom quarks," Phys. Lett. B **426** (1998) 125 [arXiv:hep-ph/9712228].

[45] J. A. M. Vermaseren, "New features of FORM," arXiv:math-ph/0010025.

[46] A. I. Davydychev and J. B. Tausk, "Two loop selfenergy diagrams with different masses and the momentum expansion," Nucl. Phys. B **397** (1993) 123.

[47] S. A. Larin, F. V. Tkachov and J. A. M. Vermaseren, "The Form Version Of Mincer,"

[48] M. Steinhauser, "MATAD: A program package for the computation of massive tadpoles," Comput. Phys. Commun. **134** (2001) 335 [arXiv:hep-ph/0009029].

[49] J. A. M. Vermaseren, "Axodraw," Comput. Phys. Commun. **83** (1994) 45.

[50] T. N. Sherry, "Higgs Potential In The SU(5) Model," J. Phys. A **13** (1980) 2205.

[51] C. Amsler *et al.* [Particle Data Group], "Review of particle physics," Phys. Lett. B **667** (2008) 1.

[52] T. Teubner, K. Hagiwara, R. Liao, A. D. Martin and D. Nomura, "Update of g-2 of the muon and Delta alpha," arXiv:1001.5401 [hep-ph].

[53] S. Bethke, "The 2009 Wolrd Average of $\alpha_s(M_Z)$," Eur. Phys. J. C **64** (2009) 689 [arXiv:0908.1135 [hep-ph]].

[54] *et al.* [CDF and D0 Collaboration], "Combination of CDF and D0 Results on the Mass of the Top Quark," arXiv:1007.3178 [hep-ex].

[55] K. G. Chetyrkin, J. H. Kuhn, A. Maier, P. Maierhofer, P. Marquard, M. Steinhauser and C. Sturm, "Charm and Bottom Quark Masses: an Update," Phys. Rev. D **80** (2009) 074010 [arXiv:0907.2110 [hep-ph]].

[56] K. G. Chetyrkin, J. H. Kuhn and M. Steinhauser, "RunDec: A Mathematica package for running and decoupling of the strong coupling and quark masses," Comput. Phys. Commun. **133** (2000) 43 [arXiv:hep-ph/0004189].

[57] D. R. T. Jones, "The Two Loop Beta Function For A G(1) X G(2) Gauge Theory," Phys. Rev. D **25** (1982) 581.

[58] C. Ford, I. Jack and D. R. T. Jones, "The Standard Model Effective Potential at Two Loops," Nucl. Phys. B **387** (1992) 373 [Erratum-ibid. B **504** (1997) 551] [arXiv:hep-ph/0111190].

[59] O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, "The Gell-Mann-Low Function Of QCD In The Three Loop Approximation," Phys. Lett. B **93** (1980) 429.

[60] S. A. Larin and J. A. M. Vermaseren, "The Three Loop QCD Beta Function And Anomalous Dimensions," Phys. Lett. B **303** (1993) 334 [arXiv:hep-ph/9302208].

[61] P. Binetruy and T. Schucker, "The Use Of Dimensional Renormalization Schemes In Unified Theories," Nucl. Phys. B **178** (1981) 307.

[62] R. van Damme, "The Two Loop Renormalization Of The Gauge Coupling And The Scalar Potential For An Arbitrary Renormalizable Field Theory," Nucl. Phys. B **227** (1983) 317 [Erratum-ibid. B **239** (1984 ERRAT,B244,549.1984) 656].

[63] T. van Ritbergen, A. N. Schellekens and J. A. M. Vermaseren, "Group theory factors for Feynman diagrams," Int. J. Mod. Phys. A **14** (1999) 41 [arXiv:hep-ph/9802376].

[64] P. Cvitanovic, "Group Theory For Feynman Diagrams In Nonabelian Gauge Theories: Exceptional Groups," Phys. Rev. D **14** (1976) 1536.