New Methods for the Calculation of Multi-Loop Amplitudes

P. Marquard

Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76131 Karlsruhe, Germany

We present a brief review of current methods for the calculation of multi-loop amplitudes including recent developments. As an example we present the calculation of the second moment of the heavy quark current correlator and the extraction of the values of the charm and bottom quark masses using the results of this calculation.

I. INTRODUCTION AND MOTIVATION

In the era of the LHC and in preparation for the ILC there is need for multi-loop calculations to match the precision of these experiments from the theory side. The classes of problems currently accessible reach from multi-leg multi-scale problems at one-loop up to no-scale propagator, and tadpole diagrams at four-loop level. Many problems fall into these categories directly while others can be cast into the required form by appropriate expansions in small parameters. A typical multi-loop calculation proceeds as follows: The needed diagrams are generated and if necessary expanded to decrease the difficulty of the problem. The resulting Feynman integrals are then reduced to a small set of simpler integrals, which have to be calculated in a further step. While the reduction of the integrals can be automatized completely and is mostly independent of the problem at hand, the calculation of the remaining integrals must still be done manually for every calculation. In the following we give a brief review of the current status of the reduction to master integrals which with recent developments is now in a good shape such that the calculation of the master integrals now again poses the most difficulties in multi-loop calculations.

II. TECHNICALITIES

One of the main ingredients of many multi-loop calculations are the so-called Integration-By-Parts (IBP) identities [1]

$$0 = \int d^n l_1 \cdots d^n l_m \frac{\partial}{\partial l_j^\mu} \frac{\{l_i^\mu , p_i^\mu\}}{P_1^{k_1} \cdots P_n^{k_n}} \,, \tag{1}$$

which hold in dimensional regularization. Here we consider typical Feynman integrals, where l_i are loop and p_i are external momenta. These identities can be used to relate different integrals. Using these relations the full set of needed integrals, which can be of the order of $10^4 - 10^6$ integrals, can be reduced to a small set of so-called master integrals. In a typical calculation there appear $\mathcal{O}(10 - 100)$ master integrals, which remain to be calculated directly. Another main advantage of the reduction to master integrals is that many properties like gauge invariance can be checked on the level of master integrals, since these are linearly independent.

To make use of the IBP identities the Laporta algorithm [2], a Gauss-elimination-like algorithm, is frequently used. In order to use the Laporta algorithm, one has to choose an ordering to ensure that more difficult integrals are expressed through simpler ones. Once this ordering has been chosen the algorithm proceeds as follows: Choose a set of integrals, which you like to solve; pick an integral from the set and generate an IBP; solve for the most difficult integral in the equation and continue with the next IBP or integral. The system of equations will become overdetermined if one chooses a sufficiently large set of integrals, since there are $N_L(N_L + 2N_E + 1)/2$ IBP identities generated by a given integral, where N_L and N_E are the number of loops and legs, respectively.

Although this method has been successfully applied in many calculations it has some drawbacks. The complexity of the Laporta algorithm is $\mathcal{O}(n^3)$, where n is the number of equations. In combination with the combinatorial properties of the problem this leads to fast growth of the size of the problem to be solved. Furthermore in typical applications the system of equations is overdetermined by a factor 3-5, so quite some time is spent just checking that the system of equations is consistent.

A different approach for solving the set of IBP identities is provided by Groebner bases. Groebner bases arose in the context of ideals of rings over multivariate polynomials. More precisely, they were introduced while trying to solve the problem, how to decide if a certain polynomial belongs to an ideal of a ring of polynomials. An ideal I is a subset of the ring R such that for every $q \in I, r \in R$, the product is an element of I, $q \cdot r \in I$.

An ideal can be generated by a set of polynomials $Q = \{q_1, \ldots, q_n\}$ using $I = \{p \in R | p = \sum r_i q_i, r_i \in R\}$. Given these generating polynomials, the question whether a given polynomial $p \in R$ is an element of the ideal, can easily be answered in the case of univariate polynomials, where the operation of polynomial division is unambiguous. If the polynomial division by the generating polynomials leaves no remainder, p is an element of the ideal.

This statement does no longer hold, when dealing with multivariate polynomials. Here the remainder of multivariate divisions by multiple polynomials depends on the chosen ordering of the polynomials. Therefore if the remainder of such divisions is not zero one may not conclude that the polynomial is not an element of the ideal.

At this point Groebner bases prove very useful since a multivariate division with respect to elements of a Groebner basis is again unambiguous. This leads to one of the definitions of a Groebner basis: A set $B \subset R$ is a Groebner basis if the reduction, i.e. multivariate division, of any $r \in R$ with respect to the elements of B yields the same remainder independent of the chosen order. Using the Buchberger algorithm a Groebner basis can be constructed from any set of polynomials generating the ideal.

This knowledge can be used for the solution of the IBP-identities. For this purpose the IBP identities have to be written in terms of shift operators [3–5] or differential operators [6, 7]. The IBP relations then form a left-ideal in the now non-commutative ring of shift operators for which a Groebner basis has to be constructed. Every integral can be obtained by applying shift operators to a basis integral, i.e., $J(a, b) = pJ(1, 1), p \in R$. Since every element p of the ring can be expressed in the form $p = \sum r_i p_i + q$, where $p_i \in I, r_i, q \in R$ the integral can be reduced via $J(a, b) = pJ(1, 1) = (\sum r_i p_i + q)J(1, 1) = qJ(1, 1)$, since $r_i p_i \in I$ and therefore $r_i p_i J(1, 1) = 0$.

Although this procedure does, in principle, generate a reduction of any integral there are still some problems. The Buchberger algorithm is guaranteed to generate a basis but it might need a very long time and a lot of resources. In practice there is no implementation of the Buchberger algorithm which succeeds in constructing a basis for non-trivial problems. Furthermore, as can be checked in simple cases, the application of Groebner bases to the reduction problem does not yield the minimal number of master integrals that can be reached using a Laporta-like kind of reduction method.

To circumvent the problematic issues of the Groebner basis approach, a modification of the Buchberger algorithm has been introduced to construct so-called s-bases. Unfortunately in this approach the construction of a basis is no longer guaranteed to succeed and is highly dependent on the chosen ordering. For certain topologies a basis cannot be found and therefore the method cannot be applied. In this cases a conventional Laporta approach still has to be used in the sectors where no basis can be found.

III. EXAMPLE AND APPLICATION

As an example for the power of the methods outlined in the previous Section we present the calculation of the second moment of the vector correlator at $\mathcal{O}(\alpha_s^3)$ and its application to the determination of the charm and bottom quark masses from experimental data.

The charm and bottom masses can be extracted from the R-ratio $R(s) = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$ [8, 9]. The extraction of the quark masses can be done by considering the moments C_n^{exp} of the ratio defined by

$$C_n^{exp} = \int \frac{R(s)}{s^{n+1}} ds \,. \tag{2}$$

These moments are related to the Taylor coefficients C_n^{theo} of the low energy expansion of the heavy quark current correlator

$$\sum_{n=0}^{\infty} C_n^{theo} \left(\frac{q^2}{4M^2} \right)^n = \Pi(q^2) \,, \tag{3}$$

where Π is the vacuum polarization function of the photon

$$\Pi_T(q^2 g^{\mu\nu} - q^\mu q^\nu) = \Pi^{\mu\nu} = \langle 0|Tj^\mu(q)j^\nu(0)|0\rangle \tag{4}$$

with the heavy quark current $j^\mu = \bar{\psi}\gamma^\mu\psi$. These theoretical moments can be calculated in an expansion in the strong coupling constant α_s in perturbative QCD

$$C_n^{theo} = C_n^{(0)} + \frac{\alpha_s}{4\pi}C_n^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 C_n^{(2)} + \left(\frac{\alpha_s}{4\pi}\right)^3 C_n^{(3)} + \cdots . \tag{5}$$

To perform the calculation the initially present propagator-type diagrams are expanded in $q^2/4M^2$ where q is the external photon momentum and M the mass of the heavy quark. This expansion leads to tadpole diagrams to which the techniques presented in the previous Section can be applied.

The Taylor coefficients have been calculated at $\mathcal{O}(\alpha_s^2)$, i.e. three-loop order, in [10–13]. At four loops the first moment has been calculated in [14–16] and the second moment in [17], see [18, 19] for higher moments for purely fermionic contributions. Using the second moment the analysis presented in [20], see also [15, 21], can be upgraded to include the now complete knowledge of the second moment. Using this newly available information the error of the bottom quark mass can be reduced to obtain

$$m_b(10\text{GeV}) = 3.607(19)\text{GeV}\,, \tag{6}$$

which has to be compared with old value of

$$m_b(10\text{GeV}) = 3.609(25)\text{GeV}. \tag{7}$$

In the case of the charm quark one finds

$$m_c(3\text{GeV}) = 0.976(16)\text{GeV} \tag{8}$$

instead of

$$m_c(3\text{GeV}) = 0.979(22)\text{GeV}\,. \tag{9}$$

Even though the shift in the absolute value is quite small the error gets reduced by around 25%.

Since the different moment weights the experimental input differently precise knowledge of the second moment allows for a consistency check of the method used for the extraction of the quark masses. Especially in the case of the bottom quark the second moment is favored over the first one since it is more sensitive to the threshold region and the narrow resonances.

IV. CONCLUSION

We presented a short review of the current status of an integral part of multi-loop calculations, namely the reduction to master integrals. Using recent Groebner basis inspired methods in combination with Laporta-like approaches most reduction problems can be solved in an efficient way and the most problematic part is again the calculation of the master integrals. As an example we presented the calculation of the second moment of the heavy quark current correlator at four-loop order and its application to the extraction of the charm- and bottom-quark masses from experimental data.

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