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Wilson Expansion of QCD Propagators at Three Loops: Operators of Dimension Two and Three

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Abstract

In this paper we construct the Wilson short distance operator product expansion for the gluon, quark and ghost propagators in QCD, including operators of dimension two and three, namely, A^2, m^2, $m\,A^2$, $\overline{\psi}\,\psi$ and m^3. We compute analytically the coefficient functions of these operators at three loops for all three propagators in the general covariant gauge. Our results, taken in the Landau gauge, should help to improve the accuracy of extracting the vacuum expectation values of these operators from lattice simulation of the QCD propagators.

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1 Introduction

Two-point correlation functions of the fundamental fields of the QCD Lagrangian – that is gluon, ghost and quark propagators – are of direct importance in any perturbative treatment of QCD. Suffice it to say that the corresponding wave function renormalization constants are vital ingredients in calculations of the QCD β-function and the quark mass anomalous dimensions (currently known at four-loop level [1–4]). Scheme-invariant versions of these propagators are presently known in NNNLO (that is up to and including three loops) in arbitrary covariant gauge [5] including the Landau one, which is distinguished from the point of view of lattice simulations.

Purely perturbative treatment essentially assumes a weak-coupling regime. QCD propagators, especially the gluon and the quark ones, have been much under examination also beyond perturbation theory (that is in the strong-coupling regime). Here one should mention at least two broad directions, namely, the use of Schwinger-Dyson equations (for reviews see e.g., [6–8]) and non-perturbative computation on the lattice by Monte Carlo simulations. In what follows we will concentrate our discussion on the latter.

It is expected — due to the asymptotic freedom — that the behavior of full QCD-propagators is to be governed at sufficiently large momentum transfers by perturbation theory *and* by the Operator Product Expansion (OPE) [9–11]. Thus, by comparing results of continuum perturbation theory calculations with those of lattice simulations one hopes to get a lot of information about the (renormalized) running coupling constant and quark masses as well as on *condensates* — Vacuum Expectation Values (VEV's) of composite operators — entering into OPE. The idea has been pursued in lattice simulations performed by various groups. (As for investigating condensates in lattice framework along these lines, see, e.g. Refs. [12–15] and also references therein for earlier results and for more lattice-specific information).

While purely perturbative contributions to the QCD propagators have been computed in NNNLO, the corresponding (power suppressed) condensate contributions are usually known only at leading order or, at best, at next-to-leading order. To be specific, let us consider the gluon and ghost propagators in Landau gauge (for space-like momentum $q^2 < 0$)

$$D^{ab}_{\mu\nu}(q) = \frac{\delta^{ab}}{-q^2}\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right] D^g(Q), \quad \Delta^{ab}(q) = \frac{\delta^{ab}}{-q^2} \; D^h(Q) \tag{1}$$

with $Q = \sqrt{-q^2}$. The *dressing functions* D^g and D^h can be decomposed in terms of the appropriate OPE as follows

$$D^?(Q^2) \underset{Q^2\to\infty}{=\!=} D_0^?(\mu/Q, a_s) + \sum_i \frac{C_i^?(\mu/Q, a_s)}{Q^{\,d_i}}\,\langle O_i\rangle, \tag{2}$$

where ? stand for g or h, $D_0^?(\mu/Q, a_s)$ is the purely perturbative contribution, μ is the renormalization scale and $a_s = \frac{\alpha_s}{\pi} = \frac{g_s^2}{4\pi^2}$ is the quark-gluon coupling constant. The sum goes over all scalar operators with vacuum quantum numbers; d_i stands for the dimension of the operator O_i in mass units.

Assuming the case of massless QCD, the leading non-perturbative corrections in (2) should come from operators with the lowest possible mass dimension $d_i = 2$, namely (to be in agreement with the commonly used in lattice publication sign convention we effectively use below the *euclidean* scalar product in the definition of A^2)

$$A^2 \equiv -A^a_\mu A^{a\,\mu} \quad \text{and} \quad i\,\overline{C}^a C^a,$$

where A^a_μ is the gauge field , $C(\overline{C})$ is the ghost (antighost) field. Within the class of covariant gauges[2] the coefficient function of the second operator is known to vanish identically in every OPE [16]. In what follows we will not consider this operator.

The first operator, the *gluon mass condensate*[3], does have nonzero coefficient functions $C^?_{A^2}$ already in the tree approximation, namely [16] (see, also [17–19])

$$C^g_{A^2} = g_s^2\,\frac{3}{32} + \mathcal{O}(a_s^2), \quad C^h_{A^2} = g_s^2\,\frac{3}{32} + \mathcal{O}(a_s^2). \tag{3}$$

From the phenomenological side, lattice simulations carried on by Boucaud et al in a series of publications [15, 17, 18, 20–22] (see, also [23]) seem to demonstrate the existence of effects of order $1/Q^2$ in both gluon and ghost dressing functions[4]. Moreover, numerical fits produce the results consistent with the OPE description of power suppressed $1/Q^2$ corrections to the gluon, ghost and quark propagators. This means, for instance, that *one and the same value* of the gluon mass condensate [15]

$$g_s^2\,\langle A^2\rangle = 5.1^{+0.7}_{-1.1}\ \text{GeV}^2 \tag{4}$$

multiplied by the *tree level* CF's (3) together with purely perturbative contributions (known to *three loops*) describe the ghost and gluon dressing functions over the whole available momentum window 2 GeV $\leq Q \leq$ 6 GeV.

On the other hand, a study of a dressing function itself could, obviously, at best result in the determination of the product of the CF and the VEV of a composite operator (even if one assumes no contamination from operators of higher mass dimension). Thus, knowledge of higher order corrections to the coefficient functions of condensates is of some importance, at least for better understanding the results of lattice simulations.

The quarks are massive. As a consequence the possible composite operators could contain powers of quark masses along with quantum fields. It is worthwhile to remember at this point that in "good" renormalization schemes like those based on the dimensional regularization [27–29] and minimal subtractions [30] the coefficient functions of any (short

[2] By a covariant gauge we mean the one generated by adding the term $-\frac{1}{2\xi_L}(\partial_\nu A^a_\mu)(d_\nu A^a_\mu)$ to the gauge invariant Yang-Mills Lagrangian; the corresponding expression for the tree-level vector boson propagator reads $\frac{\delta^{ab}}{-q^2}\left[-g_{\mu\nu} + (1-\xi_L)\frac{q_\mu q_\nu}{q^2}\right]$; the choice of the Landau gauge corresponds to limit of $\xi_L \to 0$.

[3] We use this expression as the title of just *gluon condensate* is traditionally referred to the VEV of the operator $G^a_{\mu\nu}G^a_{\mu\nu}$ starting from the seminal works by the ITEP group [11].

[4] The gluon mass condensate as well as the quark condensate also show up in the quark propagator [12, 13, 22, 24–26], see Section 3 below.

distance) OPE obey the following important property[5]: their dependence on any particle/field masses is *polynomial*. In particular, it means that any more complicated mass dependence of a correlator will be “hidden” in the corresponding VEV of composite operators. It also means that if one allows, as we do, mass factors to be used in constructing composite operators, then their coefficient functions become totally mass-independent by definition. Limiting ourselves to operators with mass dimensions not higher than three we arrive to the following list of operators which could appear in OPE for the QCD propagators:

$$A^2 \equiv A^a_\mu A^a_\mu, \quad m^2, \quad m^3, \quad m\, A^2, \quad \bar{\psi}\psi, \tag{5}$$

where m is a quark mass. and we have assumed QCD with $n_f = n_l + 1$ total number of quark flavors, one of those having a mass m, while all others are strictly massless[6].

The aim of the research we are going to present was to compute the higher order contributions (up to and including three loops) to coefficient functions of operators (5) appearing in the OPE of QCD propagators. The structure of the paper is as follows. In the next two sections we describe our results for OPE of QCD propagators. In the fourth section we consider the RG evolution equations for propagators and operators under consideration and construct the scale and scheme invariant combinations of operators and coefficient functions. Due to their scheme independence the latter should be most convenient for comparisons with the results of lattice calculations. Then we briefly discuss (in Section 5) some technical details of the calculations as well as software/hardware tools employed. Finally, a short summary of our findings is given in the Conclusion (Section 6).

We finish the introduction by adding that in recent years, starting from works [33,34], the condensates of mass dimension two, especially the gluon mass condensate, have been intensively studied in view of better understanding of confinement in Yang-Mills theories and QCD. (For example, see recent works [35–43] and references therein). Unfortunately, any discussion of these developments is beyond the scope of the present paper.

2 OPE for the gluon and ghost propagators

On dimensional grounds, the operators of dimensions three do not contribute to the OPE for the gluon and ghost propagators. The remaining coefficient functions $C^{?}_{m^2}$ and $C^{?}_{A^2}$

[5] To our knowledge it was first established in [31]; see also [32].

[6] Later, in Appendix A, we generalize our results for the case of arbitrary many massive quarks.

read:

$$
\begin{aligned}
C^g_{m^2} =& a_s\Bigg[1 + a_s\bigg(\frac{383}{24} + \frac{3}{2}\zeta_3 - \frac{5}{9}n_f + \frac{93}{16}l_{\mu Q} - \frac{1}{3}l_{\mu Q}n_f\bigg) \\
&+ a_s^2\bigg(\frac{7370507}{27648} - \frac{27}{64}\zeta_4 - \frac{22615}{864}\zeta_5 + \frac{415679}{6912}\zeta_3 - \frac{69941}{3456}n_f \\
&- \frac{113}{24}n_f\zeta_3 + \frac{25}{108}n_f^2 + \frac{7405}{48}l_{\mu Q} + \frac{411}{32}l_{\mu Q}\zeta_3 - \frac{4123}{288}l_{\mu Q}n_f \\
&- \frac{3}{4}l_{\mu Q}n_f\zeta_3 + \frac{5}{18}l_{\mu Q}n_f^2 + \frac{13263}{512}l_{\mu Q}^2 - \frac{281}{96}l_{\mu Q}^2n_f + \frac{1}{12}l_{\mu Q}^2n_f^2\bigg)\Bigg], \qquad (6) \\
C^g_{A^2} =& \frac{3}{8}\pi^2 a_s\Bigg[1 + a_s\bigg(\frac{785}{96} - \frac{11}{18}n_f + \frac{35}{16}l_{\mu Q} - \frac{1}{6}l_{\mu Q}n_f\bigg) \\
&+ a_s^2\bigg(\frac{799087}{9216} + \frac{27}{128}\zeta_3 - \frac{90371}{6912}n_f - \frac{11}{24}n_f\zeta_3 + \frac{121}{324}n_f^2 + \frac{70097}{1536}l_{\mu Q} \\
&- \frac{3719}{576}l_{\mu Q}n_f + \frac{11}{54}l_{\mu Q}n_f^2 + \frac{2765}{512}l_{\mu Q}^2 - \frac{149}{192}l_{\mu Q}^2n_f + \frac{1}{36}l_{\mu Q}^2n_f^2\bigg) \\
&+ a_s^3\bigg(\frac{985590473}{884736} - \frac{243}{4096}\zeta_4 - \frac{4545}{128}\zeta_5 - \frac{57399}{8192}\zeta_3 - \frac{159678799}{663552}n_f \\
&+ \frac{33}{256}n_f\zeta_4 + \frac{3355}{576}n_f\zeta_5 - \frac{36455}{6912}n_f\zeta_3 + \frac{1702769}{124416}n_f^2 + \frac{29}{72}n_f^2\zeta_3 \\
&- \frac{1115}{5832}n_f^3 + \frac{38346881}{49152}l_{\mu Q} + \frac{1539}{1024}l_{\mu Q}\zeta_3 - \frac{6165035}{36864}l_{\mu Q}n_f \\
&- \frac{863}{256}l_{\mu Q}n_f\zeta_3 + \frac{48095}{4608}l_{\mu Q}n_f^2 + \frac{11}{48}l_{\mu Q}n_f^2\zeta_3 - \frac{121}{648}l_{\mu Q}n_f^3 \\
&+ \frac{3082507}{16384}l_{\mu Q}^2 - \frac{238649}{6144}l_{\mu Q}^2n_f + \frac{1453}{576}l_{\mu Q}^2n_f^2 - \frac{11}{216}l_{\mu Q}^2n_f^3 \\
&+ \frac{113365}{8192}l_{\mu Q}^3 - \frac{1479}{512}l_{\mu Q}^3n_f + \frac{77}{384}l_{\mu Q}^3n_f^2 - \frac{1}{216}l_{\mu Q}^3n_f^3\bigg)\Bigg], \qquad (7) \\
C^h_{m^2} =& -\frac{3}{8}a_s^2\Bigg[1 + \frac{3}{2}l_{\mu Q} + a_s\bigg(\frac{35501}{1152} + \frac{9}{8}\zeta_4 + \frac{15}{4}\zeta_5 - \frac{35}{16}\zeta_3 - \frac{23}{24}n_f \\
&+ \frac{1847}{64}l_{\mu Q} + \frac{9}{4}l_{\mu Q}\zeta_3 - \frac{7}{12}l_{\mu Q}n_f + \frac{441}{64}l_{\mu Q}^2 - \frac{1}{4}l_{\mu Q}^2n_f\bigg)\Bigg], \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
C^h_{A^2} =&\frac{3}{8}\pi^2 a_s\Bigg[1 + a_s\left(\frac{15}{4} + \frac{9}{8}l_{\mu Q}\right)\\
&+ a_s^2\left(\frac{14853}{512} + \frac{27}{32}\zeta_3 - \frac{187}{128}n_f + \frac{2145}{128}l_{\mu Q} - \frac{25}{32}l_{\mu Q}n_f + \frac{279}{128}l^2_{\mu Q} - \frac{3}{32}l^2_{\mu Q}n_f\right)\\
&+ a_s^3\Bigg(\frac{12444649}{36864} + \frac{243}{2048}\zeta_4 - \frac{56745}{4096}\zeta_5 + \frac{53823}{4096}\zeta_3 - \frac{505459}{13824}n_f\\
&- \frac{33}{128}n_f\zeta_4 - \frac{307}{256}n_f\zeta_3 + \frac{13081}{20736}n_f^2 + \frac{1}{48}n_f^2\zeta_3 + \frac{950963}{4096}l_{\mu Q}\\
&+ \frac{5967}{1024}l_{\mu Q}\zeta_3 - \frac{72907}{3072}l_{\mu Q}n_f \frac{51}{64}l_{\mu Q}n_f\zeta_3 + \frac{263}{576}l_{\mu Q}n_f^2 + \frac{61797}{1024}l^2_{\mu Q}\\
&- \frac{757}{128}l^2_{\mu Q}n_f + \frac{25}{192}l^2_{\mu Q}n_f^2 + \frac{4929}{1024}l^3_{\mu Q} - \frac{115}{256}l^3_{\mu Q}n_f + \frac{1}{96}l^3_{\mu Q}n_f^2\Bigg)\Bigg]\,.
\end{aligned}
\tag{9}
$$

Here and everywhere in the paper the renormalization is carried out in the $\overline{\mathrm{MS}}$-scheme, n_f is the total number of quark flavours, $l_{\mu Q} = \ln\frac{\mu^2}{Q^2}$, $m = m(\mu)$ and $a_s = \frac{\alpha_s(\mu)}{\pi}$ are the running quark mass and quark-gluon coupling constant respectively. In addition, the irrational constants $\zeta_3 = 1.2020569$, $\zeta_4 = 1.0823232$, $\zeta_5 = 1.0369277$ appear. In numerical form Eqs. (6)-(9) read

$$
\begin{aligned}
C^g_{m^2} =&a_s\big[1 + a_s(17.7614 - 0.555556\,n_f)\\
&+ a_s^2(311.276 - 25.8972\,n_f + 0.231481\,n_f^2)\big]\,,
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
C^g_{A^2} =&\frac{3}{8}\,\pi^2\,a_s\big[1 + a_s(8.17708 - 0.611111\,n_f)\\
&+ a_s^2(86.9600 - 13.6255\,n_f + 0.373457\,n_f^2)\\
&+ a_s^3(1068.69 - 240.803\,n_f + 14.1703\,n_f^2 - 0.191187\,n_f^3)\big]\,,
\end{aligned}
\tag{11}
$$

$$
C^h_{m^2} = -\frac{3}{8}a_s^2\big[1 + a_s(33.2934 - 0.958333\,n_f)\big]\,,
\tag{12}
$$

$$
\begin{aligned}
C^h_{A^2} =&\frac{3}{8}\,\pi^2\,a_s\big[1 + 3.75000a_s\\
&+ a_s^2(30.0240 - 1.46094\,n_f)\\
&+ a_s^3(339.141 - 38.2844\,n_f + 0.655878\,n_f^2)\big]\,,
\end{aligned}
\tag{13}
$$

where we have set $\mu = Q$.

3 OPE for the quark propagator

The quark propagator of a quark field ψ_q with mass m is expressed in terms of the corresponding dressing functions as follows:

$$
\mathrm{i}\int \mathrm{d}x\,\mathrm{e}^{\mathrm{i}qx}\langle \mathrm{T}[\psi(x)\bar{\psi}(0)]\rangle = \frac{\not{q}}{Q^2}\,V(Q) + \frac{S(Q)}{Q^2}
\tag{14}
$$

The OPE expansions for the dressing functions (up to operators of dimension three) are

$$V(Q) = V_0(\mu/Q, a_s) + \frac{C^q_{m^2}(\mu/Q, a_s)}{Q^2}\, m^2 + \frac{C^q_{A^2}(\mu/Q, a_s)}{Q^2}\langle A^2\rangle, \tag{15}$$

$$S(Q) = S_0(\mu/Q, a_s)\, m + \frac{C^q_{m^3}(\mu/Q, a_s)}{Q^2}\, m^3 + \frac{C^q_{A^2}(\mu/Q, a_s)}{Q^2}\langle mA^2\rangle + \frac{C^q_{\bar\psi\psi}(\mu/Q, a_s)}{Q^2}\langle\bar\psi\psi\rangle, \tag{16}$$

where m is the quark mass of the quark associated with the quark field ψ.

The purely perturbative contributions V_0 and S_0 have been already discussed at three-loop level in [44], the files with results in computer readable form can be downloaded from `http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp99/ttp99-43`. The remaining coefficient functions are listed below:

$$\begin{aligned}
C^q_{m^2} =& -\Bigg[1 + a_s\left(\frac{8}{3} + 2l_{\mu Q}\right) + a_s^2\Bigg(\frac{617}{24} - \frac{10}{3}\zeta_3 - \frac{121}{144}n_f \\
&+ \frac{307}{16}l_{\mu Q} - \frac{23}{36}l_{\mu Q}n_f + \frac{19}{4}l^2_{\mu Q} - \frac{1}{6}l^2_{\mu Q}n_f\Bigg) \\
&+ a_s^3\Bigg(\frac{58211}{192} - \frac{17}{256}\zeta_4 + \frac{2165}{144}\zeta_5 - \frac{279733}{3456}\zeta_3 - \frac{173449}{7776}n_f \\
&- \frac{5}{6}n_f\zeta_4 + \frac{625}{216}n_f\zeta_3 + \frac{2999}{23328}n_f^2 + \frac{1}{27}n_f^2\zeta_3 + \frac{64803}{256}l_{\mu Q} \\
&- \frac{3217}{128}l_{\mu Q}\zeta_3 - \frac{685}{36}l_{\mu Q}n_f - \frac{5}{9}l_{\mu Q}n_f\zeta_3 + \frac{139}{648}l_{\mu Q}n_f^2 + \frac{2045}{24}l^2_{\mu Q} \\
&- \frac{1895}{288}l^2_{\mu Q}n_f + \frac{23}{216}l^2_{\mu Q}n_f^2 + \frac{95}{8}l^3_{\mu Q} - \frac{17}{18}l^3_{\mu Q}n_f + \frac{1}{54}l^3_{\mu Q}n_f^2\Bigg)
\end{aligned} \tag{17}$$

$$\begin{aligned}
C^q_{A^2} =& -\frac{1}{3}\pi^2 a_s\Bigg[1 + a_s\left(\frac{3}{4} + \frac{9}{16}l_{\mu Q}\right) + a_s^2\Bigg(\frac{32167}{9216} - \frac{117}{128}\zeta_3 \\
&- \frac{137}{768}n_f + \frac{1691}{768}l_{\mu Q} - \frac{23}{192}l_{\mu Q}n_f + \frac{477}{512}l^2_{\mu Q} - \frac{3}{64}l^2_{\mu Q}n_f\Bigg) \\
&+ a_s^3\Bigg(\frac{13735835}{663552} + \frac{1507}{4096}\zeta_4 - \frac{11205}{2048}\zeta_5 + \frac{16481}{4096}\zeta_3 - \frac{207901}{82944}n_f \\
&- \frac{33}{256}n_f\zeta_4 + \frac{419}{1152}n_f\zeta_3 + \frac{3139}{124416}n_f^2 + \frac{1}{96}n_f^2\zeta_3 + \frac{3036731}{147456}l_{\mu Q} \\
&- \frac{4921}{1024}l_{\mu Q}\zeta_3 - \frac{253765}{110592}l_{\mu Q}n_f + \frac{3}{64}l_{\mu Q}n_f\zeta_3 + \frac{113}{3456}l_{\mu Q}n_f^2 + \frac{103069}{12288}l^2_{\mu Q} \\
&- \frac{8767}{9216}l^2_{\mu Q}n_f + \frac{23}{1152}l^2_{\mu Q}n_f^2 + \frac{15423}{8192}l^3_{\mu Q} - \frac{203}{1024}l^3_{\mu Q}n_f + \frac{1}{192}l^3_{\mu Q}n_f^2\Bigg)
\end{aligned} \tag{18}$$

$$
\begin{aligned}
C^q_{m^3} =& -\Bigg[1 + a_s(4 + 2l_{\mu Q}) \\
&+ a_s^2\left(\frac{3545}{96} - \frac{2}{3}\zeta_3 - \frac{5}{4}n_f + \frac{641}{24}l_{\mu Q} - \frac{13}{18}l_{\mu Q}n_f + \frac{39}{8}l_{\mu Q}^2 - \frac{1}{12}l_{\mu Q}^2 n_f\right) \\
&+ a_s^3\left(\frac{9287323}{20736} + \frac{493}{768}\zeta_4 + \frac{1975}{54}\zeta_5 - \frac{63643}{864}\zeta_3 - \frac{523}{16}n_f\right. \\
&- \frac{5}{4}n_f\zeta_4 - \frac{55}{216}n_f\zeta_3 + \frac{383}{1944}n_f^2 + \frac{1}{6}n_f^2\zeta_3 + \frac{424327}{1152}l_{\mu Q} \\
&- \frac{241}{64}l_{\mu Q}\zeta_3 - \frac{10375}{432}l_{\mu Q}n_f - \frac{13}{9}l_{\mu Q}n_f\zeta_3 + \frac{20}{81}l_{\mu Q}n_f^2 \\
&\left.+ \frac{7401}{64}l_{\mu Q}^2 - \frac{211}{32}l_{\mu Q}^2 n_f + \frac{2}{27}l_{\mu Q}^2 n_f^2 + \frac{25}{2}l_{\mu Q}^3 - \frac{5}{9}l_{\mu Q}^3 n_f\right)
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
C^q_{mA^2} =& \frac{25}{48}\pi^2 a_s^2\Bigg[1 + a_s\left(\frac{4409}{400} - \frac{373}{900}n_f + \frac{69}{16}l_{\mu Q} - \frac{1}{6}l_{\mu Q}n_f\right) \\
&+ a_s^2\left(\frac{35490283}{230400} - \frac{72037}{4800}\zeta_3 - \frac{2219557}{172800}n_f + \frac{29}{900}n_f\zeta_3\right. \\
&+ \frac{3011}{16200}n_f^2 + \frac{282071}{3200}l_{\mu Q} - \frac{54191}{7200}l_{\mu Q}n_f + \frac{373}{2700}l_{\mu Q}n_f^2 \\
&\left.+ \frac{7797}{512}l_{\mu Q}^2 - \frac{251}{192}l_{\mu Q}^2 n_f + \frac{1}{36}l_{\mu Q}^2 n_f^2\right)
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
C^q_{\bar\psi\psi} =& -\frac{4}{3}\pi^2 a_s\Bigg[1 + a_s\left(\frac{99}{16} - \frac{5}{18}n_f + \frac{7}{4}l_{\mu Q} - \frac{1}{6}l_{\mu Q}n_f\right) \\
&+ a_s^2\left(\frac{13745}{256} - \frac{79}{128}\zeta_3 - \frac{1193}{216}n_f - \frac{5}{6}n_f\zeta_3 + \frac{25}{324}n_f^2 + \frac{2747}{96}l_{\mu Q}\right. \\
&\left.- \frac{559}{144}l_{\mu Q}n_f + \frac{5}{54}l_{\mu Q}n_f^2 + \frac{63}{16}l_{\mu Q}^2 - \frac{2}{3}l_{\mu Q}^2 n_f + \frac{1}{36}l_{\mu Q}^2 n_f^2\right) \\
&+ a_s^3\left(\frac{26331733}{41472} + \frac{79}{256}\zeta_4 - \frac{12166325}{331776}\zeta_5 - \frac{2236285}{82944}\zeta_3\right. \\
&- \frac{403157}{3888}n_f + \frac{5}{12}n_f\zeta_4 + \frac{70}{9}n_f\zeta_5 - \frac{8209}{576}n_f\zeta_3 \\
&+ \frac{722269}{186624}n_f^2 + \frac{301}{432}n_f^2\zeta_3 - \frac{125}{5832}n_f^3 + \frac{3937861}{9216}l_{\mu Q} - \frac{1975}{512}l_{\mu Q}\zeta_3 \\
&- \frac{1097999}{13824}l_{\mu Q}n_f - \frac{3763}{768}l_{\mu Q}n_f\zeta_3 + \frac{39487}{10368}l_{\mu Q}n_f^2 + \frac{5}{12}l_{\mu Q}n_f^2\zeta_3 \\
&- \frac{25}{648}l_{\mu Q}n_f^3 + \frac{28155}{256}l_{\mu Q}^2 - \frac{13255}{576}l_{\mu Q}^2 n_f + \frac{2453}{1728}l_{\mu Q}^2 n_f^2 \\
&\left.- \frac{5}{216}l_{\mu Q}^2 n_f^3 + \frac{609}{64}l_{\mu Q}^3 - \frac{653}{288}l_{\mu Q}^3 n_f + \frac{77}{432}l_{\mu Q}^3 n_f^2 - \frac{1}{216}l_{\mu Q}^3 n_f^3\right)
\end{aligned} \tag{21}
$$

Their numerical form (with $\mu = Q$) reads:

$$
\begin{aligned}
C^q_{m^2} = &- \big[1 + 2.66667 a_s + a_s^2(21.7015 - 0.840278\, n_f) \\
&+ a_s^3(221.404 - 19.7294\, n_f + 0.173079\, n_f^2)\big] &(22)\\
C^q_{A^2} = &- \frac{\pi^2}{3} a_s \big[1 + 0.750000 a_s + a_s^2(2.39159 - 0.178385\, n_f) \\
&+ a_s^3(20.2621 - 2.20883\, n_f + 0.0377513\, n_f^2)\big] &(23)\\
C^q_{m^3} = &- \big[1 + 4.00000 a_s + a_s^2(36.1257 - 1.25000\, n_f) \\
&+ a_s^3(397.959 - 34.3465\, n_f + 0.397359\, n_f^2)\big] &(24)\\
C^q_{mA^2} = &\frac{25\pi^2}{48} a_s^2 \big[1 + a_s(11.0225 - 0.414444\, n_f) \\
&+ a_s^2(135.998 - 12.8059\, n_f + 0.185864\, n_f^2)\big] &(25)\\
C^q_{\bar\psi\psi} = &- \frac{4\pi^2}{3} a_s \big[1 + a_s(6.18750 - 0.277778\, n_f) \\
&+ a_s^2(52.9495 - 6.52486\, n_f + 0.0771605\, n_f^2) \\
&+ a_s^3(564.828 - 112.308\, n_f + 4.70773\, n_f^2 - 0.0214335\, n_f^3)\big] &(26)
\end{aligned}
$$

4 Renormalization Group Improvement

4.1 Anomalous dimensions

We limit ourselves to the three-loop level. Let us start from the operators of dimension two. The corresponding matrix of anomalous dimensions is defined by the following matrix equation

$$
\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \begin{pmatrix} A^2 \\ m^2 \end{pmatrix} = \begin{pmatrix} \gamma_{A^2} & \gamma_{A^2,m^2} \\ 0 & 2\gamma_m \end{pmatrix} \begin{pmatrix} A^2 \\ m^2 \end{pmatrix} , \tag{27}
$$

where the differentiation on the lhs is carried out with fixed bare coupling and quark masses. The quark mass anomalous dimension is known since long [45,46] and the anomalous dimension of A^2 in Landau gauge was found in [47] to be

$$
\begin{aligned}
\gamma_{A^2} = &a_s \left(\frac{35}{16} - \frac{n_f}{6}\right) + a_s^2 \left(\frac{1347}{256} - \frac{137 n_f}{192}\right) \\
&+ a_s^3 \left(\frac{75607}{4096} - \frac{18221 n_f}{4608} + \frac{755 n_f^2}{6912} - \frac{243\zeta_3}{2048} + \frac{33 n_f \zeta_3}{128}\right) \\
&+ a_s^4 \left(\frac{29764511}{393216} - \frac{57858155 n_f}{2654208} + \frac{46549 n_f^2}{41472} + \frac{6613 n_f^3}{746496}\right. \\
&- \frac{99639\zeta_3}{131072} + \frac{335585 n_f\zeta_3}{110592} + \frac{8489 n_f^2\zeta_3}{41472} - \frac{n_f^3\zeta_3}{192} + \frac{8019\zeta_4}{16384} \\
&\left. - \frac{8955 n_f\zeta_4}{8192} + \frac{33 n_f^2\zeta_4}{512} + \frac{40905\zeta_5}{2048} - \frac{3355 n_f\zeta_5}{1024}\right) .
\end{aligned} \tag{28}
$$

The non-diagonal three-loop anomalous dimension γ_{A^2,m^2} reads

$$\gamma_{A^2,m^2} = \frac{a_s}{16\pi^2}\left[24 + a_s\left(\frac{971}{4} - 4\,n_f + 36\,\zeta_3\right)\right]. \tag{29}$$

Life is easier with operators of dimension three. First, the anomalous dimensions of the pair $m\,A^2$ and m^3 are, obviously, additively related to those considered above, namely:

$$\gamma_{m\,A^2} = \gamma_m + \gamma_{A^2}\,, \qquad \gamma_{m^3} = 3\gamma_m\,. \tag{30}$$

Second, in the process of renormalization the quark condensate could mix only with the unit operator (times a quark mass cubed):

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\begin{pmatrix}\bar\psi\psi\\ m^3\end{pmatrix} = \begin{pmatrix}\gamma_{\bar\psi\psi} & \gamma_{\bar\psi\psi,m^3}\\ 0 & 3\gamma_m\end{pmatrix}\begin{pmatrix}\bar\psi\psi\\ m^3\end{pmatrix}. \tag{31}$$

The fact that

$$\gamma_{\bar\psi\psi} \equiv -\gamma_m \tag{32}$$

is well-known from text-books. The non-diagonal part of the mixing was investigated in detail a long time ago [48, 49]. It is naturally expressed in terms of the so-called vacuum anomalous dimension, γ_0^d as follows [50]:

$$\mu^2\frac{\mathrm{d}}{d\mu^2}\bar\psi\psi = -\gamma_m\,\bar\psi\psi - 4m^3\,\gamma_0^d(a_s), \tag{33}$$

with

$$\gamma_0^d = -\frac{3}{16\pi^2}\left[1 + \frac{4}{3}a_s + \left(\frac{313}{72} - \frac{5}{12}n_f - \frac{2}{3}\zeta_3\right)a_s^2\right].$$

4.2 Scheme-independent correlators and operators

In general a Green function G depends on the renormalization prescription (*scheme*) and the choice of the artificial scale μ. It is, however, well-known how to define a variant $\hat G$ of G which is invariant under changes of the renormalization scheme and μ. The corresponding renormalization group equation (RGE)

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\hat G = 0 \tag{34}$$

has the formal solution

$$\hat G = G(a_s,\mu)/f(a_s), \qquad f(a_s) = \exp\left(\int\frac{\mathrm{d}a_s}{a_s}\frac{\gamma_G}{\beta}\right). \tag{35}$$

The OPE of a (suitable) scale-invariant Green function can be rewritten in terms of scale-invariant operators $\hat{\mathcal{O}}_i$ and Wilson coefficients $\hat C_i$, which again obey RGEs of the form (34).

Here, we consider the OPEs of the scheme-independent dressing functions in the limit of massless quarks. In this limit — aside from the perturbative contributions — only the operators A^2 and $\bar{\psi}\psi$ contribute. For the operator A^2 and its coefficient functions we obtain (in the three-loop approximation)

$$\widehat{A^2}\big|_{n_f=0} = a_s^{-\frac{35}{44}}(1 + 0.0693440\, a_s + 0.0240863\, a_s^2 + 0.405494\, a_s^3)\, A^2\,, \tag{36}$$

$$\widehat{A^2}\big|_{n_f=2} = a_s^{-\frac{89}{116}}(1 + 0.0654912\, a_s + 0.0933818\, a_s^2 + 0.508904\, a_s^3)\, A^2\,, \tag{37}$$

$$\widehat{A^2}\big|_{n_f=3} = a_s^{-\frac{3}{4}}(1 + 0.0538194\, a_s + 0.136131\, a_s^2 + 0.570436\, a_s^3)\, A^2\,, \tag{38}$$

$$\hat{C}^g_{A^2}\big|_{n_f=0} = \frac{3}{8}\,\pi^2 a_s^{\frac{9}{44}}(a_s + 8.24643\, a_s^2 + 87.5512\, a_s^3 + 1075.32\, a_s^4)\,, \tag{39}$$

$$\hat{C}^g_{A^2}\big|_{n_f=2} = \frac{3}{8}\,\pi^2 a_s^{\frac{27}{116}}\left(a_s + 7.02035\, a_s^2 + 61.7518\, a_s^3 + 647.400\, a_s^4\right)\,, \tag{40}$$

$$\hat{C}^g_{A^2}\big|_{n_f=3} = \frac{3}{8}\,\pi^2 a_s^{\frac{1}{4}}(a_s + 6.39757\, a_s^2 + 49.9224\, a_s^3 + 472.744\, a_s^4)\,, \tag{41}$$

$$\hat{C}^h_{A^2}\big|_{n_f=0} = \frac{3}{8}\,\pi^2 a_s^{\frac{13}{22}}(a_s + 3.61131\, a_s^2 + 29.4702\, a_s^3 + 334.048\, a_s^4)\,, \tag{42}$$

$$\hat{C}^h_{A^2}\big|_{n_f=2} = \frac{3}{8}\,\pi^2 a_s^{\frac{31}{58}}\left(a_s + 3.61902\, a_s^2 + 26.4370\, a_s^3 + 260.012\, a_s^4\right)\,, \tag{43}$$

$$\hat{C}^h_{A^2}\big|_{n_f=3} = \frac{3}{8}\,\pi^2\,\sqrt{a_s}(a_s + 3.64236\, a_s^2 + 24.9740\, a_s^3 + 225.345\, a_s^4)\,, \tag{44}$$

$$\hat{C}^q_{A^2}\big|_{n_f=0} = -\,\frac{\pi^2}{3} a_s^{\frac{35}{44}}(a_s + 0.173080\, a_s^2 + 1.19104\, a_s^3 + 15.9766\, a_s^4)\,, \tag{45}$$

$$\hat{C}^q_{A^2}\big|_{n_f=2} = -\,\frac{\pi^2}{3} a_s^{\frac{89}{116}}\left(a_s + 0.175888\, a_s^2 + 0.859625\, a_s^3 + 12.6022\, a_s^4\right)\,, \tag{46}$$

$$\hat{C}^q_{A^2}\big|_{n_f=3} = -\,\frac{\pi^2}{3} a_s^{\frac{3}{4}}(a_s + 0.186921\, a_s^2 + 0.689610\, a_s^3 + 11.0348\, a_s^4) \tag{47}$$

by inserting the corresponding anomalous dimensions into Eq. (35). The scale- and scheme-independent versions of $\bar{\psi}\psi$ and its coefficient function in the OPE of the quark propagator read

$$\widehat{\bar{\psi}\psi}\big|_{n_f=0} = a_s^{-\frac{4}{11}}(1 + 0.687328\, a_s + 1.51211\, a_s^2 + 4.05787\, a_s^3)\, \bar{\psi}\psi\,, \tag{48}$$

$$\widehat{\bar{\psi}\psi}\big|_{n_f=2} = a_s^{-\frac{12}{29}}(1 + 0.805985\, a_s + 1.40095\, a_s^2 + 2.72916\, a_s^3)\, \bar{\psi}\psi\,, \tag{49}$$

$$\widehat{\bar{\psi}\psi}\big|_{n_f=3} = a_s^{-\frac{4}{9}}(1 + 0.895062\, a_s + 1.37143\, a_s^2 + 1.95168\, a_s^3)\, \bar{\psi}\psi\,, \tag{50}$$

$$\hat{C}^q_{\bar{\psi}\psi}\big|_{n_f=0} = -\,\frac{4\pi^2}{3} a_s^{\frac{4}{11}}(a_s + 4.99260\, a_s^2 + 44.0815\, a_s^3 + 489.206\, a_s^4)\,, \tag{51}$$

$$\hat{C}^q_{\bar{\psi}\psi}\big|_{n_f=2} = -\,\frac{4\pi^2}{3} a_s^{\frac{12}{29}}\left(a_s + 4.31734\, a_s^2 + 31.7744\, a_s^3 + 298.894\, a_s^4\right)\,, \tag{52}$$

$$\hat{C}^q_{\bar{\psi}\psi}\big|_{n_f=3} = -\,\frac{4\pi^2}{3} a_s^{\frac{4}{9}}(a_s + 3.94985\, a_s^2 + 25.7972\, a_s^3 + 217.583\, a_s^4)\,. \tag{53}$$

Another useful scheme-invariant object is the so-called "effective quark mass" $m_P(Q)$ which is defined as follows [10].

$$\mathrm{i}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx}\langle \mathrm{T}[\psi(x)\bar{\psi}(0)]\rangle = \frac{1}{B - A\ \not{q}}, \quad m_P(Q) = \frac{B(q)}{A(q)} = \frac{S(Q)}{V(Q)}, \tag{54}$$

where we have used Eq. (54) to express $m_P(Q)$ in terms of the dressing functions $V(Q)$ and $S(Q)$.

In the chiral limit the leading contribution to $m_P(Q)$ comes from the quark condensate; in explicit form we get

$$m_P(Q) = C^q_{\bar{\psi}\psi} V_0^{-1} \frac{\langle\bar{\psi}\psi\rangle}{Q^2}\,. \tag{55}$$

Using the results of Section 3 we arrive at:

$$m_P(Q)|_{n_f=0} = -\frac{4\pi^2}{3}a_s\Bigg[1 + a_s\left(\frac{99}{16}\right) + a_s^2\left(\frac{129449}{2304} - \frac{175}{128}\zeta_3\right) \tag{56}$$

$$+ a_s^3\left(\frac{28729643}{41472} - \frac{10153205}{331776}\zeta_5 - \frac{4351141}{82944}\zeta_3\right)\Bigg]\frac{\langle\bar{\psi}\psi\rangle}{Q^2}\,, \tag{57}$$

$$m_P(Q)|_{n_f=1} = -\frac{4\pi^2}{3}a_s\Bigg[1 + a_s\left(\frac{851}{144}\right) + a_s^2\left(\frac{1049089}{20736} - \frac{845}{384}\zeta_3\right) \tag{58}$$

$$+ a_s^3\left(\frac{72992597}{124416} - \frac{7572725}{331776}\zeta_5 + \frac{5}{12}\zeta_4 - \frac{5394805}{82944}\zeta_3\right)\Bigg]\frac{\langle\bar{\psi}\psi\rangle}{Q^2}\,, \tag{59}$$

$$m_P(Q)|_{n_f=2} = -\frac{4\pi^2}{3}a_s\Bigg[1 + a_s\left(\frac{811}{144}\right) + a_s^2\left(\frac{936337}{20736} - \frac{1165}{384}\zeta_3\right) \tag{60}$$

$$+ a_s^3\left(\frac{182335471}{373248} - \frac{4992245}{331776}\zeta_5 + \frac{5}{6}\zeta_4 - \frac{6322885}{82944}\zeta_3\right)\Bigg]\frac{\langle\bar{\psi}\psi\rangle}{Q^2}\,, \tag{61}$$

$$m_P(Q)|_{n_f=3} = -\frac{4\pi^2}{3}a_s\Bigg[1 + a_s\left(\frac{257}{48}\right) + a_s^2\left(\frac{91865}{2304} - \frac{495}{128}\zeta_3\right) \tag{62}$$

$$+ a_s^3\left(\frac{611489}{1536} - \frac{2411765}{331776}\zeta_5 + \frac{5}{4}\zeta_4 - \frac{7135381}{82944}\zeta_3\right)\Bigg]\frac{\langle\bar{\psi}\psi\rangle}{Q^2}\,, \tag{63}$$

where we have set the renormalization scale $\mu = Q$. Numerically these equations read:

$$m_P(Q)|_{n_f=0} = -\frac{4\pi^2}{3}a_s(1 + 6.1875\, a_s + 54.541\, a_s^2 + 597.957\, a_s^3)\frac{\langle\bar{\psi}\psi\rangle}{Q^2}\,, \tag{64}$$

$$m_P(Q)|_{n_f=1} = -\frac{4\pi^2}{3}a_s(1 + 5.90972\, a_s + 47.9475\, a_s^2 + 485.281\, a_s^3)\frac{\langle\bar{\psi}\psi\rangle}{Q^2}\,, \tag{65}$$

$$m_P(Q)|_{n_f=2} = -\frac{4\pi^2}{3}a_s(1 + 5.63194\, a_s + 41.5083\, a_s^2 + 382.176\, a_s^3)\frac{\langle\bar{\psi}\psi\rangle}{Q^2}\,, \tag{66}$$

$$m_P(Q)|_{n_f=3} = -\frac{4\pi^2}{3}a_s(1 + 5.35417\, a_s + 35.2234\, a_s^2 + 288.511\, a_s^3)\frac{\langle\bar{\psi}\psi\rangle}{Q^2}\,. \tag{67}$$

In analogy to the effective quark mass one can also define effective masses for gluon and ghost fields, which are induced by the gluon mass condensate $\langle A^2\rangle$. An explicit formula can be derived by considering the ghost propagator (or the gluon propagator) in the chiral limit:

$$\Delta^{ab}(q) = \frac{\delta^{ab}}{Q^2} \quad \left(D_0^h(Q) + C_{A^2}^h(Q)\frac{\langle A^2\rangle}{Q^2}\right) \approx \frac{\delta^{ab}}{Q^2 - C_{A^2}^h(Q)/D_0^h(Q)\langle A^2\rangle} \quad D_0^h(Q)\,. \tag{68}$$

The effective masses are then given by

$$m_?^2(Q) = -\frac{C_{A^2}^?(Q)}{D_0^?(Q)}\langle A^2\rangle\,, \tag{69}$$

where ? stands for g or h. The analytic results for $\mu = Q$ and $n_f = 1,\,2,\,3$ are

$$\begin{aligned}
m_g^2|_{n_f=0} =& -\frac{3}{8}\pi^2 a_s\Bigg[1 + a_s\left(\frac{197}{32}\right) + a_s^2\left(\frac{480587}{9216} + \frac{243}{128}\zeta_3\right)\\
&+ a_s^3\left(\frac{520248245}{884736} - \frac{82215}{4096}\zeta_5 + \frac{243}{4096}\zeta_4 + \frac{331077}{8192}\zeta_3\right)\\
&+ a_s^4\Bigg(- \frac{38766561211}{7077888} + \frac{25948995}{131072}\zeta_5\\
&+ \frac{35721}{32768}\zeta_4 + \frac{57028439}{131072}\zeta_3 + \frac{729}{2048}\zeta_3^2\Bigg)\Bigg]\langle A^2\rangle\,,
\end{aligned}\tag{70}$$

$$\begin{aligned}
m_g^2|_{n_f=1} =& -\frac{3}{8}\pi^2 a_s\Bigg[1 + a_s\left(\frac{559}{96}\right) + a_s^2\left(\frac{3836375}{82944} + \frac{195}{128}\zeta_3\right)\\
&+ a_s^3\left(\frac{11737602763}{23887872} - \frac{74855}{4096}\zeta_5 - \frac{285}{4096}\zeta_4 + \frac{7127207}{221184}\zeta_3\right)\\
&+ a_s^4\Bigg(- \frac{2182125807293}{573308928} + \frac{1465676035}{10616832}\zeta_5\\
&- \frac{38475}{32768}\zeta_4 + \frac{12243145349}{31850496}\zeta_3 - \frac{8075}{18432}\zeta_3^2\Bigg)\Bigg]\langle A^2\rangle\,,
\end{aligned}\tag{71}$$

$$\begin{aligned}
m_g^2|_{n_f=2} =& -\frac{3}{8}\pi^2 a_s\Bigg[1 + a_s\left(\frac{527}{96}\right) + a_s^2\left(\frac{3368459}{82944} + \frac{147}{128}\zeta_3\right)\\
&+ a_s^3\left(\frac{9610131071}{23887872} - \frac{67495}{4096}\zeta_5 - \frac{813}{4096}\zeta_4 + \frac{5402375}{221184}\zeta_3\right)\\
&+ a_s^4\Bigg(- \frac{477599196341}{191102976} + \frac{916103635}{10616832}\zeta_5\\
&- \frac{99999}{32768}\zeta_4 + \frac{3506053415}{10616832}\zeta_3 - \frac{24119}{18432}\zeta_3^2\Bigg)\Bigg]\langle A^2\rangle\,,
\end{aligned}\tag{72}$$

$$
\begin{aligned}
m_g^2|_{n_f=3} = &-\frac{3}{8}\pi^2 a_s\bigg[1+a_s\left(\frac{165}{32}\right)+a_s^2\left(\frac{108205}{3072}+\frac{99}{128}\zeta_3\right)\\
&+a_s^3\left(\frac{31529153}{98304}-\frac{60135}{4096}\zeta_5-\frac{1341}{4096}\zeta_4+\frac{139429}{8192}\zeta_3\right)\\
&+a_s^4\bigg(-\frac{10684895233}{7077888}+\frac{16783385}{393216}\zeta_5-\\
&\frac{148851}{32768}\zeta_4+\frac{322973567}{1179648}\zeta_3-\frac{4619}{2048}\zeta_3^2\bigg)\bigg]\langle A^2\rangle\,,
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
m_h^2|_{n_f=0} = &-\frac{3}{8}\pi^2 a_s\bigg[1+3\,a_s+a_s^2\left(\frac{20997}{1024}+\frac{351}{256}\zeta_3\right)\\
&+a_s^3\left(\frac{4345483}{18432}-\frac{53235}{4096}\zeta_5+\frac{243}{4096}\zeta_4+\frac{24945}{1024}\zeta_3\right)\\
&+a_s^4\left(-\frac{1004316139}{1572864}+\frac{222885}{16384}\zeta_5-\frac{5103}{16384}\zeta_4+\frac{4937751}{131072}\zeta_3+\frac{3645}{8192}\zeta_3^2\right)\bigg]\langle A^2\rangle\,,
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
m_h^2|_{n_f=1} = &-\frac{3}{8}\pi^2 a_s\bigg[1+3\,a_s+a_s^2\left(\frac{19881}{1024}+\frac{351}{256}\zeta_3\right)\\
&+a_s^3\left(\frac{34790023}{165888}-\frac{53235}{4096}\zeta_5-\frac{285}{4096}\zeta_4+\frac{69767}{3072}\zeta_3\right)\\
&+a_s^4\left(-\frac{8001655387}{14155776}+\frac{222885}{16384}\zeta_5+\frac{5985}{16384}\zeta_4+\frac{4761971}{131072}\zeta_3+\frac{3645}{8192}\zeta_3^2\right)\bigg]\langle A^2\rangle\,,
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
m_h^2 = |_{n_f=2} &-\frac{3}{8}\pi^2 a_s\bigg[1+3\,a_s+a_s^2\left(\frac{18765}{1024}+\frac{351}{256}\zeta_3\right)\\
&+a_s^3\left(\frac{30638707}{165888}-\frac{53235}{4096}\zeta_5-\frac{813}{4096}\zeta_4+\frac{64763}{3072}\zeta_3\right)\\
&+a_s^4\left(-\frac{7006421587}{14155776}+\frac{222885}{16384}\zeta_5+\frac{17073}{16384}\zeta_4+\frac{4571855}{131072}\zeta_3+\frac{3645}{8192}\zeta_3^2\right)\bigg]\langle A^2\rangle\,,
\end{aligned}
\tag{76}
$$

$$
\begin{aligned}
m_h^2|_{n_f=3} = &-\frac{3}{8}\pi^2 a_s\bigg[1+3\,a_s+a_s^2\left(\frac{17649}{1024}+\frac{351}{256}\zeta_3\right)\\
&+a_s^3\left(\frac{329079}{2048}-\frac{53235}{4096}\zeta_5-\frac{1341}{4096}\zeta_4+\frac{19941}{1024}\zeta_3\right)\\
&+a_s^4\left(-\frac{224190513}{524288}+\frac{222885}{16384}\zeta_5+\frac{28161}{16384}\zeta_4+\frac{4367403}{131072}\zeta_3+\frac{3645}{8192}\zeta_3^2\right)\bigg]\langle A^2\rangle\,.
\end{aligned}
\tag{77}
$$

In numerical form we obtain

$$m_g^2|_{n_f=0} = -\frac{3}{8}\pi^2 a_s[1 + 6.15625\, a_s + 54.4291\, a_s^2 + 615.858\, a_s^3 - 4747.15\, a_s^4]\langle A^2\rangle\,, \quad (78)$$

$$m_g^2|_{n_f=1} = -\frac{3}{8}\pi^2 a_s[1 + 5.82292\, a_s + 48.0839\, a_s^2 + 511.071\, a_s^3 - 3202.89\, a_s^4]\langle A^2\rangle\,, \quad (79)$$

$$m_g^2|_{n_f=2} = -\frac{3}{8}\pi^2 a_s[1 + 5.48958\, a_s + 41.9917\, a_s^2 + 414.36\, a_s^3 - 2017.93\, a_s^4]\langle A^2\rangle\,, \quad (80)$$

$$m_g^2|_{n_f=3} = -\frac{3}{8}\pi^2 a_s[1 + 5.15625\, a_s + 36.1527\, a_s^2 + 325.612\, a_s^3 - 1144.42\, a_s^4]\langle A^2\rangle\,, \quad (81)$$

$$m_h^2|_{n_f=0} = -\frac{3}{8}\pi^2 a_s[1 + 3\, a_s + 22.153\, a_s^2 + 251.628\, a_s^3 - 578.831\, a_s^4]\langle A^2\rangle\,, \quad (82)$$

$$m_h^2|_{n_f=1} = -\frac{3}{8}\pi^2 a_s[1 + 3\, a_s + 21.0632\, a_s^2 + 223.467\, a_s^3 - 506.441\, a_s^4]\langle A^2\rangle\,, \quad (83)$$

$$m_h^2|_{n_f=2} = -\frac{3}{8}\pi^2 a_s[1 + 3\, a_s + 19.9733\, a_s^2 + 196.345\, a_s^3 - 437.146\, a_s^4]\langle A^2\rangle\,, \quad (84)$$

$$m_h^2|_{n_f=3} = -\frac{3}{8}\pi^2 a_s[1 + 3\, a_s + 18.8835\, a_s^2 + 170.26\, a_s^3 - 370.947\, a_s^4]\langle A^2\rangle\,. \quad (85)$$

5 Theoretical and hardware tools

In our work we have heavily used the software packages QGRAF [51], EXP [52] and a modern version of MINCER [53,54] written in the algebraic computer language FORM 3 [55] for the generation and calculation of the required diagrams.

The calculations have have been performed in a general covariant gauge and for the gauge group SU(n). The total number of diagrams contributing to different channels (according to QGRAF) are displayed in Table 1. The files with all results (in a computer readable form) can be downloaded from `http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp09/ttp09-40/`

Any coefficient function of any operator entering into the OPE of two local operators can be expressed in terms of massless propagators. The reduction to massless propagators is conveniently done with the well-known method of projectors [56,57].

Finally, the package MINCER is able to compute very effectively massless propagators up to (and including) three loop level.

6 Conclusion

We have computed the coefficient functions of the operators of dimension two and three in the OPE for the gluon, ghost and quark propagators. The higher order corrections are essential as one could see by inspecting eqs. (36) -(53). They are most important in two cases: gluon condensate contributions to ghost and gluon propagators as well as the quark condensate one to the quark propagator. In general the terms proportional to n_f tend to significantly stabilize the perturbative series by decreasing the value of higher order terms (cmp. e.g. eq.(39) and (42)).

	tree	one loop	two loops	three loops
$C^g_{m^2}$	1	5	59	1148
$C^g_{A^2}$	6	222	7407	264399
$C^h_{m^2}$	1	1	9	148
$C^h_{A^2}$	2	23	595	19419
$C^q_{m^n}$	1	1	9	148
$C^q_{m^n A^2}$	2	23	657	23251
$C^q_{\bar\psi\psi}$	1	11	234	6641

Table 1: Number of diagrams contributing to the various coefficient functions

Specific numerical analysis should be made with specific lattice data. Still, we observe that the higher order corrections to the coeffcient functions display relatively good (apparent) convergency pattern with basically positive coefficients which, presumably, should lead to a noticeable decrease of the value of the A^2 condensate once the lattice data are reanalyzed with an account of newly computed terms in the corresponding OPE.

Note that such dependence of the numerical value of the gluon mass condensate on the number of perturbative terms accounted in the corresponding OPE gives an extra support to the the hypothesis of duality between perturbative and non-perturbative contributions (see a very recent work [58] and references therein).

Finally, we hope that our results will be of use for better understanding of the present and future data coming from lattice simulations of QCD propagators.

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A Several massive quarks

Until now, all OPEs in this work have been formulated for the case of (at most) one massive quark with mass m. From a physical point of view, this is a valid approximation: corrections from u and d masses are in general negligible, while all other quark masses show a strong hierarchy.

Nevertheless, it is also possible to generalize our results to the case of n_f massive quarks with masses m_1, m_2, $\ldots, m_{n_f}$. The generalization of the OPEs of the gluon and ghost propagators is straightforward: We replace m^2 by the sum over m_i^2:

$$D^?(Q^2) = D_0^?(\mu/Q, a_s) + \frac{C^?_{A^2}(\mu/Q, a_s)}{Q^2}\langle A^2\rangle + \sum_{i=1}^{n_f} \frac{C^?_{m_i^2}(\mu/Q, a_s)}{Q^2} m_i^2\,, \qquad (86)$$

where $C^?_{m_i^2}$ is the same as $C^?_{m^2}$ from Eqs. (6) and (8).

The case of the quark propagator is a bit more complicated. Without loss of generality we assume that the external quark has the mass m_1. The correspondingly generalized OPE for the dressing functions $V(Q)$ and $S(Q)$ defined in Eq. (14) then read

$$V(Q) = V_0(\mu/Q, a_s) + \sum_{i=1}^{n_f} \frac{C^q_{m_i^2}(\mu/Q, a_s)}{Q^2} m_i^2 + \frac{C^q_{A^2}(\mu/Q, a_s)}{Q^2}\langle A^2\rangle \tag{87}$$

and

$$\begin{aligned} S(Q) = S_0(\mu/Q, a_s)m_1 + \sum_{i=1}^{n_f} \frac{C^q_{m_i^2 m_1}(\mu/Q, a_s)}{Q^2} m_i^2 m_1 \\ + \frac{C^q_{m_1 A^2}(\mu/Q, a_s)}{Q^2}\langle m_1 A^2\rangle + \frac{C^q_{\bar\psi\psi}(\mu/Q, a_s)}{Q^2}\langle\bar\psi\psi\rangle \,. \end{aligned} \tag{88}$$

Note that for $i \neq 1$ only even powers of m_i may appear in the OPEs.

The Wilson Coefficients $C^q_{m_1^2}$, $C^q_{m_1^3}$ and $C^q_{m_1 A^2}$ are identical to $C^q_{m^2}$, $C^q_{m^3}$ and $C^q_{mA^2}$ as given in Eqs. (17), (19) and (20). For $i \neq 1$, the anomalous dimension matrix of (31) has to be extended to account for the mixing of $\bar\psi\psi$ with $m_i^2 m_1$. The corresponding anomalous dimension $\gamma_{\bar\psi\psi, m_i^2 m_1}$ can be found in Ref. [50]. The results for the new coefficient functions are (note that below $i \neq 1$!)

$$\begin{aligned} C^q_{m_i^2} &= a_s^2\left(\frac{5}{6} - \frac{1}{2}L_Q\right) \\ &\quad + a_s^3\left(\frac{63659}{3456} - \frac{2243}{192}L_Q + \frac{129}{64}L_Q^2 - \frac{5}{2}\zeta_5 + \frac{3}{8}\zeta_4\right. \\ &\quad\quad \left. -\frac{1}{8}\zeta_3 - \frac{3}{4}\zeta_3 L_Q - \frac{37}{72}n_f + \frac{13}{36}n_f L_Q - \frac{1}{12}n_f L_Q^2\right) \,, \end{aligned} \tag{89}$$

$$\begin{aligned} C^q_{m_i^2 m_1} &= a_s^2\left(\frac{3}{2}\right) + a_s^3\left(\frac{17143}{576} - \frac{319}{32}L_Q\right. \\ &\quad\quad \left. -5\zeta_5 + \frac{41}{4}\zeta_3 - \frac{3}{4}n_f + \frac{1}{2}n_f L_Q\right) \,. \end{aligned} \tag{90}$$

References

[1] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, Phys. Lett. B400 (1997) 379, hep-ph/9701390.

[2] M. Czakon, Nucl. Phys. B710 (2005) 485, hep-ph/0411261.

[3] J.A.M. Vermaseren, S.A. Larin and T. van Ritbergen, Phys. Lett. B405 (1997) 327, hep-ph/9703284.

[4] K.G. Chetyrkin, Phys. Lett. B404 (1997) 161, hep-ph/9703278.

[5] K.G. Chetyrkin, Nucl. Phys. B710 (2005) 499, hep-ph/0405193.

[6] C.D. Roberts and A.G. Williams, Prog. Part. Nucl. Phys. 33 (1994) 477, hep-ph/9403224.

[7] P. Maris and C.D. Roberts, Int. J. Mod. Phys. E12 (2003) 297, nucl-th/0301049.

[8] R. Alkofer and L. von Smekal, Phys. Rept. 353 (2001) 281, hep-ph/0007355.

[9] K.G. Wilson, Phys. Rev. 179 (1969) 1499.

[10] H.D. Politzer, Nucl. Phys. B117 (1976) 397.

[11] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B147 (1979) 385.

[12] E. Ruiz Arriola, P.O. Bowman and W. Broniowski, Phys. Rev. D70 (2004) 097505, hep-ph/0408309.

[13] V. Gimenez et al., Eur. Phys. J. C41 (2005) 535, hep-lat/0503001.

[14] A. Cucchieri and T. Mendes, Braz. J. Phys. 37 (2007) 484, hep-ph/0605224.

[15] P. Boucaud et al., Phys. Rev. D79 (2009) 014508, 0811.2059.

[16] M.J. Lavelle and M. Schaden, Phys. Lett. B208 (1988) 297.

[17] P. Boucaud et al., Phys. Lett. B493 (2000) 315, hep-ph/0008043.

[18] P. Boucaud et al., Phys. Rev. D63 (2001) 114003, hep-ph/0101302.

[19] P. Boucaud et al., JHEP 01 (2006) 037, hep-lat/0507005.

[20] P. Boucaud et al., Phys. Rev. D66 (2002) 034504, hep-ph/0203119.

[21] P. Boucaud et al., Phys. Rev. D70 (2004) 114503, hep-ph/0312332.

[22] P. Boucaud et al., Phys. Rev. D74 (2006) 034505, hep-lat/0504017.

[23] F. De Soto and J. Rodriguez-Quintero, Phys. Rev. D64 (2001) 114003, hep-ph/0105063.

[24] M. Lavelle and M. Oleszczuk, Mod. Phys. Lett. A7 (1992) 3617.

[25] P.O. Bowman et al., Nucl. Phys. Proc. Suppl. 128 (2004) 23, hep-lat/0403002.

[26] P.O. Bowman et al., Nucl. Phys. Proc. Suppl. 161 (2006) 27.

[27] G.M. Cicuta and E. Montaldi, Nuovo Cim. Lett. 4 (1972) 329.

[28] J.F. Ashmore, Lett. Nuovo Cim. 4 (1972) 289.

[29] G. 't Hooft and M.J.G. Veltman, Nucl. Phys. B44 (1972) 189.

[30] G. 't Hooft, Nucl. Phys. B61 (1973) 455.

[31] K.G. Chetyrkin, F.V. Tkachov and S.G. Gorishnii, Phys. Lett. B119 (1982) 407.

[32] F.V. Tkachov, Phys. Lett. B124 (1983) 212.

[33] F.V. Gubarev, L. Stodolsky and V.I. Zakharov, Phys. Rev. Lett. 86 (2001) 2220, hep-ph/0010057.

[34] F.V. Gubarev and V.I. Zakharov, Phys. Lett. B501 (2001) 28, hep-ph/0010096.

[35] K.I. Kondo, Phys. Lett. B619 (2005) 377, hep-th/0504088.

[36] K.I. Kondo, Phys. Rev. D74 (2006) 125003, hep-th/0609166.

[37] R.N. Baranov, D.V. Bykov and A.A. Slavnov, Theor. Math. Phys. 148 (2006) 1168, hep-th/0601142.

[38] E.R. Arriola and W. Broniowski, Eur. Phys. J. A31 (2007) 739, hep-ph/0609266.

[39] D. Dudal et al., Braz. J. Phys. 37 (2007) 232, hep-th/0609184.

[40] S.P. Sorella et al., PoS IC2006 (2006) 026, hep-th/0610306.

[41] J.A. Gracey, JHEP 05 (2006) 052, hep-ph/0605077.

[42] M.A.L. Capri et al., Phys. Rev. D74 (2006) 045008, hep-th/0605288.

[43] M.N. Chernodub and E.M. Ilgenfritz, Phys. Rev. D78 (2008) 034036, 0805.3714.

[44] K.G. Chetyrkin and A. Retey, Nucl. Phys. B583 (2000) 3, hep-ph/9910332.

[45] O.V. Tarasov, JINR-P2-82-900.

[46] S.A. Larin and J.A.M. Vermaseren, Phys. Lett. B303 (1993) 334, hep-ph/9302208.

[47] J.A. Gracey, Phys. Lett. B552 (2003) 101, hep-th/0211144.

[48] V.P. Spiridonov and K.G. Chetyrkin, Sov. J. Nucl. Phys. 47 (1988) 522.

[49] K.G. Chetyrkin et al., Phys. Rev. D51 (1995) 5090, hep-ph/9409371.

[50] K.G. Chetyrkin and J.H. Kühn, Nucl. Phys. B432 (1994) 337, hep-ph/9406299.

[51] P. Nogueira, J. Comput. Phys. 105 (1993) 279.

[52] T. Seidensticker, (1999), hep-ph/9905298.

[53] S.G. Gorishnii et al., Comput. Phys. Commun. 55 (1989) 381.

[54] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, NIKHEF-H-91-18.

[55] J.A.M. Vermaseren, (2000), math-ph/0010025.

[56] S.G. Gorishnii, S.A. Larin and F.V. Tkachov, Phys. Lett. B124 (1983) 217.

[57] S.G. Gorishnii and S.A. Larin, Nucl. Phys. B283 (1987) 452.

[58] S. Narison and V.I. Zakharov, (2009), 0906.4312.