Adler Function, DIS sum rules and Crewther Relations [*]

P. A. Baikov[a], K.G.Chetyrkin[b†] and J. H. Kühn[b]

[a]Skobeltsyn Institute of Nuclear Physics, Moscow State University,
Moscow 119991, Russia

[b] Institut für Theoretische Teilchenphysik, KIT, D-76128 Karlsruhe, Germany

The current status of the Adler function and two closely related Deep Inelastic Scattering (DIS) sum rules, namely, the Bjorken sum rule for polarized DIS and the Gross-Llewellyn Smith sum rule are briefly reviewed. A new result is presented: an analytical calculation of the coefficient function of the latter sum rule in a generic gauge theory in order $\mathcal{O}(\alpha_s^4)$. It is demonstrated that the corresponding Crewther relation allows to fix two of three colour structures in the $\mathcal{O}(\alpha_s^4)$ contribution to the singlet part of the Adler function.

1. Introduction

Exactly ten years ago at "Loops and Legs 2000" [1] one of the present authors discussed the perspectives of computing the famous ratio $R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$ or, equivalently, the Adler function of the correlator of the EM vector currents at order α_s^4 in massless QCD (for a general review see [2]). The main conclusion was that " ... a better understanding of all kinds of relations connecting various p-integrals could eventually result to the reduction of an arbitrary 5-loop p-integral to a combination of some limited number (a few dozens?) of master p-integrals[3]. Once it is done it should be not very difficult to evaluate the latter analytically or numerically ..." Now, ten years later, we can rightfully state that the program has been successfully worked out and almost (see below) completed. [3–8].

Below we summarize the current status of the calculations of the Adler function and of two other, through generalized Crewther relations [9,10] closely related, physical observables: the Bjorken and the Gross-Llewellyn Smith DIS sum rules [11,12] at order α_s^4 in massless QCD.

2. Adler Function

It is convenient to start with the polarization operator of the flavor singlet vector current:

$$3Q^2\Pi(Q^2) = i \int \mathrm{d}^4 x e^{iq\cdot x}\langle 0|\mathrm{T}j_\mu(x)j^\mu(0)|0\rangle, \quad (1)$$

with $j_\mu = \sum_i \overline{\psi}_i\gamma_\mu\psi_i$ and $Q^2 = -q^2$. The corresponding Adler function

$$D(Q^2) = -12\,\pi^2 Q^2\,\frac{\mathrm{d}}{\mathrm{d}Q^2}\Pi(Q^2) \quad (2)$$

is naturally decomposed into a sum of the non-singlet (NS) and singlet (SI) components (see Fig. 1):

$$D(Q^2) = n_f\,D^{NS}(Q^2) + n_f^2\,D^{SI}(Q^2). \quad (3)$$

Here n_f stands for the total number of quark flavours; all quarks are considered as massless.

Note that the Adler function D^{EM} corresponding to the electromagnetic vector current $j_\mu^{EM} = \sum_i Q_i\overline{\psi}_i\gamma_\mu\psi_i$ (Q_i stands for the electric charge of the quark field ψ_i) is given by the expression:

$$D^{EM} = \left(\sum_i Q_i^2\right)D^{NS} + \left(\sum_i Q_i\right)^2 D^{SI}. \quad (4)$$

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[3]By "p-integrals" we understand massless propagator-like (that is dependent on only one external momentum) Feynman integrals.
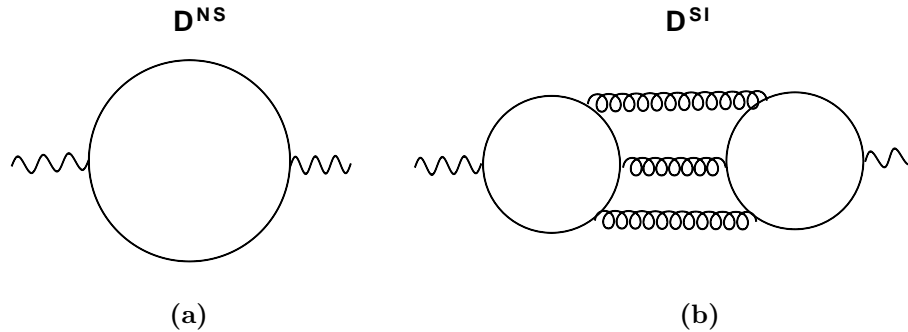
Figure 1. (a),(b): lowest order nonsinglet and singlet diagrams contributing to the Adler function.

The perturbative expansions of nonsinglet and singlet parts read ($a_s \equiv \frac{\alpha_s}{\pi}$):

$$D^{NS}(Q^2) = d_R \left(1 + \sum_{i=1}^{\infty} d_i^{NS} a_s^i(Q^2) \right), \quad (5)$$

$$D^{SI}(Q^2) = d_R \left(\sum_{i=3}^{\infty} d_i^{SI} a_s^i(Q^2) \right), \quad (6)$$

where for future convenience the parameter d_R (the dimension of the quark color representation, $d_R = 3$ in QCD) is factorized in *both* nonsinglet and singlet components.

At order α_s^3 both components of the Adler function are known since long [13–15] for the case of a general colour gauge group. The corresponding α_s^4 calculation has been recently finished [5,6] for the nonsinglet function D^{NS} *only* .

The singlet component has the following structure at orders α_s^3 and α_s^4:

$$d_3^{SI} = \frac{d^{abc} d^{abc}}{d_R} \left(\frac{11}{192} - \frac{1}{8} \zeta_3 \right), \quad (7)$$

$$d_4^{SI} = \quad (8)$$
$$\frac{d^{abc} d^{abc}}{d_R} \left(C_F d_{4,1}^{SI} + C_A d_{4,2}^{SI} + T n_f d_{4,3}^{SI} \right).$$

Here C_F and C_A are the quadratic Casimir operators of the fundamental and the adjoint representation of the Lie algebra, $d^{abc} = 2 \operatorname{Tr}(\{\frac{\lambda^a}{2} \frac{\lambda^b}{2}\} \frac{\lambda^c}{2})$, T is the trace normalization of the fundamental representation. For QCD (colour gauge group SU(3)):

$$C_F = 4/3 \,, C_A = 3 \,, T = 1/2 \,, d^{abc} d^{abc} = \frac{40}{3}.$$

From general considerations we expect that the missing α_s^4 contribution to the singlet component of the Adler function should be numerically inessential (see the next section for an explicit argument in favour of this assumption). The corresponding direct calculation is under way and should be finished in the near future.

3. DIS sum rules

The Bjorken sum rule expresses the integral over the spin distributions of quarks inside of the nucleon in terms of its axial charge times a coefficient function C^{Bjp}:

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)] dx$$
$$= \frac{g_A}{6} C^{Bjp}(a_s) + \sum_{i=2}^{\infty} \frac{\mu_{2i}^{p-n}(Q^2)}{Q^{2i-2}}, (9)$$

where g_1^{ep} and g_1^{en} are the spin-dependent proton and neutron structure functions, g_A is the nucleon axial charge as measured in neutron β-decay. The coefficient function $C^{Bjp}(a_s) = 1 + \mathcal{O}(a_s)$ is proportional to the flavour-nonsinglet axial vector current $\bar{\psi} \gamma^\mu \gamma_5 t^a \psi$ in the corresponding short distance Wilson expansion. The sum in the second line of (9) describes the nonperturbative power
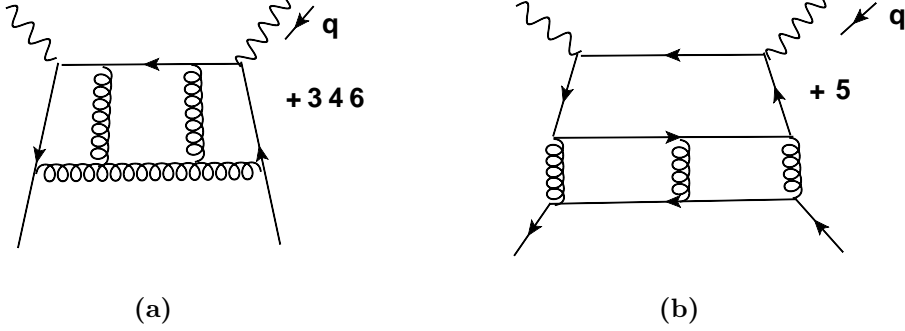
(a)   **(b)**

Figure 2. (a),(b): $\mathcal{O}(\alpha_s^3)$ nonsinglet and singlet diagrams contributing to the Gross-Llewellyn Smith sum rule; note that the coefficient function C^{Bjp} is contributed by only nonsinglet diagrams.

corrections (higher twist) which are inaccessible for pQCD.

Another, closely related sum rule, the Gross-Llewellyn Smith one, reads (we do not write explicitly the higher twist corrections below)

$$\frac{1}{2}\int_0^1 F_3^{\nu p + \overline{\nu}p}(x, Q^2)dx = 3\,C^{GLS}(a_s), \qquad (10)$$

where $F_3^{\nu p + \overline{\nu}p}(x, Q^2)$ is the isospin singlet structure function. The function $C^{GLS}(a_s)$ comes from operator-product expansion of the axial and vector nonsinglet currents

$$i\int T A_\mu^a(x)V_\nu^b(0)e^{iqx}dx|_{q^2\to\infty} \approx C_{\mu\nu\alpha}^{V,ab}\,V_\alpha(0)+\ldots(11)$$

where

$$C_{\mu\nu\alpha}^{V,ab} \sim i\delta^{ab}\epsilon_{\mu\nu\alpha\beta}\frac{q^\beta}{Q^2}C^{GLS}(a_s)$$

and $V_\alpha = \overline{\psi}\gamma_\alpha\psi$ is a flavour singlet quark current. At last $A_\mu^a = \overline{\psi}\gamma_\mu\gamma_5 t^a\psi$, $V_\nu^b = \overline{\psi}\gamma_\nu t^b\psi$ are axial vector and vector nonsinglet quark currents, with the t^a, t^b being being the generators of the flavour group $SU(n_f)$.

All diagrams contributing to $C^{GLS}(a_s)$ can be separated in two groups: nonsinglet and singlet ones (see Fig. 2):

$$C^{GLS} = C^{NS} + C^{SI}, \qquad (12)$$

$$C^{NS}(Q^2) = 1 + \sum_{i=1}^{\infty} c_i^{NS}\,a_s^i(Q^2), \qquad (13)$$

$$C^{SI}(Q^2) = \sum_{i=3}^{\infty} c_i^{SI}\,a_s^i(Q^2). \qquad (14)$$

The results for both functions C^{Bjp} and C^{GLS} at order α_s^3 are known since early 90-ties [17]. Note that there is a remarkable connection (following from the chiral invariance [17])

$$C^{Bjp} \equiv C^{NS} \qquad (15)$$

valid in all orders of the perturbation theory.

The $\mathcal{O}(\alpha_s^4)$ contribution to C^{Bjp} has been recently computed and published in [6]. The calculation of the $\mathcal{O}(\alpha_s^4)$ contribution to C^{SI} has been just finished. The results for both functions are given below.

We start from their numerical form.

$$C^{NS} = 1 - a_s + (-4.583 + 0.3333\,n_f)\,a_s^2 \quad (16)$$
$$+ \quad a_s^3 \left(-41.44 + 7.607\,n_f - 0.1775\,n_f^2\right) a_s^3$$
$$+ \quad \left(-479.4 + 123.4\,n_f - 7.697\,n_f^2 + 0.104\,n_f^3\right) a_s^4,$$

$$C^{SI} = 0.4132\,n_f\,a_s^3 + a_s^4\,n_f\,(5.802 - 0.2332\,n_f)\,.(17)$$

For a typical value of $n_f = 3$ the above relations read:

$$C^{NS}(n_f = 3) = 1 - 1.a_s - 3.583a_s^2 \qquad (18)$$
$$- 20.22a_s^3 - 175.7\,a_s^4,$$

$$C^{SI} = 1.2396\,a_s^3 + 15.3072\,a_s^4. \qquad (19)$$

As expected the singlet contributions are less than the nonsinglet ones by at least one order in

magnitude for each from two available orders in α_s.

The result for C^{NS} valid for a generic gauge group is given in [6]. For the singlet coefficient function the generalization of eq. (17) to a generic gauge group read

$$c_3^{SI} = n_f \frac{d^{abc}d^{abc}}{d_R} \left(c_{3,1}^{SI} \equiv -\frac{11}{192} + \frac{1}{8}\zeta_3 \right), \quad (20)$$

$$c_4^{SI} = \quad (21)$$
$$n_f \frac{d^{abc}d^{abc}}{d_R} \left(C_F\, c_{4,1}^{SI} + C_A\, c_{4,2}^{SI} + T\, n_f\, c_{4,3}^{SI} \right),$$

$$c_{4,1}^{SI} = \frac{37}{128} + \frac{1}{16}\zeta_3 - \frac{5}{8}\zeta_5, \quad (22)$$

$$c_{4,2}^{SI} = -\frac{481}{1152} + \frac{971}{1152}\zeta_3 - \frac{295}{576}\zeta_5 + \frac{11}{32}\zeta_3^2, \quad (23)$$

$$c_{4,3}^{SI} = \frac{119}{1152} - \frac{67}{288}\zeta_3 + \frac{35}{144}\zeta_5 - \frac{1}{8}\zeta_3^2. \quad (24)$$

4. Crewther relations

There exist two (generalized) Crewther relations which connect the nonsinglet and the full Adler functions to the coefficient functions C^{Bjp} and C^{GLS} respectively [10]. The relations state that

$$D^{NS}(a_s)\, C^{Bjp}(a_s) = d_R \left[1 + \frac{\beta(a_s)}{a_s}\, K^{NS} \right], \quad (25)$$
$$K^{NS} \equiv K^{NS}(a_s) = a_s\, K_1^{NS}$$
$$+ a_s^2\, K_2^{NS} + a_s^3\, K_3^{NS} + \dots$$

and

$$\frac{D(a_s)\, C^{GLS}(a_s)}{d_R\, n_f} = 1 + \frac{\beta(a_s)}{a_s}\, K(a_s), \quad (26)$$
$$K(a_s) = a_s\, K_1$$
$$+ a_s^2\, K_2 + a_s^3\, K_3 + \dots$$

Here $\beta(a_s) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} a_s(\mu) = -\sum_{i\geq0} \beta_i a_s^{i+2}$ is the QCD β-function with its first term $\beta_0 = \frac{11}{12} C_A - \frac{T}{3} n_f$. The term proportional to the β-function describes the deviation from the limit of exact conformal invariance, with the deviations starting in order α_s^2. The relations (25) and (26) were suggested in [10] on the basis of

[9,18] and the $\mathcal{O}(\alpha_s^3)$ calculations of the functions C^{Bjp}, C^{GLS} and D carried out in[13,14,17]. Formal proofs were considered in [19,20].

Relation (25) has been investigated in work [6]. Here it was demonstrated that at orders α_s^2, α_s^3 and α_s^4 eq. (25) produces as many as 2, 3 and, finally, 6 constraints on the combinations $d_2^{NS} + c_2^{NS}, d_3^{NS} + c_3^{NS}$ and $d_4^{NS} + c_4^{NS}$ respectively (for a very detailed discussion at orders α_s^2 and α_s^3 see also [10]).

The fulfillment of these constraints has provided us with a powerful test of the correctness of the calculations of $D^{NS}(a_s)$ and $C^{Bjp}(a_s)$. It also fixes unambiguously the (nonsinglet) Crewther parameters K_1^{NS}, K_2^{NS} ans K_3^{NS} (for explicit expressions see [6]).

Let us consider now eq. (26) (assuming that (25) is fulfilled). Combining eqs. (3,12,15) and (25) leads to the following relations between coefficients K_i^{NS} and K_i:

$$K_1 = K_1^{NS}, \quad K_2 = K_2^{NS}, \quad (27)$$
$$K_3 = K_3^{NS} + K_3^{SI}, \quad (28)$$
$$K_3^{SI} = k_{3,1}^{SI}\, n_f \frac{d^{abc}d^{abc}}{d_R}. \quad (29)$$

Thus, we conclude that eq. (26) puts $3 - 1 = 2$ constraints between two triplets of (purely numerical) parameters $\{d_{4,1}^{SI}, d_{4,2}^{SI}, d_{4,3}^{SI}\}$ and $\{c_{4,1}^{SI}, c_{4,2}^{SI}, c_{4,3}^{SI}\}$ appearing in eqs. (8) and (21)) and completely describing the order α_s^4 singlet contributions to the Adler function and the Gross-Llewellyn Smith sum rule respectively.

The solution of the constraints and eqs. (22-24) produces the following result for d_4^{SI}:

$$d_{4,1}^{SI} = -\frac{3}{2}c_{3,1}^{SI} - c_{4,1}^{SI} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}, \quad (30)$$

$$d_{4,2}^{SI} = -c_{4,2}^{SI} - \frac{11}{12}k_{3,1}^{SI}, \quad (31)$$

$$d_{4,3}^{SI} = -c_{4,3}^{SI} + \frac{1}{3}k_{3,1}^{SI}. \quad (32)$$

5. Conclusion

We have analytically computed the $\mathcal{O}(\alpha_s^4)$ contribution to the Gross-Llewellyn Smith sum rule.

The result taken together with the corresponding (generalized) Crewther relation leads to a prediction for the (still unknown) $\mathcal{O}(\alpha_s^4)$ term in the singlet component of the Adler function. The prediction depends on only one unknown numerical parameter — the Crewther coefficient $k_{3,1}^{SI}$.

The direct calculation of the three coefficients $\{d_{4,1}^{SI}, d_{4,2}^{SI}, d_{4,3}^{SI}\}$ parameterizing the $\mathcal{O}(\alpha_s^4)$ contribution to the singlet component of the Adler function will be finished soon. Then will get another strong check of the complicated machinery employed in performing the calculations.

The calculation of the coefficient function C^{GLS} has been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers and on the HP XC4000 supercomputer of the federal state Baden-Württemberg using parallel MPI-based [21] as well as thread-based [22] versions of FORM [23]. For the evaluation of color factors we have used the FORM program $COLOR$ [24]. The diagrams have been generated with QGRAF [25]. The figures have been drawn with the the help of Axodraw [26] and JaxoDraw [27].

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