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The four-loop DRED gauge β-function and fermion mass anomalous dimension for general gauge groups

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Abstract

We present four-loop results for the gauge β-function and the fermion mass anomalous dimension for a gauge theory with a general gauge group and a multiplet of fermions transforming according to an arbitrary representation, calculated using the dimensional reduction scheme. In the special case of a supersymmetric theory we confirm previous calculations of both the gauge β-function and the gaugino mass β-function.

KEYWORDS: Renormalisation Group, Supersymmetric Gauge Theory, QCD

1 Introduction

In recent papers some of us presented calculations of the QCD β-function, β_s, and the fermion mass anomalous dimension (or mass β-function), γ_m through three loops [1] and four loops [2] using the DRED (or $\overline{\rm DR}$) scheme, which is based on regularisation by dimensional reduction [3,4]. An interesting feature of these calculations is the dependence of β_s on the evanescent couplings: ε-scalar interactions that do not renormalise like the gauge coupling. At three loops β_s depends on the ε-scalar Yukawa coupling, and at four loops it also depends on the ε-scalar quartic interaction. The first explicit calculations of the one loop corrections to this quartic interaction appeared in Ref. [5] (for a particular $SU(2)$ model), and in Ref. [2] (for QCD). Here we generalise the calculation to $SU(N)$, $SO(N)$ and $Sp(N)$. This involves some quite interesting and (relatively) little known group theory. We also similarly generalise the result from Ref. [2] for the ε-scalar Yukawa coupling.

Of course it is the $SU(3)$ case described in the previous papers which is most obviously currently useful, but the general result is also of interest, for possible future applications to other symmetry groups, and if only as a further test of the validity of the DRED procedure. Our confidence in this is reinforced by once again comparing our results for the special case of supersymmetry (when there is a single fermion multiplet in the adjoint representation). The result for β_s for a renormalisable $\mathcal{N} = 1$ supersymmetric theory was given through four loops in Ref. [6], the derivation being based on the completion of a construction of the coupling constant redefinition connecting the DRED scheme to the NSVZ scheme developed in Ref. [7]. Here we not only verify this result through four loops (in the special case of a theory with no superpotential) but also we verify the result for the gaugino β-function through the same order. This is of interest because of course the gaugino mass breaks supersymmetry, and the issue of regularisation and renormalisation of softly-broken supersymmetric theories present additional subtleties. The exact formula for the gaugino β-function (expressing it in terms of β_s), as first derived in Ref. [8] (inspired by an observation by Hisano and Shifman in Ref. [9]) relied heavily on the spurion formalism, as developed in particular by Yamada [10]; it is reassuring to find that the relationship between the two β-functions indeed holds in an explicit DRED calculation.

In section 2 we review the renormalisation procedure for a gauge theory using DRED; then in section 3 we describe the one loop renormalisation of the ε-scalar self-interaction. We first give results for the $SU(N)$ case, explaining how to reduce to the special cases $N = 2$ and $N = 3$. We then generalise to expressions valid for an arbitrary groups.

In section 4 we give the full four-loop results for β_s and γ_m for the general case, and in section 5 we reduce to the special case of supersymmetry for comparison with earlier results, as described above. Finally in the Appendices we explain some of the group theory involved in the calculations and give explicit results for $SO(N)$ and $Sp(N)$ for the one-loop ε-scalar quartic interaction β-functions.

2 Gauge theory with fermions

Consider a non-abelian gauge theory with a multiplet of two-component fermions $\psi^\alpha(x)$ transforming according to a representation R of the gauge group $\mathcal{G}$.

The Lagrangian density (in terms of bare fields) is

$$L_B = -\tfrac{1}{4}G_{\mu\nu}^2 - \tfrac{1}{2\alpha}(\partial^\mu W_\mu)^2 + C^{a*}\partial^\mu D_\mu^{ab}C^b + i\overline{\psi}^\alpha\overline{\sigma}^\mu D_\mu^{\alpha\beta}\psi^\beta \tag{1}$$

where

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + gf^{abc}W_\mu^b W_\nu^c \tag{2}$$

and

$$D_\mu^{\alpha\beta} = \delta^{\alpha\beta}\partial_\mu - ig(R^a)^{\alpha\beta}W_\mu^a \tag{3}$$

and the usual covariant gauge fixing and ghost terms have been introduced.

For the case when the theory admits a gauge invariant fermion mass term we will have $L_B \to L_B + L_B^m$, where

$$L_B^m = \tfrac{1}{2}m_{\alpha\beta}\psi^\alpha\psi^\beta + \text{c.c.} \tag{4}$$

Dimensional reduction amounts to imposing that all field variables depend only on a subset of the total number of space-time dimensions; in this case d out of 4 where $d = 4 - 2\epsilon$. We can then make the decomposition

$$W_\mu^a(x^j) = \{W_i^a(x^j), W_\sigma^a(x^j)\} \tag{5}$$

where

$$\delta_i^i = \delta_j^j = d \qquad \text{and} \qquad \delta_{\sigma\sigma} = 2\epsilon. \tag{6}$$

It is then easy to show that

$$L_B = L_B^d + L_B^\epsilon \tag{7}$$

where

$$L_B^d = -\tfrac{1}{4}G_{ij}^2 - \tfrac{1}{2\alpha}(\partial^i W_i)^2 + C^{a*}\partial^i D_i^{ab}C^b + i\overline{\psi}^\alpha\overline{\sigma}^i D_i^{\alpha\beta}\psi^\beta \tag{8}$$

and

$$L_B^\epsilon = \tfrac{1}{2}(D_i^{ab}W_\sigma^b)^2 - g\overline{\psi}\overline{\sigma}_\sigma R^a\psi W_\sigma^a - \tfrac{1}{4}g^2 f^{abc}f^{ade}W_\sigma^b W_{\sigma'}^c W_\sigma^d W_{\sigma'}^e. \tag{9}$$

Conventional dimensional regularisation (DREG) amounts to using Eq. (8) and discarding Eq. (9).

We would now like to rewrite Eq. (8) and Eq. (9) in terms of renormalised quantities. It is clear, however, from the dimensionally reduced form of the gauge transformations:

$$\delta W_i^a = \partial_i\Lambda^a + gf^{abc}W_i^b\Lambda^c \tag{10a}$$
$$\delta W_\sigma^a = gf^{abc}W_\sigma^b\Lambda^c \tag{10b}$$
$$\delta\psi^\alpha = ig(R^a)^{\alpha\beta}\psi^\beta\Lambda^a \tag{10c}$$

that each term in Eq. (9) is separately invariant under gauge transformations. The W_σ-fields behave exactly like scalar fields, and are hence known as ε-scalars. There is therefore no reason to expect the $\overline{\psi}\psi W_\sigma$ vertex to renormalise in the same way as the $\overline{\psi}\psi W_i$ vertex (except in the case of supersymmetric theories). In the case of the quartic ε-scalar interaction it is evident that more than one such coupling is permitted by Eq. (10b). In other words, we cannot in general expect the $f-f$ tensor structure present in Eq. (9) to be preserved under renormalisation. This is clear from the abelian case, where there is no quartic interaction in L_B^ϵ but there is a divergent graph at one loop from a fermion loop.

We are therefore led to consider the following expressions [1] for renormalised quantities L^d and L^ε :

$$\begin{aligned} L^d = & - \tfrac{1}{4}Z^{WW}(\partial_i W_j - \partial_j W_i)^2 - \tfrac{1}{2\alpha}(\partial^i W_i)^2 \\ & - Z^{WWW} g f^{abc}\partial_i W_j^a W^{bi} W^{cj} - \tfrac{1}{4}Z^{4W} g^2 f^{abc} f^{ade} W_i^b W_j^c W^{di} W^{ej} \\ & + Z^{C^*C}\partial^i C^* \partial_i C + Z^{C^*CW} g f^{abc}\partial^i C^{a*} W_i^b C^c \\ & + Z^{\overline{\psi}\psi} i\overline{\psi}\overline{\sigma}^i \partial_i \psi + Z^{\overline{\psi}\psi W} g\overline{\psi} R^a \overline{\sigma}^i \psi W_i^a \end{aligned} \tag{11}$$

and

$$\begin{aligned} L^\varepsilon = & \tfrac{1}{2}Z^{\varepsilon\varepsilon}(\partial_i W_\sigma)^2 + Z^{\varepsilon\varepsilon W} g f^{abc}\partial_i W_\sigma^a W^{bi} W_\sigma^c \\ & + Z^{\varepsilon\varepsilon WW} g^2 f^{abc} f^{ade} W_i^b W_\sigma^c W^{di} W_\sigma^e - Z^{\overline{\psi}\psi\varepsilon} g_e \overline{\psi} R^a \overline{\sigma}^\sigma \psi W_\sigma^a \\ & - \tfrac{1}{4}\sum_{r=1}^{p} Z_r^{4\varepsilon}\lambda_r H_r^{abcd} W_\sigma^a W_{\sigma'}^c W_\sigma^b W_{\sigma'}^d . \end{aligned} \tag{12}$$

In the case when we have a fermion mass term we would also have

$$L^m = \tfrac{1}{2}Z_m Z^{\overline{\psi}\psi} m\psi\psi + \text{c.c.} \tag{13}$$

Eq. (11) is the usual expression for the Lagrangian in terms of renormalised parameters. In Eq. (12) we have introduced a "Yukawa" coupling g_e and a set of p quartic couplings λ_r. (Strictly speaking, Eq. (12) should also have a mass term for the ε-scalars; but since this mass term does not affect β_s or γ_m we omit it here.) The number p is given by the number of independent rank four tensors H^{abcd} which are non-vanishing when symmetrised with respect to (ab) and (cd) interchange. In the next section we discuss the quartic vertex and its renormalisation in more detail.

[1]Since ε-scalars are present only on internal lines we could, in fact, choose the wave function renormalisation of W_σ and W_i to be the same; or, indeed, have no wave function renormalisation for W_σ at all. The crucial thing is correct treatment of sub-graphs, which means recognition that vertices with ε-scalars renormalise in a different way from their gauge counterparts. However, we choose to renormalise the ε-scalar conventionally.

3 The ε-scalar self coupling

Let us discuss the structure of the quartic ε-scalar couplings for an arbitrary gauge group. These interactions are invariant under the symmetry $\mathcal{G} \otimes O(2\epsilon)$, where only the $\mathcal{G}$ is gauged. The renormalisation properties of scalar theories with invariances of the type $G_1 \otimes G_2$ have been studied in considerable detail, for example $O(m) \otimes O(n)$ in the theory of critical phenomena and $U(m) \otimes U(n)$ in the context of QCD. In these cases, however, the scalars transform as vector (fundamental) representations of the gauge group factors whereas for us they transform as adjoints.

This raises an interesting group theory question: how many independent couplings are there for a given gauge group $\mathcal{G}$? Evidently the question of how many independent tensors of the form K^{abcd} there are is the question of how many times the singlet representation occurs in the reduction to irreducible representations of the product of four adjoint representations. (Neither this question, nor the obvious generalisation to n-tensors $K^{a_1 \cdots a_n}$ has been much studied in the literature; an exception being the classic work of Cvitanovic [11], to which we will return presently). The set of tensors relevant to our problem is the subset of such tensors H^{abcd} which is invariant with respect to (a,b) and (c,d) exchange, because of the $O(2\epsilon)$ invariance.

If we have an irreducible basis of dimensionality $\gamma(n)$ for the n-tensors of the form

$$K_\alpha^{a_1 \cdots a_n}, \quad 1 \le \alpha \le \gamma(n) \tag{14}$$

then a general n-tensor $K^{a_1 \cdots a_n}$ can be expressed in terms of the basis as

$$K^{a_1 \cdots a_n} = x_\beta K_\beta^{a_1 \cdots a_n}, \tag{15}$$

where x_β are determined by the equation

$$Q^n_{\alpha\beta} x_\beta = y_\alpha = K^{a_1 \cdots a_n} K_\alpha^{a_1 \cdots a_n} \tag{16}$$

and

$$Q^n_{\alpha\beta} = K_\alpha^{a_1 \cdots a_n} K_\beta^{a_1 \cdots a_n}. \tag{17}$$

Thus construction of the Q^n-matrix permits reduction of an arbitrary n-tensor to the basis.

3.1 The case $\mathcal{G} = SU(N)$

The fundamental representation T^a of the generators R^a of $SU(N)$ satisfies

$$\begin{aligned} \left[T^a, T^b\right] &= i f^{abc} T^c \\ \left\{T^a, T^b\right\} &= d^{abc} T^c + \frac{b}{N}\delta^{ab} \\ \mathrm{Tr}\left(T^a T^b\right) &= \frac{b}{2}\delta^{ab}, \end{aligned} \tag{18}$$

n	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
2	1	1	1	1	1
3	1	2	2	2	2
4	3	8	9	9	9
5	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 1: Basis dimensionality for n-tensors in $SU(N)$.

where b is a constant. For the rest of this section we will adopt the usual convention whereby $b=1$.

In table 1 we present some results for the dimensionality $\gamma(n)$ for $SU(N)$ as a function of N. It is interesting that Cvitanovic [11] remarks that a formula for the dimensionality of a basis (in general over-complete) is provided by the subfactorial $\beta(n)$ where

$$\beta(n) = n!(1 - \frac{1}{1!} + \frac{1}{2!} + \ldots (-1)^n \frac{1}{n!}). \tag{19}$$

It appears that for sufficiently large N we have $\gamma(n) = \beta(n)$.

A natural choice for the basis for the case $n=4$ when $N \geq 4$ is given by[2]

$$\begin{aligned} K_1 &= \delta^{ab}\delta^{cd} & K_4 &= d^{abe}d^{cde} & K_7 &= d^{abe}f^{cde} \\ K_2 &= \delta^{ac}\delta^{bd} & K_5 &= d^{ace}d^{bde} & K_8 &= d^{ace}f^{bde} \\ K_3 &= \delta^{ad}\delta^{bc} & K_6 &= d^{ade}d^{bde} & K_9 &= d^{ade}f^{bce}. \end{aligned} \tag{20}$$

The reduction of the basis to $\gamma = 8$ in the case $SU(3)$ is achieved via the relation [12,13]

$$K_4 + K_5 + K_6 = \frac{1}{3}(K_1 + K_2 + K_3) \tag{21}$$

which is not valid for $N \geq 4$. The corresponding identity for general N reduces a symmetrised $(N+1)$-tensor consisting of $N-1$ d-tensors; for an elegant derivation see Ref. [14].

[2]An alternative way to define a basis which has the virtue of being immediately generalisable to any group [11] is in terms of traces of products of the generators in the defining representation, thus $\mathrm{Tr}\left(T^aT^bT^cT^d\right)$, $\mathrm{Tr}\left(T^aT^b\right)\mathrm{Tr}\left(T^cT^d\right)$ etc.

For the ε-scalar interactions a possible basis for $N \geq 4$ is therefore

$$\begin{aligned} H_1 &= \tfrac{1}{2} K_1 \\ H_2 &= \tfrac{1}{2}(K_2 + K_3) \\ H_3 &= \tfrac{1}{2} K_4 \\ H_4 &= \tfrac{1}{2}(K_5 + K_6). \end{aligned} \tag{22}$$

Note that the absence of a $d - f$ type term from the basis follows from the identity

$$K_8 + K_9 = -f^{abe} d^{cde}. \tag{23}$$

Let us introduce the couplings

$$\alpha_s = \frac{g^2}{4\pi}, \quad \alpha_e = \frac{g_e^2}{4\pi} \quad \text{and} \quad u_r = \frac{\lambda_r}{4\pi}, \tag{24}$$

and define the corresponding β functions for the u_r couplings

$$\beta_{u_r} = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \frac{u_r}{\pi}. \tag{25}$$

If we write (with the normalisation of Eq. (12))

$$\lambda_r H_r^{abcd} \to \sum_{r=1}^{4} 4\pi u_r H_r \tag{26}$$

then the β-functions for the u_r couplings are given at one loop by[3]

$$\begin{aligned} \beta_{u_1} &= 8u_1^2 + 4N^2 u_1 u_2 + 12 u_2^2 \\ &+ \frac{4(N^2-4)}{N} \left\{ u_1 u_3 + u_1 u_4 + 2u_2 u_4 + \frac{1}{N}(u_3^2 + 2u_3 u_4 + 3u_4^2) \right\} \\ \beta_{u_2} &= 12 u_1 u_2 + (2N^2+6) u_2^2 \\ &+ \frac{2(N^2-4)}{N} \left\{ 2u_2 u_3 + 2u_2 u_4 + \frac{1}{N}(u_3^2 + 6u_3 u_4 + 3u_4^2) \right\} \\ \beta_{u_3} &= 12 u_1 u_3 + 4u_2 u_3 + 16 u_2 u_4 \\ &+ \frac{1}{N} \left\{ (3N^2 - 40) u_3^2 + 6(N^2 - 12) u_3 u_4 + (7N^2 - 96) u_4^2 \right\} \\ \beta_{u_4} &= 12 u_1 u_4 + 8 u_2 u_3 + 12 u_2 u_4 \\ &+ \frac{1}{N} \left\{ 4(N^2 - 14) u_4^2 + 4(N^2 - 18) u_3 u_4 - 8 u_3^2 \right\} \end{aligned} \tag{27}$$

where for the moment we suppress contributions from the gauge coupling α_s and ε-scalar Yukawa coupling α_e.

[3] Here and for the rest of this section we suppress a factor of $1/8\pi^2$ in every one-loop β-function.

Because of the nature of the bare theory, and to explore more easily the supersymmetric case, it is natural to consider alternative bases, for example:

$$\begin{aligned}
\overline{H}_1 &= H_1 \\
\overline{H}_2 &= H_2 \\
\overline{H}_3 &= \tfrac{1}{2}(f^{ace}f^{bde} + f^{ade}f^{bce}) \\
\overline{H}_4 &= \tfrac{1}{2}(f^{aef}f^{bfg}f^{cgh}f^{dhe} + f^{aef}f^{bfg}f^{dgh}f^{che}).
\end{aligned} \tag{28}$$

We shall also see that it is this kind of basis (avoiding use of the d-tensor) that generalises most easily to other groups.

We have

$$\begin{aligned}
\overline{H}_3 &= \frac{4}{N}H_1 - \frac{2}{N}H_2 + 2H_3 - H_4 \\
\overline{H}_4 &= 2H_1 + H_2 + \frac{N}{2}H_3,
\end{aligned} \tag{29}$$

so that if we write

$$\sum_{r=1}^{4} u_r H_r = \sum_{r=1}^{4} v_r \overline{H}_r \tag{30}$$

then

$$\begin{aligned}
v_1 &= u_1 - \frac{4}{N}(u_3 + u_4) \\
v_2 &= u_2 - \frac{2}{N}u_3 - \frac{6}{N}u_4 \\
v_3 &= -u_4 \\
v_4 &= \frac{2}{N}u_3 + \frac{4}{N}u_4.
\end{aligned} \tag{31}$$

The β-functions for the v_r couplings are given at one loop by

$$\begin{aligned}
\beta_{v_1} &= 8v_1^2 + 4N^2v_1v_2 + 12v_2^2 - 4Nv_1v_3 + 6N^2v_1v_4 + 8Nv_2v_3 \\
&+ 8Nv_3v_4 + 8N^2v_2v_4 + 10N^2v_4^2 - 12Nv_1\alpha_s \\
\beta_{v_2} &= 12v_1v_2 + (2N^2+6)v_2^2 - 4Nv_2v_3 + 6N^2v_2v_4 \\
&- 4Nv_3v_4 + 3N^2v_4^2 - 12Nv_2\alpha_s \\
\beta_{v_3} &= 4Nv_3^2 + 12v_1v_3 - 4v_2v_3 - 4Nv_2v_4 + (2N^2+8)v_3v_4 \\
&- 2Nv_4^2 - 12Nv_3\alpha_s \\
\beta_{v_4} &= (\tfrac{3}{2}N^2 + 16)v_4^2 + 12v_1v_4 + 20v_2v_4 - 2v_3^2 \\
&- 2Nv_3v_4 - 12Nv_4\alpha_s + 6\alpha_s^2
\end{aligned} \tag{32}$$

where we have now included the gauge coupling contribution (note that the α_s^2 terms contribute only to β_{v_4}).

Another choice of basis (in fact the one employed in Ref. [1]) is

$$\begin{aligned}
\tilde{H}_1 &= \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc} + \delta^{ab}\delta^{cd} \\
\tilde{H}_2 &= \tfrac{1}{2}(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) - \delta^{ab}\delta^{cd} \\
\tilde{H}_3 &= \overline{H}_3 \\
\tilde{H}_4 &= \overline{H}_4
\end{aligned} \tag{33}$$

so that

$$\begin{aligned}
\tilde{H}_1 &= 2(\overline{H}_1 + \overline{H}_2) \\
\tilde{H}_2 &= \overline{H}_2 - 2\overline{H}_1
\end{aligned} \tag{34}$$

and if we write

$$\sum_{r=1}^{4} v_r \overline{H}_r = \sum_{r=1}^{4} w_r \tilde{H}_r \tag{35}$$

then

$$\begin{aligned}
w_1 &= \tfrac{1}{6}(v_1 + 2v_2) \\
w_2 &= \tfrac{1}{3}(-v_1 + v_2) \\
w_3 &= v_3 \\
w_4 &= v_4.
\end{aligned} \tag{36}$$

In this basis the β-functions become

$$\begin{aligned}
\beta_{w_1} &= \tfrac{1}{3}[(112 + 16N^2)w_1^2 + (4N^2 - 8)w_1w_2 - 4Nw_1w_3 + 26N^2w_1w_4 \\
&- (2N^2 - 4)w_2^2 + 4Nw_2w_3 + 4N^2w_2w_4 + 8N^2w_4^2] - 12Nw_1\alpha_s \\
\beta_{w_2} &= \tfrac{1}{3}[-8(N^2 + 1)w_1^2 + 16(N^2 + 1)w_1w_2 - 16Nw_1w_3 - 16N^2w_1w_4 \\
&+ (10N^2 - 62)w_2^2 - 20Nw_2w_3 + 10N^2w_2w_4 - 12Nw_3w_4 - 7N^2w_4^2] \\
&- 12Nw_2\alpha_s \\
\beta_{w_3} &= 16w_1w_3 - 8Nw_1w_4 - 28w_2w_3 - 4Nw_2w_4 + 4Nw_3^2 \\
&+ (2N^2 + 8)w_3w_4 - 2Nw_4^2 - 12Nw_3\alpha_s \\
\beta_{w_4} &= 64w_1w_4 - 4w_2w_4 - 2w_3^2 - 2Nw_3w_4 + (\tfrac{3}{2}N^2 + 16)w_4^2 \\
&- 12Nw_4\alpha_s + 6\alpha_s^2.
\end{aligned} \tag{37}$$

For the rest of the paper we will use the v-basis; as already remarked the use of $\overline{H}_{3,4}$ means the generalisation to other groups can be carried out easily.

3.1.1 The fermion contribution

The contribution of the fermion loop to the scalar anomalous dimension results in a contribution of

$$\Delta\beta_{u_i} = 8n_f I_2(R)\alpha_e u_i \tag{38}$$

to each β-function in Eq. (27), with corresponding contributions to Eq. (32) and Eq. (37).

In Eq. (38) and subsequently we follow the following convention: our fermion representation consists of n_f sets of Dirac fermions or $2n_f$ sets of two-component fermions, in irreducible representations with identical Casimirs; and the whole representation must of course be anomaly free. We will pay particular attention to the case of an adjoint representation with $n_f = \frac{1}{2}$, which is supersymmetric, and to the case of n_f flavours, that is n_f sets of fundamental two component fermions with n_f sets of anti-fundamental two component fermions, which is QCD. For the definition of $I_2(R)$ and more details on group theoretic considerations see Appendix A.

The 1PI fermion box diagram makes a contribution to the β-functions (appropriately normalised) of the form

$$\begin{aligned}\overline{H}_i\Delta\beta_{v_i} &= -4n_f\alpha_e^2\big[\mathrm{Tr}(R^aR^bR^cR^d)+\mathrm{Tr}(R^aR^bR^dR^c)+\mathrm{Tr}(R^aR^dR^cR^b)\\ &+ \mathrm{Tr}(R^aR^cR^dR^b)-\mathrm{Tr}(R^aR^dR^bR^c)-\mathrm{Tr}(R^aR^cR^bR^d)\big].\end{aligned} \tag{39}$$

For a general representation this is not easily expressed in terms of one of our choice of bases. In the special case of an adjoint representation (with $n_f = \frac{1}{2}$), we find that

$$\overline{H}_i\Delta\beta_{v_i} = \alpha_e^2(-2N\overline{H}_3 - 4\overline{H}_4) \tag{40}$$

so that the complete set of β-functions for the case of an $SU(N)$ gauge theory with an adjoint fermion multiplet is:

$$\begin{aligned}\beta_{v_1} &= 8v_1^2+4N^2v_1v_2+12v_2^2-4Nv_1v_3+6N^2v_1v_4+8Nv_2v_3\\ &+ 8Nv_3v_4+8N^2v_2v_4+10N^2v_4^2-12Nv_1\alpha_s+4Nv_1\alpha_e\\ \beta_{v_2} &= 12v_1v_2+(2N^2+6)v_2^2-4Nv_2v_3+6N^2v_2v_4\\ &- 4Nv_3v_4+3N^2v_4^2-12Nv_2\alpha_s+4Nv_2\alpha_e\\ \beta_{v_3} &= 4Nv_3^2+12v_1v_3-4v_2v_3-4Nv_2v_4+(2N^2+8)v_3v_4\\ &- 2Nv_4^2-12Nv_3\alpha_s+4Nv_3\alpha_e-2N\alpha_e^2\\ \beta_{v_4} &= (\tfrac{3}{2}N^2+16)v_4^2+12v_1v_4+20v_2v_4-2v_3^2\\ &- 2Nv_3v_4-12Nv_4\alpha_s+6\alpha_s^2+4Nv_4\alpha_e-4\alpha_e^2.\end{aligned} \tag{41}$$

If we now set $v_1 = v_2 = v_4 = 0$ and $v_3 = \alpha_e = \alpha_s$ the theory becomes supersymmetric; and substituting these values in Eq. (41) we indeed find $\beta_{v_1} = \beta_{v_2} = \beta_{v_4} = 0$ and

$$\beta_{v_3} = -\frac{6N}{8\pi^2}\alpha_s^2, \tag{42}$$

(restoring the $8\pi^2$ factor) which is identical to the one-loop gauge β-function β_s in the supersymmetric case.

Let us consider now the special case of $SU(3)$. In $SU(3)$, $(\overline{H}_1, \overline{H}_2, \overline{H}_3)$ form a basis; however if we set $v_4 = 0$ in Eq. (41) then we nevertheless have

$$\beta_{v_4} = -2v_3^2 + 6\alpha_s^2 - 4\alpha_e^2. \tag{43}$$

This represents a set of contributions to $\beta_{v_{1,2,3}}$ which we can identify by using the identity

$$\overline{H}_4 = \tfrac{3}{2}(\overline{H}_1 + \overline{H}_2) + \tfrac{1}{2}\overline{H}_3. \tag{44}$$

Incorporating these contributions into Eq. (41) we thus find for in $SU(3)$

$$\begin{aligned}
\beta_{v_1} &= 8v_1^2 + 36v_1v_2 + 12v_2^2 - 12v_1v_3 + 24v_2v_3 - 36v_1\alpha_s \\
&+ 12v_1\alpha_e - 3v_3^2 + 9\alpha_s^2 - 6\alpha_e^2 \\
\beta_{v_2} &= 12v_1v_2 + 24v_2^2 - 12v_2v_3 - 36v_2\alpha_s \\
&+ 12v_2\alpha_e - 3v_3^2 + 9\alpha_s^2 - 6\alpha_e^2 \\
\beta_{v_3} &= 11v_3^2 + 12v_1v_3 - 4v_2v_3 - 36v_3\alpha_s \\
&+ 12v_3\alpha_e + 3\alpha_s^2 - 8\alpha_e^2.
\end{aligned} \tag{45}$$

It is easy to check that this set still reduces correctly in the supersymmetric limit.

For the special case of $SU(2)$, the basis is two dimensional and we have the identities

$$\begin{aligned}
2\overline{H}_1 - \overline{H}_2 - \overline{H}_3 &= 0 \\
2\overline{H}_1 + \overline{H}_2 - \overline{H}_4 &= 0.
\end{aligned} \tag{46}$$

If we choose the basis $(\overline{H}_1, \overline{H}_2)$ then we find

$$\begin{aligned}
\beta_{v_1} &= 8v_1^2 + 16v_1v_2 + 12v_2^2 - 24v_1\alpha_s + 8v_1\alpha_e - 16\alpha_e^2 + 12\alpha_s^2 \\
\beta_{v_2} &= 12v_1v_2 + 14v_2^2 - 24v_2\alpha_s + 8v_2\alpha_e + 6\alpha_s^2.
\end{aligned} \tag{47}$$

Alternatively we could choose the basis $(\overline{H}_3, \overline{H}_4)$ when we find

$$\begin{aligned}
\beta_{v_3} &= 8v_3^2 + 24v_3v_4 - 24v_3\alpha_s + 8v_3\alpha_e - 4\alpha_e^2 \\
\beta_{v_4} &= 38v_4^2 - 2v_3^2 - 4v_3v_4 - 24v_4\alpha_s + 6\alpha_s^2 + 8v_4\alpha_e - 4\alpha_e^2.
\end{aligned} \tag{48}$$

With this basis the supersymmetric limit is again apparent; setting $v_4 = 0$ and $v_3 = \alpha_e = \alpha_s$ we obtain $\beta_{v_4} = 0$ and $\beta_{v_3} = -12\alpha_s^2$ as expected.

For the fundamental representation of $SU(N)$ we find

$$\mathrm{Tr}(R^aR^bR^cR^d) = \frac{1}{4N}\left[K_1 - K_2 + K_3\right] + \frac{1}{8}\left[K_4 - K_5 + K_6\right] + \frac{i}{8}\left[K_7 + K_8 + K_9\right] \tag{49}$$

and hence for the case of $2n_f$ sets of fermions in the fundamental representation of $SU(N)$ (corresponding to QCD with n_f flavours),

$$\overline{H}_i\Delta\beta_{v_i} = 2n_f\alpha_e^2\left[\frac{2}{N}(\overline{H}_1 + \overline{H}_2 - \overline{H}_4) - \overline{H}_3\right] \tag{50}$$

so that the complete set of β-functions for this case is:

$$
\begin{aligned}
\beta_{v_1} &= 8v_1^2+4N^2v_1v_2+12v_2^2-4Nv_1v_3+6N^2v_1v_4+8Nv_2v_3\\
&+ 8Nv_3v_4+8N^2v_2v_4+10N^2v_4^2-12Nv_1\alpha_s+4n_fv_1\alpha_e+4\frac{n_f}{N}\alpha_e^2\\
\beta_{v_2} &= 12v_1v_2+(2N^2+6)v_2^2-4Nv_2v_3+6N^2v_2v_4\\
&- 4Nv_3v_4+3N^2v_4^2-12Nv_2\alpha_s+4n_fv_2\alpha_e+4\frac{n_f}{N}\alpha_e^2\\
\beta_{v_3} &= 4Nv_3^2+12v_1v_3-4v_2v_3-4Nv_2v_4+(2N^2+8)v_3v_4\\
&- 2Nv_4^2-12Nv_3\alpha_s+4n_fv_3\alpha_e-2n_f\alpha_e^2\\
\beta_{v_4} &= (\tfrac{3}{2}N^2+16)v_4^2+12v_1v_4+20v_2v_4-2v_3^2\\
&- 2Nv_3v_4-12Nv_4\alpha_s+6\alpha_s^2+4n_fv_4\alpha_e-4\frac{n_f}{N}\alpha_e^2. \qquad (51)
\end{aligned}
$$

It is straightforward to incorporate the fermion contributions in our other choices of bases involving u_i or w_i.

Turning again to the special case of $SU(3)$, and setting $v_4=0$ in Eq. (51) we have

$$\beta_{v_4} = -2v_3^2+6\alpha_s^2-4\frac{n_f}{3}\alpha_e^2 \qquad (52)$$

and incorporating these contributions into Eq. (51) we thus find for $SU(3)$:

$$
\begin{aligned}
\beta_{v_1} &= 8v_1^2+36v_1v_2+12v_2^2-12v_1v_3+24v_2v_3-36v_1\alpha_s\\
&+ 4n_fv_1\alpha_e-3v_3^2+9\alpha_s^2-2\frac{n_f}{3}\alpha_e^2\\
\beta_{v_2} &= 12v_1v_2+24v_2^2-12v_2v_3-36v_2\alpha_s\\
&+ 4n_fv_2\alpha_e-3v_3^2+9\alpha_s^2-2\frac{n_f}{3}\alpha_e^2\\
\beta_{v_3} &= 11v_3^2+12v_1v_3-4v_2v_3-36v_3\alpha_s\\
&+ 4n_fv_3\alpha_e+3\alpha_s^2-8\frac{n_f}{3}\alpha_e^2. \qquad (53)
\end{aligned}
$$

The special case of $SU(2)$ in the fundamental fermion case we leave as an exercise for the reader.

3.2 The general case

In this subsection we give the results for β_{v_i} for a general gauge group. The various group invariants are defined in Appendix A, where results for them for the fundamental representations of $SU(N)$, $SO(N)$ and $Sp(N)$ also appear.

We have derived these results both by substituting in the general expressions that follow and by direct calculations with each class of group in the manner described in the previous section.

We find

$$
\begin{aligned}
\beta_{v_1} &= -32n_f\frac{5C_A^2D_2(RA)+(C_A-6C_R)D_2(A)I_2(R)}{25C_A^4N_A-12D_2(A)(2+N_A)}\alpha_e^2-12C_Av_1\alpha_s\\
&+ 8I_2(R)n_fv_1\alpha_e+8v_1^2+12v_2^2-\frac{192D_2(A)-80C_A^4N_A}{9C_AN_A(N_A-3)}v_3v_4\\
&+ \frac{4}{27N_A}\Big\{7\frac{-12D_2(A)+5C_A^4N_A}{N_A-3}\\
&- 24\frac{72D_2(A)^2-90C_A^2D_3(A)N_A+25C_A^4D_2(A)N_A}{25C_A^4N_A-12D_2(A)(2+N_A)}\Big\}v_4^2\\
&+ v_1[4(1+N_A)v_2-4C_Av_3+6C_A^2v_4]+v_2(8C_Av_3+8C_A^2v_4)\\
\beta_{v_2} &= -32n_f\frac{5C_A^2D_2(RA)+(C_A-6C_R)D_2(A)I_2(R)}{25C_A^4N_A-12D_2(A)(2+N_A)}\alpha_e^2\\
&- 12C_Av_2\alpha_s+8I_2(R)n_fv_2\alpha_e+12v_1v_2+2(4+N_A)v_2^2\\
&+ \frac{96D_2(A)-40C_A^4N_A}{9C_AN_A(N_A-3)}v_3v_4+\frac{2}{27N_A}\Big\{7\frac{12D_2(A)-5C_A^4N_A}{(N_A-3)}\\
&- 48\frac{72D_2(A)^2-90C_A^2D_3(A)N_A+25C_A^4D_2(A)N_A}{25C_A^4N_A-12D_2(A)(2+N_A)}\Big\}v_4^2\\
&+ v_2(-4C_Av_3+6C_A^2v_4)\\
\beta_{v_3} &= -\frac{4n_f}{25C_A^4N_A-12D_2(A)(2+N_A)}\Big\{35C_A^4I_2(R)N_A-10C_A^3C_RI_2(R)N_A\\
&+ 4C_AD_2(RA)(2+N_A)-16D_2(A)I_2(R)(2+N_A)\Big\}\alpha_e^2\\
&- 12C_Av_3\alpha_s+8I_2(R)n_fv_3\alpha_e+12v_1v_3+4C_Av_3^2\\
&+ 2\frac{48D_2(A)(-1+N_A)+C_A^4N_A(-61+7N_A)}{9C_A^2N_A(N_A-3)}v_3v_4\\
&- \frac{4}{27C_A(N_A-3)N_A[25C_A^4N_A-12D_2(A)(2+N_A)]}\\
&\quad \Big\{144D_2(A)^2(2+N_A)(1+2N_A)\\
&+ 12C_A^4D_2(A)N_A[-191+(-56+N_A)N_A]\\
&+ C_A^2N_A[-216D_3(A)(-3+N_A)(2+N_A)+25C_A^6N_A(23+4N_A)]\Big\}v_4^2\\
&- v_2(4v_3+4C_Av_4)\\
\beta_{v_4} &= 6\alpha_s^2+8n_f\frac{5C_A^2(C_A-6C_R)I_2(R)N_A+12D_2(RA)(2+N_A)}{25C_A^4N_A-12D_2(A)(2+N_A)}\alpha_e^2\\
&- 2v_3^2-12C_Av_4\alpha_s+8I_2(R)n_fv_4\alpha_e+12v_1v_4+20v_2v_4-2C_Av_3v_4\\
&- \frac{1152D_3(A)(2+N_A)-5C_A^2[125C_A^4N_A+4D_2(A)(98+N_A)]}{6[25C_A^4N_A-12D_2(A)(2+N_A)]}v_4^2. \qquad (54)
\end{aligned}
$$

The forms taken by $C_{A,R}$, $I_2(R)$ and the various invariants $D_2(A)$ etc for $SU(N)$, $SO(N)$

and $Sp(N)$ are given in Tables 2-4 in the Appendix. Using Table 2, it is easy to show that the results in Eq. (54) reduce to the results in Eq. (51) for the case of $SU(N)$.

4 The general four-loop results

The renormalisation constants for the various couplings are defined through

$$\begin{aligned} g_s^0 &= \mu^\epsilon Z_s g_s \,, & g_e^0 &= \mu^\epsilon Z_e g_e \,, & \sqrt{v_r^0} &= \mu^\epsilon Z_{v_r} \sqrt{v_r} \,, \\ \varepsilon_\sigma^{0,a} &= \sqrt{Z^{\varepsilon\varepsilon}}\, \varepsilon_\sigma^a \,, & \Gamma^0_{\overline{\psi}\psi\varepsilon} &= Z^{\overline{\psi}\psi\varepsilon} \Gamma_{\overline{\psi}\psi\varepsilon} \,, & \Gamma^{r,0}_{\varepsilon\varepsilon\varepsilon\varepsilon} &= Z_r^{4\varepsilon} \Gamma^r_{\varepsilon\varepsilon\varepsilon\varepsilon} \,, \end{aligned} \tag{55}$$

where $\Gamma_{\overline{\psi}\psi\varepsilon}$ and $\Gamma_{\varepsilon\varepsilon\varepsilon\varepsilon}$ are the one-particle irreducible ε-scalar–fermion and four-ε-scalar Green functions, respectively, the superscript "0" denotes bare quantities, and μ is the renormalisation scale. The renormalisation constants associated with the various couplings satisfy the following relations

$$Z_s = \frac{Z^{\overline{\psi}\psi W}}{Z^{\overline{\psi}\psi} \sqrt{Z^{WW}}} \,, \qquad Z_e = \frac{Z^{\overline{\psi}\psi\varepsilon}}{Z^{\overline{\psi}\psi} \sqrt{Z^{\varepsilon\varepsilon}}} \,, \qquad Z_{v_r} = \frac{\sqrt{Z_r^{4\varepsilon}}}{Z^{\varepsilon\varepsilon}} \,, \tag{56}$$

with renormalisation constants as defined in Eq. (11) and Eq. (12).

Let us define the β functions for the corresponding couplings in the $\overline{\mathrm{DR}}$ scheme:

$$\begin{aligned}
\beta_s^{\overline{\mathrm{DR}}}(\alpha_s,\alpha_e,\{v_r\}) &= \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\frac{\alpha_s}{\pi} \\
&= -\left[\epsilon\frac{\alpha_s}{\pi}+2\frac{\alpha_s}{Z_s^{\overline{\mathrm{DR}}}}\left(\frac{\partial Z_s^{\overline{\mathrm{DR}}}}{\partial\alpha_e}\beta_e+\sum_r\frac{\partial Z_s^{\overline{\mathrm{DR}}}}{\partial v_r}\beta_{v_r}\right)\right]\left(1+2\frac{\alpha_s}{Z_s^{\overline{\mathrm{DR}}}}\frac{\partial Z_s^{\overline{\mathrm{DR}}}}{\partial\alpha_s}\right)^{-1} \\
&= -\epsilon\frac{\alpha_s}{\pi}-\sum_{i,j,k,l,m,n}\beta_{ijklmn}^{\overline{\mathrm{DR}}}\left(\frac{\alpha_s}{\pi}\right)^i\left(\frac{\alpha_e}{\pi}\right)^j\left(\frac{v_1}{\pi}\right)^k\left(\frac{v_2}{\pi}\right)^l\left(\frac{v_3}{\pi}\right)^m\left(\frac{v_4}{\pi}\right)^n, \qquad (57)\\
\beta_e(\alpha_s,\alpha_e,\{v_r\}) &= \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\frac{\alpha_e}{\pi} \\
&= -\left[\epsilon\frac{\alpha_e}{\pi}+2\frac{\alpha_e}{Z_e}\left(\frac{\partial Z_e}{\partial\alpha_s}\beta_s^{\overline{\mathrm{DR}}}+\sum_r\frac{\partial Z_e}{\partial v_r}\beta_{v_r}\right)\right]\left(1+2\frac{\alpha_e}{Z_e}\frac{\partial Z_e}{\partial\alpha_e}\right)^{-1} \\
&= -\epsilon\frac{\alpha_e}{\pi}-\sum_{i,j,k,l,m,n}\beta_{ijklmn}^{e}\left(\frac{\alpha_s}{\pi}\right)^i\left(\frac{\alpha_e}{\pi}\right)^j\left(\frac{v_1}{\pi}\right)^k\left(\frac{v_2}{\pi}\right)^l\left(\frac{v_3}{\pi}\right)^m\left(\frac{v_4}{\pi}\right)^n, \qquad (58)\\
\beta_{v_r}(\alpha_s,\alpha_e,\{v_r\}) &= \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\frac{v_r}{\pi} \\
&= -\left[\epsilon\frac{v_r}{\pi}+2\frac{v_r}{Z_{\lambda_r}}\left(\frac{\partial Z_{\lambda_r}}{\partial\alpha_s}\beta_s^{\overline{\mathrm{DR}}}+\frac{\partial Z_{\lambda_r}}{\partial\alpha_e}\beta_e+\sum_{r'\neq r}\frac{\partial Z_{\lambda_r}}{\partial v_{r'}}\beta_{v_{r'}}\right)\right] \\
&\quad\left(1+2\frac{v_r}{Z_{\lambda_r}}\frac{\partial Z_{\lambda_r}}{\partial v_r}\right)^{-1} \\
&= -\epsilon\frac{v_r}{\pi}-\sum_{i,j,k,l,m,n}\beta_{ijklmn}^{v_r}\left(\frac{\alpha_s}{\pi}\right)^i\left(\frac{\alpha_e}{\pi}\right)^j\left(\frac{v_1}{\pi}\right)^k\left(\frac{v_2}{\pi}\right)^l\left(\frac{v_3}{\pi}\right)^m\left(\frac{v_4}{\pi}\right)^n. \qquad (59)
\end{aligned}$$

Here and in the following we do not explicitly display the dependence on the renormalisation scale μ, i.e., $\alpha_s\equiv\alpha_s(\mu)$ etc. Note that in the second line of Eq. (57), the $\mathcal{O}(\epsilon)$ terms of β_e and β_{v_r} contribute to the finite part of $\beta_s^{\overline{\mathrm{DR}}}$, and similarly for Eqs. (58) and (59). As we will see below, in order to compute the four-loop term of $\beta^{\overline{\mathrm{DR}}}$ one needs β_e to two loops and β_{v_r} $(r=1,\cdots 4)$ to one loop.

For the cases when the fermion representation allows a mass term we introduce the fermion mass anomalous dimension, which is defined through

$$\begin{aligned}
\gamma_m^{\overline{\mathrm{DR}}} &= \frac{\mu^2}{m^{\overline{\mathrm{DR}}}}\frac{\mathrm{d}}{\mathrm{d}\mu^2}m^{\overline{\mathrm{DR}}} \\
&= -\pi\beta_s^{\overline{\mathrm{DR}}}\frac{\partial\ln Z_m^{\overline{\mathrm{DR}}}}{\partial\alpha_s}-\pi\beta_e\frac{\partial\ln Z_m^{\overline{\mathrm{DR}}}}{\partial\alpha_e}-\pi\sum_r\beta_{v_r}\frac{\partial\ln Z_m^{\overline{\mathrm{DR}}}}{\partial v_r} \\
&= -\sum_{i\cdots n}\gamma_{i\cdots n}^{\overline{\mathrm{DR}}}\left(\frac{\alpha_s}{\pi}\right)^i\left(\frac{\alpha_e}{\pi}\right)^j\left(\frac{v_1}{\pi}\right)^k\left(\frac{v_2}{\pi}\right)^l\left(\frac{v_3}{\pi}\right)^m\left(\frac{v_4}{\pi}\right)^n. \qquad (60)
\end{aligned}$$

From this equation one can see that for the four-loop term of $\gamma_m^{\overline{\mathrm{DR}}}$, the beta functions β_e

and β_{v_r} are needed to three loops and one loop, respectively, since the dependence of $Z_m^{\overline{\rm DR}}$ on α_e (v_r) starts at one loop (three loops) [1]. The general result for the two-loop order β_e is known [1] and for the three-loop order contributions we find (generalising the QCD results from Ref. [2]):

$$
\begin{aligned}
\beta^e_{022000} &= \frac{1}{256}[-6I_2(R)N_An_f - 12N_AC_R + 6N_AC_A - 10C_R + 35C_A + 15I_2(R)n_f] \\
\beta^e_{010030} &= -\frac{63}{1024}C_A^3 \\
\beta^e_{030100} &= -\frac{3}{32}[C_A^2 + 6I_2(R)C_Rn_f - 10C_AC_R + 2C_AI_2(R)n_f + 16C_R^2] \\
\beta^e_{020200} &= -\frac{1}{256}[25N_AC_A - 3I_2(R)N_An_f - 30N_AC_R - 5C_A - 33I_2(R)n_f + 118C_R] \\
\beta^e_{021010} &= \frac{1}{128}[-27I_2(R)n_f + 53C_A - 30C_R]C_A \\
\beta^e_{210010} &= -\frac{3}{1024N_AI_2(R)}[5C_A^3I_2(R)N_A + 128D_2(RA)] \\
\beta^e_{110101} &= -\frac{17}{64}C_A^3 \\
\beta^e_{211000} &= -\frac{15}{512}C_A^2 \\
\beta^e_{012100} &= \frac{3}{64}(N_A - 1) \\
\beta^e_{021100} &= \frac{1}{128}[3N_AC_A + 15I_2(R)N_An_f + 6N_AC_R - 9C_A + 3I_2(R)n_f - 50C_R] \\
\beta^e_{010003} &= \frac{1}{24576N_A}(96D_2(A) - 7C_A^4N_A)C_A^2 \\
\beta^e_{020110} &= \frac{1}{128}[35C_A + 27I_2(R)n_f - 18C_R]C_A \\
\beta^e_{110200} &= \frac{1}{64}(N_A - 9)C_A \\
\beta^e_{010300} &= -\frac{1}{256}(3N_A + 2)(N_A - 1) \\
\beta^e_{111001} &= -\frac{3}{64}C_A^3 \\
\beta^e_{010120} &= \frac{3}{512}C_A^2 \\
\beta^e_{010021} &= -\frac{9}{2048N_A}(32D_2(A) + 7C_A^4N_A) \\
\beta^e_{110020} &= \frac{33}{128}C_A^3
\end{aligned}
$$

$$
\begin{aligned}
\beta^e_{011020} &= -\frac{93}{512}C_A^2 \\
\beta^e_{020020} &= \frac{1}{512}[71C_A - 42C_R - 81I_2(R)n_f]C_A^2 \\
\beta^e_{010012} &= -\frac{21}{4096}C_A^5 \\
\beta^e_{010111} &= \frac{81}{512}C_A^3 \\
\beta^e_{110011} &= \frac{11}{128}C_A^4 \\
\beta^e_{011011} &= -\frac{51}{512}C_A^3 \\
\beta^e_{011200} &= \frac{3}{256}(N_A-1)(N_A-2) \\
\beta^e_{020011} &= \frac{1}{1536N_AI_2(R)}\{-384C_AD_2(RA) + 768D_2(A)I_2(R) \\
&+ N_AI_2(R)[71C_A^4 - 42C_RC_A^3 - 81I_2(R)n_fC_A^3]\} \\
\beta^e_{121000} &= -\frac{1}{32}(11C_A^2 - 8C_AC_R + 8C_R^2) \\
\beta^e_{010210} &= \frac{3}{256}(5N_A-2)C_A \\
\beta^e_{110110} &= -\frac{11}{32}C_A^2 \\
\beta^e_{011110} &= -\frac{3}{128}(-10+N_A)C_A \\
\beta^e_{012010} &= -\frac{9}{64}C_A \\
\beta^e_{111010} &= \frac{11}{32}C_A^2 \\
\beta^e_{120010} &= -\frac{1}{32}(14C_A + 5C_R)C_A^2 \\
\beta^e_{030010} &= -\frac{3}{64}(C_A - 2C_R - 9I_2(R)n_f)C_A^2 \\
\beta^e_{031000} &= -\frac{3}{64}[-12C_AC_R + 6I_2(R)C_Rn_f - 7C_AI_2(R)n_f + 2C_A^2 + 16C_R^2] \\
\beta^e_{210100} &= -\frac{1}{512}(-192C_R + 47C_A)C_A \\
\beta^e_{111100} &= -\frac{1}{32}(5N_A-1)C_A \\
\beta^e_{010102} &= \frac{1}{2048N_A}(96D_2(A) - 13C_A^4N_A)
\end{aligned}
$$

$$
\begin{aligned}
\beta^e_{013000} &= -\frac{1}{64}(-1+N_A) \\
\beta^e_{110002} &= -\frac{1}{1536N_A}[-11C_A^4N_A+192D_2(A)]C_A \\
\beta^e_{112000} &= \frac{1}{64}(3N_A-5)C_A \\
\beta^e_{011002} &= -\frac{1}{2048N_A}(96D_2(A)-43C_A^4N_A) \\
\beta^e_{010201} &= -\frac{3}{512}(5N_A-6)C_A^2 \\
\beta^e_{011101} &= \frac{9}{256}N_AC_A^2 \\
\beta^e_{020101} &= -\frac{1}{768N_AI_2(R)}\{N_AI_2(R)[185C_A^3-207C_A^2I_2(R)n_f-342C_A^2C_R] \\
&+ 960D_2(RA)\} \\
\beta^e_{210001} &= -\frac{1}{2048N_AI_2(R)}[5C_A^4I_2(R)N_A-384C_AD_2(RA)+96D_2(A)I_2(R)] \\
\beta^e_{012001} &= -\frac{3}{128}C_A^2 \\
\beta^e_{021001} &= -\frac{1}{768N_AI_2(R)}\{N_AI_2(R)[90C_A^2C_R-143C_A^3-63C_A^2I_2(R)n_f] \\
&+ 192D_2(RA)\} \\
\beta^e_{120100} &= \frac{1}{16}(3C_A^2-8C_R^2+13C_AC_R) \\
\beta^e_{030001} &= -\frac{1}{128N_AI_2(R)}\{(C_A-2C_R)[-96D_2(RA)+C_A^3I_2(R)N_A] \\
&- 9I_2(R)[-8D_2(RA)+C_A^3I_2(R)N_A]n_f\} \\
\beta^e_{020002} &= -\frac{1}{18432N_AI_2(R)[-25C_A^4N_A+12D_2(A)(2+N_A)]} \\
&\quad \{-3981312D_3(A)D_2(RA)+1132800C_A^6D_2(RA)N_A \\
&- 829440C_A^3D_3(A)I_2(R)N_A+12C_A^5D_2(A)I_2(R)(18058-71N_A)N_A \\
&+ 1775C_A^9I_2(R)N_A^2+1152C_AD_2(A)^2I_2(R)(586+5N_A) \\
&+ 179712D_3(RAA)[-25C_A^4N_A+12D_2(A)(2+N_A)] \\
&+ 9216C_A^2[540C_RD_3(A)I_2(R)N_A+D_2(A)D_2(RA)(362+N_A)] \\
&- 75C_A^8I_2(R)N_A^2[14C_R+27I_2(R)n_f] \\
&+ 36C_A^4D_2(A)I_2(R)N_A[2C_R(-16586+7N_A) \\
&+ 9I_2(R)(206+3N_A)n_f]-1990656D_3(A)D_2(RA)N_A \\
&- 3456D_2(A)^2I_2(R)[2C_R(602+13N_A)+9I_2(R)(2+N_A)n_f]\} \\
\beta^e_{120001} &= -\frac{1}{192}C_A^3(14C_A+5C_R)+\frac{1}{8N_AI_2(R)}(5C_A-4C_R)D_2(RA)
\end{aligned}
$$

$$
\begin{aligned}
\beta^e_{130000} &= -\frac{1}{64N_AI_2(R)}\Big(48D_2(RA)+I_2(R)N_A\{11C_A^3-242C_A^2C_R\\
&+ 640C_AC_R^2-416C_R^3-28C_A^2I_2(R)n_f+144C_AC_RI_2(R)n_f\\
&- 104C_R^2I_2(R)n_f-12C_AI_2(R)^2n_f^2\\
&+ 48(C_A-2C_R)(C_A-C_R)[2C_R-C_A+I_2(R)n_f]\zeta_3\}\Big)\\
\beta^e_{220000} &= -\frac{1}{1536N_AI_2(R)}\Big[3456D_2(R)n_f+I_2(R)N_A[-335C_A^3-642C_A^2C_R\\
&- 2148C_AC_R^2+3336C_R^3+3(247C_A^2+896C_AC_R-1180C_R^2)I_2(R)n_f\\
&+ 24(C_A-16C_R)I_2(R)^2n_f^2]-384D_2(RA)-288\Big(24D_2(RA)\\
&+ 24D_2(R)n_f+I_2(R)N_A\{-22C_A^3+6C_R^2[6C_R-I_2(R)n_f]\\
&+ 3C_AC_R[-32C_R+I_2(R)n_f]+C_A^2[81C_R+2I_2(R)n_f]\}\Big)\zeta_3\Big]\\
\beta^e_{310000} &= -\frac{1}{13824N_AI_2(R)}\Big(-15552D_2(RA)+I_2(R)N_A\{13755C_A^3\\
&- 4C_A^2[13819C_R+1389I_2(R)n_f]\\
&- 8C_R[3483C_R^2-280I_2(R)^2n_f^2+108C_RI_2(R)n_f(-23+24\zeta_3)]\\
&+ 4C_A[12339C_R^2+120I_2(R)^2n_f^2+4C_RI_2(R)n_f(157+1296\zeta_3)]\}\Big)\\
\beta^e_{040000} &= -\frac{1}{192N_AI_2(R)[12D_2(A)(2+N_A)-25C_A^4N_A]}\\
&\quad\Big[8640C_A^3D_2(RA)I_2(R)N_An_f-51840C_A^2C_RD_2(RA)I_2(R)N_An_f\\
&+ 10368D_2(RA)^2(2+N_A)n_f+100C_A^7I_2(R)N_A^2(-4+111\zeta_3)\\
&- 150C_A^4N_A[C_RI_2(R)^3N_An_f^2+2C_R^2I_2(R)^2N_An_f(23-6\zeta_3)\\
&+ 24D_2(R)n_f(-7+2\zeta_3)+8C_R^3I_2(R)N_A(7+9\zeta_3)\\
&+ 16D_2(RA)(-1+12\zeta_3)]+75C_A^5I_2(R)N_A^2[7I_2(R)^2n_f^2\\
&- 4C_RI_2(R)n_f(-31+9\zeta_3)+8C_R^2(17+60\zeta_3)]\\
&- 75C_A^6I_2(R)N_A^2[I_2(R)n_f(47-16\zeta_3)+4C_R(8+117\zeta_3)]\\
&+ 12D_2(A)\Big(96D_2(RA)(2+N_A)(-1+12\zeta_3)\\
&- 4C_A^3I_2(R)N_A(2+N_A)(-4+111\zeta_3)\\
&+ 3C_A^2I_2(R)N_A\{4C_R(2+N_A)(8+117\zeta_3)\\
&+ I_2(R)n_f[118+47N_A-16(2+N_A)\zeta_3]\}\\
&- 3C_AI_2(R)N_A\{7I_2(R)^2(2+N_A)n_f^2+8C_R^2(2+N_A)(17+60\zeta_3)\\
&+ 4C_RI_2(R)n_f[134+31N_A-9(2+N_A)\zeta_3]\}\\
&+ 6C_RI_2(R)^3N_A(2+N_A)n_f^2+144D_2(R)(2+N_A)n_f(-7+2\zeta_3)\\
&+ 48C_R^3I_2(R)N_A(2+N_A)(7+9\zeta_3)\\
&+ 12C_R^2I_2(R)^2N_An_f[262+23N_A-6(2+N_A)\zeta_3]\Big)\Big]. \qquad (61)
\end{aligned}
$$

(Here and elsewhere we denote $\zeta(n)$ by ζ_n.) We computed the four-loop DRED quantities from their DREG counterparts using the indirect method discussed in Refs. [1, 15]. It is based on the following formulæ:

$$\begin{aligned}
\beta_s^{\overline{\rm DR}} &= \beta_s^{\overline{\rm MS}} \frac{\partial \alpha_s}{\partial \alpha_s^{\overline{\rm MS}}} + \beta_e \frac{\partial \alpha_s}{\partial \alpha_e} + \sum_r \beta_{v_r} \frac{\partial \alpha_s}{\partial v_r}\,, \\
\gamma_m^{\overline{\rm DR}} &= \gamma_m^{\overline{\rm MS}} \frac{\partial \ln m^{\overline{\rm DR}}}{\partial \ln m^{\overline{\rm MS}}} + \frac{\pi \beta_s^{\overline{\rm MS}}}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial \alpha_s^{\overline{\rm MS}}} + \frac{\pi \beta_e}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial \alpha_e} + \sum_r \frac{\pi \beta_{v_r}}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial v_r}\,. \qquad (62)
\end{aligned}$$

Let us briefly discuss the order in perturbation theory up to which the individual building blocks are needed. Of course, the $\overline{\rm MS}$ quantities are needed to four-loop order; they can be found in Refs. [16–19]. The dependence of α_s and $m^{\overline{\rm DR}}$ on α_e starts at two- and one-loop order [1], respectively. Thus, β_e is needed up to the three-loop level (cf. Eq. (62)). On the other hand, both α_s and $m^{\overline{\rm DR}}$ depend on v_r starting from three loops and consequently only the one-loop term of β_{v_r} enters in Eq. (62). It is given in Eq. (54).

For the four-loop analysis we also require the three-loop relations between α_s and $\alpha_s^{\overline{\rm MS}}$ and between $m^{\overline{\rm DR}}$ and $m^{\overline{\rm MS}}$. The two-loop results were presented in Ref. [1], and the three-loop results for the special case of QCD in Ref. [2]. Parametrising the three-loop terms by $\delta_\alpha^{(3)}$ and $\delta_m^{(3)}$, we have

$$\begin{aligned}
\alpha_s &= \alpha_s^{\overline{\rm MS}} \left[1 + \frac{\alpha_s^{\overline{\rm MS}}}{\pi} \frac{1}{12} C_A + \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi} \right)^2 \frac{11}{72} C_A^2 - \frac{\alpha_s^{\overline{\rm MS}}}{\pi} \frac{\alpha_e}{\pi} \frac{1}{8} C_R I_2(R) n_f + \delta_\alpha^{(3)} + \ldots \right], \\
m^{\overline{\rm DR}} &= m^{\overline{\rm MS}} \Bigg[1 - \frac{\alpha_e}{\pi} \frac{1}{4} C_R + \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi} \right)^2 \frac{11}{192} C_A C_R - \frac{\alpha_s^{\overline{\rm MS}}}{\pi} \frac{\alpha_e}{\pi} \frac{1}{32} C_R (3C_A + 8C_R) \\
&\quad + \left(\frac{\alpha_e}{\pi} \right)^2 \frac{1}{32} [3C_R + I_2(R) n_f] + \delta_m^{(3)} + \ldots \Bigg] \qquad (63)
\end{aligned}$$

where the dots denote higher orders in $\alpha_s^{\overline{\rm MS}}$, α_e, and v_r. We find

$$\begin{aligned}
\pi^3 \delta_\alpha^{(3)} &= \frac{1}{96} \alpha_s^{\overline{\rm MS}} \alpha_e^2 I_2(R) n_f [2C_A^2 - 3C_A C_R + 2C_R^2 - C_A I_2(R) n_f] \\
&+ 7 C_R I_2(R) n_f) - \frac{1}{192} (\alpha_s^{\overline{\rm MS}})^2 \alpha_e I_2(R) n_f (5C_A^2 + 60 C_A C_R + 6C_R^2) \\
&+ \frac{1}{9216} \alpha_s^{\overline{\rm MS}} (-168 C_A^3 v_4 v_2 - 72 C_A^3 v_4 v_1 + 12 v_3 v_4 C_A^4 - 48 v_2 v_3 C_A^2 \\
&+ 48 v_1 v_3 C_A^2 - 48 C_A v_1 v_2 - 48 C_A N_A v_1 v_2 + 36 C_A^3 v_3^2 + C_A^5 v_4^2 - 72 C_A v_2^2 \\
&- 24 C_A N_A v_2^2 - 24 C_A v_1^2) - \frac{1}{96 N_A} \alpha_s^{\overline{\rm MS}} v_4^2 C_A D_2(A) + \frac{1}{48} (\alpha_s^{\overline{\rm MS}})^2 v_4 D_2(A) \\
&+ \frac{1}{4608} (\alpha_s^{\overline{\rm MS}})^2 (-6 C_A^3 v_3 + 84 C_A^2 v_2 + 36 C_A^2 v_1 - v_4 C_A^4) \\
&+ \frac{1}{10368} (\alpha_s^{\overline{\rm MS}})^3 [3049 C_A^3 - 416 C_A^2 I_2(R) n_f - 138 C_A C_R I_2(R) n_f] \qquad (64)
\end{aligned}$$

$$
\begin{aligned}
\pi^3\delta_m^{(3)} &= -\frac{1}{384}\alpha_e^3 C_R[-10C_A^2+14C_AC_R+27C_R^2-7C_AI_2(R)n_f \\
&+ 39C_RI_2(R)n_f-10I_2(R)^2n_f^2+12C_A^2\zeta_3-36C_AC_R\zeta_3+24C_R^2\zeta_3] \\
&- \alpha_e^2C_R\Big(\frac{1}{192}[6C_Rv_1+12C_Rv_2-2C_Av_2-C_Av_1] \\
&+ \frac{1}{16I_2(R)N_A}D_2(RA)v_4+\frac{1}{384}\alpha_s^{\overline{\mathrm{MS}}}[47C_A^2+10C_R^2 \\
&- 3I_2(R)C_An_f-19I_2(R)C_Rn_f-165C_AC_R+144C_R^2\zeta_3 \\
&- 48I_2(R)C_An_f\zeta_3+48I_2(R)C_Rn_f\zeta_3+72C_A^2\zeta_3-216C_AC_R\zeta_3]\Big) \\
&+ \alpha_eC_R\Big(\frac{1}{12288}[200v_2^2+88N_Av_2^2+56v_1^2+16N_Av_1^2 \\
&+ 112N_Av_2v_1-C_A^4v_4^2-12C_A^3v_3v_4+176v_2v_1+48v_3C_Av_2 \\
&- 36C_A^2v_3^2+488C_A^2v_4v_2+232C_A^2v_4v_1-48C_Av_1v_3] \\
&+ \frac{1}{3072}(\alpha_s^{\overline{\mathrm{MS}}})^2[2880C_R^2\zeta_3-168C_AI_2(R)n_f-1544C_AC_R-52C_R^2 \\
&- 128I_2(R)C_Rn_f+1440C_A^2\zeta_3-4320C_AC_R\zeta_3-79C_A^2]\Big) \\
&+ \frac{1}{20736}(\alpha_s^{\overline{\mathrm{MS}}})^3C_RC_A[4354C_A+135C_R+304I_2(R)n_f] \\
&+ \frac{3}{128N_A}D_2(A)v_4^2.
\end{aligned}
\tag{65}
$$

4.1 The β function and anomalous dimension

Inserting Eqs. (64) and (65) into Eq. (62), we obtain

$$
\begin{aligned}
\beta_{500000}^{\overline{\rm DR}} &= b_3 - \frac{1}{165888 N_A}\{2592 D_2(A) \\
&+ C_A N_A [27648 b_2 - C_A(1152 b_1 + 85280 b_0 C_A + 27 C_A^2) \\
&+ 64 b_0 (208 C_A + 69 C_R) I_2(R) n_f]\} \\
\beta_{203000}^{\overline{\rm DR}} &= \frac{1}{192} C_A \\
\beta_{302000}^{\overline{\rm DR}} &= -\frac{1}{64} C_A^2 \\
\beta_{311000}^{\overline{\rm DR}} &= -\frac{1}{128} n_f I_2(R) C_A^2 \\
\beta_{300002}^{\overline{\rm DR}} &= -\frac{1}{663552 N_A}[19872 C_A^2 D_2(A) - 227 C_A^6 N_A + 27648 D_3(A)] \\
\beta_{400010}^{\overline{\rm DR}} &= -\frac{1}{3072}(4 b_0 + 9 C_A) C_A^3 \\
\beta_{200030}^{\overline{\rm DR}} &= -\frac{11}{3072} C_A^4 \\
\beta_{310001}^{\overline{\rm DR}} &= -\frac{1}{4608 N_A}[96 D_2(A) - C_A^4 N_A] I_2(R) n_f \\
\beta_{220100}^{\overline{\rm DR}} &= -\frac{1}{48}(C_A + 3 C_R) n_f I_2(R) C_R \\
\beta_{301100}^{\overline{\rm DR}} &= -\frac{1}{256}(5 N_A + 12) C_A^2 \\
\beta_{210011}^{\overline{\rm DR}} &= -\frac{1}{1536}(4 C_A + 3 C_R) n_f C_A^3 I_2(R) \\
\beta_{400001}^{\overline{\rm DR}} &= \frac{1}{18432 N_A}(4 b_0 + 9 C_A)[96 D_2(A) - C_A^4 N_A] \\
\beta_{300011}^{\overline{\rm DR}} &= \frac{1}{55296 N_A}[384 D_2(A) + 227 C_A^4 N_A] C_A \\
\beta_{400100}^{\overline{\rm DR}} &= \frac{7}{1536}(4 b_0 + 9 C_A) C_A^2 \\
\beta_{401000}^{\overline{\rm DR}} &= \frac{1}{512}(4 b_0 + 9 C_A) C_A^2 \\
\beta_{200120}^{\overline{\rm DR}} &= -\frac{3}{512} C_A^3 \\
\beta_{201020}^{\overline{\rm DR}} &= -\frac{7}{512} C_A^3 \\
\beta_{310100}^{\overline{\rm DR}} &= -\frac{7}{384} n_f I_2(R) C_A^2
\end{aligned}
$$

$$
\begin{aligned}
\beta^{\overline{\mathrm{DR}}}_{220001} &= -\frac{1}{1152N_A[-25C_A^4N_A+12D_2(A)(2+N_A)]} \\
&\quad \{C_A^3(C_A+3C_R)(25C_A^4-12D_2(A))I_2(R)N_A^2 \\
&- 24(C_A+3C_R)[25C_A^4D_2(RA)-36D_2(A)D_2(RA) \\
&+ C_A^3D_2(A)I_2(R)]N_A\} \\
\beta^{\overline{\mathrm{DR}}}_{202100} &= \frac{1}{64}(N_A+1)C_A \\
\beta^{\overline{\mathrm{DR}}}_{410000} &= -\frac{1}{1536}I_2(R)n_f\{8b_0(5C_A^2+56C_AC_R+6C_R^2) \\
&+ C_R[-192b_1+39C_A^2+892C_AC_R+108C_R^2+24C_AI_2(R)n_f \\
&- 80C_RI_2(R)n_f]\} \\
\beta^{\overline{\mathrm{DR}}}_{301001} &= -\frac{1}{3072}[96D_2(A)+89C_A^4N_A] \\
\beta^{\overline{\mathrm{DR}}}_{220010} &= \frac{1}{192}(C_A+3C_R)n_fI_2(R)C_A^2 \\
\beta^{\overline{\mathrm{DR}}}_{300020} &= \frac{1}{18432N_A}[96D_2(A)+227C_A^4N_A] \\
\beta^{\overline{\mathrm{DR}}}_{210110} &= \frac{1}{384}(4C_A+3C_R)n_fI_2(R)C_A \\
\beta^{\overline{\mathrm{DR}}}_{320000} &= -\frac{1}{1152N_A}n_f\Big(-24D_2(RA) \\
&+ I_2(R)N_A\{16C_A^3+6C_AC_R[25C_R-22I_2(R)n_f] \\
&+ 3C_A^2[4C_R-5I_2(R)n_f]-72C_R^2[7C_R+5I_2(R)n_f]\}\Big) \\
\beta^{\overline{\mathrm{DR}}}_{230000} &= -\frac{1}{192}I_2(R)n_f[-4C_A^3+23C_A^2C_R-46C_AC_R^2+32C_R^3 \\
&+ (6C_A^2-33C_AC_R+50C_R^2)I_2(R)n_f-2(C_A-7C_R)I_2(R)^2n_f^2] \\
\beta^{\overline{\mathrm{DR}}}_{300200} &= -\frac{1}{1536}(19N_A+82)C_A^2 \\
\beta^{\overline{\mathrm{DR}}}_{200111} &= -\frac{1}{512}C_A^4 \\
\beta^{\overline{\mathrm{DR}}}_{201011} &= -\frac{9}{512}C_A^4 \\
\beta^{\overline{\mathrm{DR}}}_{300110} &= -\frac{23}{1536}C_A^3 \\
\beta^{\overline{\mathrm{DR}}}_{301010} &= \frac{11}{512}C_A^3 \\
\beta^{\overline{\mathrm{DR}}}_{211100} &= \frac{1}{384}(N_A+1)(4C_A+3C_R)I_2(R)n_f \\
\beta^{\overline{\mathrm{DR}}}_{300101} &= -\frac{1}{9216N_A}[480D_2(A)+703C_A^4N_A]
\end{aligned}
$$

$$
\begin{aligned}
\beta^{\overline{\rm DR}}_{210020} &= -\frac{1}{512}(4C_A+3C_R)n_f I_2(R)C_A^2 \\
\beta^{\overline{\rm DR}}_{310010} &= \frac{1}{768}I_2(R)C_A^3 n_f \\
\beta^{\overline{\rm DR}}_{200003} &= -\frac{1}{663552N_A}[864C_A^2D_2(A)+11C_A^6N_A-27648D_3(A)]C_A \\
\beta^{\overline{\rm DR}}_{200102} &= \frac{1}{6144N_A}[480D_2(A)+199C_A^4N_A]C_A \\
\beta^{\overline{\rm DR}}_{201002} &= \frac{1}{2048N_A}[96D_2(A)+17C_A^4N_A]C_A \\
\beta^{\overline{\rm DR}}_{202001} &= \frac{3}{128}C_A^3 \\
\beta^{\overline{\rm DR}}_{200300} &= \frac{1}{768}(N_A^2+13N_A+18)C_A \\
\beta^{\overline{\rm DR}}_{201101} &= \frac{3}{256}(8+N_A)C_A^3 \\
\beta^{\overline{\rm DR}}_{200021} &= -\frac{1}{6144N_A}[96D_2(A)+11C_A^4N_A]C_A \\
\beta^{\overline{\rm DR}}_{202010} &= -\frac{1}{64}C_A^2 \\
\beta^{\overline{\rm DR}}_{210002} &= \frac{1}{18432N_A}(4C_A+3C_R)[96D_2(A)-C_A^4N_A]I_2(R)n_f \\
\beta^{\overline{\rm DR}}_{212000} &= \frac{1}{768}(4C_A+3C_R)I_2(R)n_f \\
\beta^{\overline{\rm DR}}_{211010} &= -\frac{1}{384}(4C_A+3C_R)n_f I_2(R)C_A \\
\beta^{\overline{\rm DR}}_{201200} &= \frac{1}{256}(N_A^2+5N_A+10)C_A \\
\beta^{\overline{\rm DR}}_{200210} &= \frac{1}{256}(N_A-2)C_A^2 \\
\beta^{\overline{\rm DR}}_{201110} &= -\frac{1}{128}(N_A-2)C_A^2 \\
\beta^{\overline{\rm DR}}_{221000} &= \frac{1}{192}(C_A+3C_R)(C_A-2C_R)I_2(R)n_f \\
\beta^{\overline{\rm DR}}_{200201} &= \frac{1}{512}(7N_A+46)C_A^3 \\
\beta^{\overline{\rm DR}}_{211001} &= \frac{1}{256}(4C_A+3C_R)n_f I_2(R)C_A^2 \\
\beta^{\overline{\rm DR}}_{200012} &= -\frac{1}{36864N_A}[384D_2(A)+11C_A^4N_A]C_A^2 \\
\beta^{\overline{\rm DR}}_{210200} &= \frac{1}{768}(N_A+3)(4C_A+3C_R)I_2(R)n_f \\
\beta^{\overline{\rm DR}}_{210101} &= \frac{7}{768}(4C_A+3C_R)n_f I_2(R)C_A^2
\end{aligned} \tag{66}
$$

where

$$
\begin{aligned}
b_0 &= \frac{1}{4}\Big(\frac{11}{3}C_A - \frac{4}{3}I_2(R)n_f\Big)\\
b_1 &= \frac{1}{16}\Big(\frac{34}{3}C_A^2 - 4C_RI_2(R)n_f - \frac{20}{3}C_AI_2(R)n_f\Big)\\
b_2 &= \frac{1}{64}\Big(\frac{2857}{54}C_A^3 + 2C_R^2I_2(R)n_f - \frac{205}{9}C_RC_AI_2(R)n_f\\
&- \frac{1415}{27}C_A^2I_2(R)n_f + \frac{44}{9}C_RI_2(R)^2n_f^2 + \frac{158}{27}C_AI_2(R)^2n_f^2\Big)\\
b_3 &= \frac{1}{256}\Big[\Big(\frac{150653}{486} - \frac{44}{9}\zeta_3\Big)C_A^4 + C_A^3I_2(R)n_f\Big(-\frac{39143}{81} + \frac{136}{3}\zeta_3\Big)\\
&+ C_A^2C_RI_2(R)n_f\Big(\frac{7073}{243} - \frac{656}{9}\zeta_3\Big) + C_AC_R^2I_2(R)n_f\Big(-\frac{4204}{27} + \frac{352}{9}\zeta_3\Big)\\
&+ 46C_R^3I_2(R)n_f + C_A^2I_2(R)^2n_f^2\Big(\frac{7930}{81} + \frac{224}{9}\zeta_3\Big)\\
&+ C_R^2I_2(R)^2n_f^2\Big(\frac{1352}{27} - \frac{704}{9}\zeta_3\Big) + C_AC_RI_2(R)^2n_f^2\Big(\frac{17152}{243} + \frac{448}{9}\zeta_3\Big)\\
&+ \frac{424}{243}C_AI_2(R)^3n_f^3 + \frac{1232}{243}C_RI_2(R)^3n_f^3\\
&+ \frac{D_2(A)}{N_A}\Big(-\frac{80}{9} + \frac{704}{3}\zeta_3\Big) + \frac{n_fD_2(RA)}{N_A}\Big(\frac{512}{9} - \frac{1664}{3}\zeta_3\Big)\\
&+ \frac{n_f^2D_2(R)}{N_A}\Big(-\frac{704}{9} + \frac{512}{3}\zeta_3\Big)\Big]
\end{aligned}
\tag{67}
$$

are the one, two, three and four-loop gauge β-function coefficients calculated in DREG.

For the fermion mass anomalous dimension, we find

$$
\begin{aligned}
\gamma^{\overline{\rm DR}}_{400000} &= \gamma_3 + \frac{91}{768}C_R^2C_A^2 - \frac{129}{512}C_AC_R^3 - \frac{3}{16}I_2(R)C_AC_R^2n_f\zeta_3 \\
&+ \frac{89}{576}I_2(R)C_AC_R^2n_f + \frac{29}{5184}I_2(R)^2C_AC_Rn_f^2 + \frac{3}{16}I_2(R)C_A^2C_Rn_f\zeta_3 \\
&- \frac{53}{1296}C_RC_A^2I_2(R)n_f - \frac{19003}{82944}C_A^3C_R \\
\gamma^{\overline{\rm DR}}_{110002} &= \frac{1}{24576N_A}[2784C_AD_2(A) + 1632C_RD_2(A) - 53C_A^5N_A \\
&- 11C_RC_A^4N_A]C_R \\
\gamma^{\overline{\rm DR}}_{110011} &= -\frac{1}{2048}(53C_A + 11C_R)C_A^3C_R \\
\gamma^{\overline{\rm DR}}_{120010} &= \frac{1}{256}(31C_A + 18C_R)C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{210010} &= \frac{3}{2048N_AI_2(R)}[3C_A^3I_2(R)N_A + 64D_2(RA)]C_R \\
\gamma^{\overline{\rm DR}}_{111001} &= \frac{1}{1024}(53C_R + 79C_A)C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{020110} &= -\frac{1}{256}[17C_A + 17I_2(R)n_f - 6C_R]C_AC_R \\
\gamma^{\overline{\rm DR}}_{020011} &= -\frac{1}{3072N_AI_2(R)}\{384D_2(A)I_2(R) - 144C_AD_2(RA) \\
&+ N_AI_2(R)[-51C_A^3I_2(R)n_f - 30C_A^3C_R + 37C_A^4]\}C_R \\
\gamma^{\overline{\rm DR}}_{310000} &= -\frac{1}{165888I_2(R)N_A}C_R\Big(46656D_2(RA) \\
&+ I_2(R)N_A\{-26505C_A^3 + C_A^2[355107C_R + 23544I_2(R)n_f] \\
&+ 2C_A[11916C_R^2 - 65508C_RI_2(R)n_f - 3744I_2(R)^2n_f^2] \\
&+ 12C_R[7965C_R^2 - 7212C_RI_2(R)n_f - 224I_2(R)^2n_f^2] \\
&+ 2592(C_A - C_R)[5(-11C_A - 9C_R)(C_A - 2C_R) \\
&+ 4(5C_A - 16C_R)I_2(R)n_f]\zeta_3\}\Big) \\
\gamma^{\overline{\rm DR}}_{110020} &= -\frac{3}{2048}(53C_A + 11C_R)C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{211000} &= -\frac{7}{1024}C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{020020} &= -\frac{1}{1024N_AI_2(R)}\{N_AI_2(R)[-51C_A^2I_2(R)n_f - 30C_A^2C_R \\
&+ 37C_A^3] + 16D_2(RA)\}C_R \\
\gamma^{\overline{\rm DR}}_{020101} &= \frac{1}{1536N_AI_2(R)}\{N_AI_2(R)[103C_A^3 - 154C_A^2C_R \\
&- 277C_A^2I_2(R)n_f] + 720D_2(RA)\}C_R
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{\overline{\rm DR}}_{110200} &= \frac{1}{1024}(19C_RN_A + 21C_AN_A + 95C_A + 49C_R)C_R \\
\gamma^{\overline{\rm DR}}_{021100} &= -\frac{1}{768}[43I_2(R)N_An_f + 10C_RN_A + 3C_AN_A + 35I_2(R)n_f - 142C_R \\
&- 5C_A]C_R \\
\gamma^{\overline{\rm DR}}_{021010} &= -\frac{1}{768}[-51I_2(R)n_f - 42C_R + 79C_A]C_AC_R \\
\gamma^{\overline{\rm DR}}_{110101} &= \frac{1}{1024}(117C_R + 211C_A)C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{120100} &= -\frac{1}{32}(C_A^2 + 6C_R^2 + 10C_AC_R)C_R \\
\gamma^{\overline{\rm DR}}_{200011} &= -\frac{1}{1024}C_A^4C_R \\
\gamma^{\overline{\rm DR}}_{200101} &= \frac{7}{512}C_A^3C_R \\
\gamma^{\overline{\rm DR}}_{010003} &= \frac{1}{442368N_A}[-41472D_3(A) + 480C_A^2D_2(A) + 37C_A^6N_A]C_R \\
\gamma^{\overline{\rm DR}}_{021001} &= \frac{1}{1536N_AI_2(R)}\{N_AI_2(R)[-63C_A^3 - 113C_A^2I_2(R)n_f + 34C_A^2C_R] \\
&+ 240D_2(RA)\}C_R \\
\gamma^{\overline{\rm DR}}_{120001} &= \frac{1}{1536}C_R[C_A^3(31C_A + 18C_R) - \frac{24}{I_2(R)N_A}(19C_A + 12C_R)D_2(RA)] \\
\gamma^{\overline{\rm DR}}_{010120} &= \frac{29}{3072}C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{210100} &= -\frac{1}{1024}(96C_R + 7C_A)C_AC_R \\
\gamma^{\overline{\rm DR}}_{010111} &= -\frac{97}{3072}C_A^3C_R \\
\gamma^{\overline{\rm DR}}_{300010} &= \frac{1}{1024}C_A^3C_R \\
\gamma^{\overline{\rm DR}}_{300100} &= -\frac{7}{512}C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{301000} &= -\frac{3}{512}C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{011002} &= -\frac{1}{36864N_A}[3360D_2(A) + 1033C_A^4N_A]C_R \\
\gamma^{\overline{\rm DR}}_{010012} &= \frac{1}{24576N_A}[384D_2(A) + 37C_A^4N_A]C_RC_A \\
\gamma^{\overline{\rm DR}}_{010300} &= -\frac{1}{1536}(N_A^2 + 57N_A + 86)C_R \\
\gamma^{\overline{\rm DR}}_{012001} &= -\frac{1}{1536}(3N_A + 74)C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{011011} &= \frac{49}{1024}C_A^3C_R
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{\overline{\rm DR}}_{111010} &= -\frac{1}{512}(53C_A + 11C_R)C_AC_R \\
\gamma^{\overline{\rm DR}}_{200020} &= -\frac{3}{1024}C_A^3C_R \\
\gamma^{\overline{\rm DR}}_{201010} &= -\frac{1}{256}C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{200002} &= -\frac{1}{12288N_A}[-96D_2(A) + C_A^4N_A]C_RC_AN_A \\
\gamma^{\overline{\rm DR}}_{202000} &= \frac{1}{512}C_AC_R \\
\gamma^{\overline{\rm DR}}_{201001} &= \frac{3}{512}C_A^3C_R \\
\gamma^{\overline{\rm DR}}_{220000} &= \frac{1}{18432I_2(R)N_A}C_R\{-288D_2(RA)(1 + 72\zeta_3) \\
&+ 2C_A^3I_2(R)N_A(-1295 + 10080\zeta_3) \\
&+ 4[3C_R^3I_2(R)N_A(1544 - 5760\zeta_3) + 2592D_2(R)n_f(1 - 2\zeta_3) \\
&+ 64C_RI_2(R)^3N_An_f^2(-10 + 3\zeta_3) - 2C_R^2I_2(R)^2N_An_f(571 + 1008\zeta_3)] \\
&+ C_A^2I_2(R)N_A[2354I_2(R)n_f - 2784I_2(R)n_f\zeta_3 - 12C_R(721 + 7704\zeta_3)] \\
&+ 2C_AI_2(R)N_A[24I_2(R)^2n_f^2(17 - 16\zeta_3) \\
&+ C_RI_2(R)n_f(7444 + 5856\zeta_3) + C_R^2(9428 + 71136\zeta_3)]\} \\
\gamma^{\overline{\rm DR}}_{011020} &= \frac{181}{3072}C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{010021} &= \frac{1}{4096N_A}[224D_2(A) + 37C_A^4N_A]C_R \\
\gamma^{\overline{\rm DR}}_{200200} &= \frac{1}{512}(N_A + 3)C_AC_R \\
\gamma^{\overline{\rm DR}}_{201100} &= \frac{1}{256}(N_A + 1)C_AC_R \\
\gamma^{\overline{\rm DR}}_{300001} &= \frac{1}{6144N_A}[-96D_2(A) + C_A^4N_A]C_R \\
\gamma^{\overline{\rm DR}}_{200110} &= \frac{1}{256}C_A^2C_R \\
\gamma^{\overline{\rm DR}}_{111100} &= \frac{1}{512}(37C_AN_A + 15C_RN_A + 19C_R + 21C_A)C_R \\
\gamma^{\overline{\rm DR}}_{010030} &= \frac{37}{2048}C_A^3C_R \\
\gamma^{\overline{\rm DR}}_{020200} &= \frac{1}{1536}[-35I_2(R)N_An_f + 39C_AN_A - 38C_RN_A + 302C_R \\
&- 121I_2(R)n_f - 23C_A]C_R \\
\gamma^{\overline{\rm DR}}_{121000} &= \frac{1}{256}(-24C_R^2 - 22C_AC_R + 27C_A^2)C_R
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{\overline{\rm DR}}_{040000} &= -\frac{1}{1536 I_2(R) N_A[-25C_A^4 N_A + 12 D_2(A)(2+N_A)]} \\
& \quad C_R\Big[-16320 C_A^3 D_2(RA) I_2(R) N_A n_f \\
&+ 97920 C_A^2 C_R D_2(RA) I_2(R) N_A n_f - 19584 D_2(RA)^2 (2+N_A) n_f \\
&+ 100 C_A^5 I_2(R) N_A^2[-288 C_R I_2(R) n_f - 15 I_2(R)^2 n_f^2 \\
&+ 2C_R^2(-126 - 432\zeta_3)] - 100 C_A^7 I_2(R) N_A^2 (-23 + 240\zeta_3) \\
&+ 100 C_A^6 I_2(R) N_A^2\{15 C_R + 75 I_2(R) n_f - 6[-132 C_R + I_2(R) n_f]\zeta_3\} \\
&+ 25 C_A^4 N_A\{1344 C_R^2 I_2(R)^2 N_A n_f + 180 C_R I_2(R)^3 N_A n_f^2 \\
&- 12 n_f[5 I_2(R)^4 N_A n_f^2 + 24 D_2(R)(7 - 2\zeta_3)] \\
&+ 192 D_2(RA)(-1 + 12\zeta_3) + 3 C_R^3 I_2(R) N_A (368 + 384\zeta_3)\} \\
&- 12 D_2(A)\Big(-4032 D_2(R) n_f + N_A\{1104 C_R^3 I_2(R)(2+N_A) \\
&- 6[336 D_2(R) - 16 C_R^2 I_2(R)^2 (79 + 14 N_A)] n_f \\
&+ 180 C_R I_2(R)^3 (2+N_A) n_f^2 - 60 I_2(R)^4 (2+N_A) n_f^3\} \\
&+ 4 C_A I_2(R) N_A[-24 C_R I_2(R)(41 + 12 N_A) n_f - 15 I_2(R)^2 (2+N_A) n_f^2 \\
&+ 2C_R^2 (2+N_A)(-126 - 432\zeta_3)] + 4 C_A^3 I_2(R) N_A (2+N_A)(23 - 240\zeta_3) \\
&- 96(2+N_A)[-12 C_R^3 I_2(R) N_A - 6 D_2(R) n_f]\zeta_3 \\
&+ 192 D_2(RA)(2+N_A)(-1 + 12\zeta_3) \\
&+ 4 C_A^2 I_2(R) N_A\{C_R(2+N_A)(15 + 792\zeta_3) \\
&+ I_2(R) n_f[184 + 75 N_A - 6(2+N_A)\zeta_3]\}\Big)\Big] \\
\gamma^{\overline{\rm DR}}_{010201} &= -\frac{1}{1024}(23 N_A + 218) C_A^2 C_R \\
\gamma^{\overline{\rm DR}}_{010210} &= -\frac{3}{512}(3 N_A - 2) C_A C_R \\
\gamma^{\overline{\rm DR}}_{112000} &= -\frac{1}{1024}(8 C_A N_A - 2 C_R N_A - 37 C_A - 15 C_R) C_R \\
\gamma^{\overline{\rm DR}}_{110110} &= \frac{1}{512}(53 C_A + 11 C_R) C_A C_R \\
\gamma^{\overline{\rm DR}}_{130000} &= -\frac{1}{3072 I_2(R) N_A} C_R\Big(-576 D_2(RA) + I_2(R) N_A\{352 C_A^3 + 616 C_A^2 C_R \\
&- 6040 C_A C_R^2 + 6992 C_R^3 - 100 C_A^2 I_2(R) n_f - 1388 C_A C_R I_2(R) n_f \\
&+ 3920 C_R^2 I_2(R) n_f + 168 C_A I_2(R)^2 n_f^2 - 88 C_R I_2(R)^2 n_f^2 \\
&+ 48(C_A - C_R)[3(8 C_A - 13 C_R)(C_A - 2 C_R) \\
&- 2(16 C_A - 32 C_R) I_2(R) n_f + 8 I_2(R)^2 n_f^2]\zeta_3\}\Big) \\
\gamma^{\overline{\rm DR}}_{022000} &= -\frac{1}{1536}[8 C_A N_A - 16 C_R N_A - 4 I_2(R) N_A n_f + 43 I_2(R) n_f \\
&- 50 C_R + 57 C_A] C_R
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{\overline{\rm DR}}_{020002} &= \frac{1}{36864 I_2(R) N_A[-25C_A^4 N_A + 12D_2(A)(2+N_A)]} \\
& \quad C_R\Big(-1880064 D_3(A) D_2(RA) + 536400 C_A^6 D_2(RA) N_A \\
&- 391680 C_A^3 D_3(A) I_2(R) N_A + 4C_A^5 D_2(A) I_2(R)(22778 - 111N_A) N_A \\
&+ 925 C_A^9 I_2(R) N_A^2 + 1152 C_A D_2(A)^2 I_2(R)(286 + 7N_A) \\
&+ 89856 D_3(RAA)[-25C_A^4 N_A + 12D_2(A)(2+N_A)] \\
&+ 192 C_A^2[12240 C_R D_3(A) I_2(R) N_A + D_2(A) D_2(RA)(8198 + 19N_A)] \\
&- 75 C_A^8 I_2(R) N_A^2[10C_R + 17 I_2(R) n_f] \\
&+ 12 C_A^4 D_2(A) I_2(R) N_A[-C_R(43540 - 30N_A) \\
&+ 3I_2(R)(2634 + 17N_A) n_f] + 3456\{-272 D_3(A) D_2(RA) N_A \\
&+ D_2(A)^2 I_2(R)[-2C_R(290 + 9N_A) - 13 I_2(R)(2+N_A) n_f]\}\Big) \\
\gamma^{\overline{\rm DR}}_{030010} &= \frac{1}{256}[3C_A - 10C_R - 30 I_2(R) n_f] C_A^2 C_R \\
\gamma^{\overline{\rm DR}}_{011110} &= \frac{1}{768}(11N_A - 50) C_A C_R \\
\gamma^{\overline{\rm DR}}_{012010} &= \frac{1}{768}(N_A + 38) C_A C_R \\
\gamma^{\overline{\rm DR}}_{030001} &= -\frac{1}{1536 N_A I_2(R)}\{384 C_A D_2(RA) - 864 C_R D_2(RA) \\
&+ N_A I_2(R)[10C_A^3 C_R - 3C_A^4 + 30 C_A^3 I_2(R) n_f] \\
&- 528 D_2(RA) I_2(R) n_f\} C_R \\
\gamma^{\overline{\rm DR}}_{010102} &= -\frac{1}{36864 N_A}[7008 D_2(A) + 2537 C_A^4 N_A] C_R \\
\gamma^{\overline{\rm DR}}_{030100} &= \frac{1}{384}[13C_A^2 + 8C_A I_2(R) n_f + 216 C_R^2 + 132 C_R I_2(R) n_f \\
&- 122 C_A C_R] C_R \\
\gamma^{\overline{\rm DR}}_{031000} &= \frac{1}{384}[11C_A^2 + 108C_R^2 - 41 C_A I_2(R) n_f - 76 C_A C_R + 66 C_R I_2(R) n_f] C_R \\
\gamma^{\overline{\rm DR}}_{210001} &= -\frac{3}{4096 N_A I_2(R)}(64 C_A D_2(RA) + 32 D_2(A) I_2(R) - C_A^4 I_2(R) N_A) C_R \\
\gamma^{\overline{\rm DR}}_{011101} &= -\frac{1}{1536}(53N_A + 320) C_A^2 C_R \\
\gamma^{\overline{\rm DR}}_{011200} &= -\frac{1}{512}(5N_A^2 + 21N_A + 46) C_R \\
\gamma^{\overline{\rm DR}}_{012100} &= -\frac{1}{768}(N_A^2 + 31N_A + 22) C_R \\
\gamma^{\overline{\rm DR}}_{013000} &= \frac{1}{768}(N_A - 10) C_R
\end{aligned}
\tag{68}
$$

where

$$
\begin{aligned}
\gamma_3 &= \frac{1}{256}\Big[C_R^4\Big(-\frac{1261}{8}-336\zeta_3\Big)+C_R^3C_A\Big(\frac{15349}{12}+316\zeta_3\Big) \\
&+ C_R^2C_A^2\Big(-\frac{34045}{36}-152\zeta_3+440\zeta_5\Big)+C_RC_A^3\Big(\frac{70055}{72}+\frac{1418}{9}\zeta_3-440\zeta_5\Big) \\
&+ C_R^3I_2(R)n_f\Big(-\frac{280}{3}+552\zeta_3-480\zeta_5\Big) \\
&+ C_R^2C_AI_2(R)n_f\Big(-\frac{8819}{27}+368\zeta_3-264\zeta_4+80\zeta_5\Big) \\
&+ C_RC_A^2I_2(R)n_f\Big(-\frac{65459}{162}-\frac{2684}{3}\zeta_3+264\zeta_4+400\zeta_5\Big) \\
&+ C_R^2I_2(R)^2n_f^2\Big(\frac{304}{27}-160\zeta_3+96\zeta_4\Big) \\
&+ C_RI_2(R)^3n_f^3\Big(-\frac{664}{81}+\frac{128}{9}\zeta_3\Big)+\frac{D_2(RA)}{d_R}\Big(-32+240\zeta_3\Big) \\
&+ C_RC_AI_2(R)^2n_f^2\Big(\frac{1342}{81}+160\zeta_3-96\zeta_4\Big)+\frac{n_fD_2(R)}{d_R}\Big(64-480\zeta_3\Big)\Big]. \qquad (69)
\end{aligned}
$$

5 The four-loop supersymmetric case

Our conventions are such that substituting in the above equations the results of Tables 2-4 corresponds to a gauge theory with n_f sets of Dirac fermions transforming according to the fundamental representation, or n_f sets of fundamental two component fermions with n_f sets of anti-fundamental two component fermions.

To extract the supersymmetric case we must make the replacements

$$
\begin{aligned}
C_R &\rightarrow C_A \\
I_2(R) &\rightarrow C_A \\
n_f &\rightarrow \tfrac{1}{2} \\
D_2(R) &\rightarrow D_2(A) \\
D_2(RA) &\rightarrow D_2(A) \\
D_3(RAA) &\rightarrow D_3(A) \\
\alpha_e &= v_3 = \alpha_s \\
v_1 &= v_2 = v_4 = 0. \qquad (70)
\end{aligned}
$$

With these substitutions we can compare our results for β_s with the four-loop results for the gauge β-function $\beta_s^{\rm SYM}$ of SQCD which was presented in Ref. [6]:

$$
\beta_s^{\rm SYM} = -\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{3}{4}C_A+\frac{3}{8}C_A^2\frac{\alpha_s}{\pi}+\frac{21}{64}C_A^3\left(\frac{\alpha_s}{\pi}\right)^2+\frac{51}{128}C_A^4\left(\frac{\alpha_s}{\pi}\right)^3\right]+\mathcal{O}(\alpha_s^6)\,. \qquad (71)
$$

We indeed find that using Eq. (70) in Eq. (57) precisely reproduces Eq. (71).

We have also checked that in the same supersymmetric limit, Eq. (61) reproduces the three-loop SQCD β-function.

Turning now to the case of softly-broken supersymmetry, there exists an exact result for γ_m [8]:

$$\gamma_m^{\mathrm{SYM}} = \pi\alpha_s \frac{\mathrm{d}}{\mathrm{d}\alpha_\mathrm{s}} \left[\frac{\beta_s^{\mathrm{SYM}}}{\alpha_s}\right], \tag{72}$$

whence it follows that

$$\gamma_m^{\mathrm{SYM}} = -\left(\frac{\alpha_s}{\pi}\right)\left[\frac{3}{4}C_A + \frac{3}{4}C_A^2\frac{\alpha_s}{\pi} + \frac{63}{64}C_A^3\left(\frac{\alpha_s}{\pi}\right)^2 + \frac{51}{32}C_A^4\left(\frac{\alpha_s}{\pi}\right)^3\right] + \mathcal{O}(\alpha_s^5)\,. \tag{73}$$

Using Eq. (70) in Eq. (60) precisely reproduces Eq. (73) in similar fashion.

The invariant $D_3(A)$ does not feature in either calculation, and the dependence on $D_2(A)$, N_A, ζ_3, ζ_4 and ζ_5 all cancel, although they appear in individual terms. It is tempting to speculate that this absence of higher order invariants and transcendental numbers (other than π) is related to the existence of the NSVZ scheme, in which the gauge β-function for any simple gauge group is given (in the supersymmetric case without matter fields) by the expression [20], [21]

$$\beta_s^{\mathrm{NSVZ}} = -\frac{3}{4}C_A\left(\frac{\alpha_s}{\pi}\right)^2\left(1 - \frac{C_A\alpha_s}{2\pi}\right)^{-1} \tag{74}$$

which is manifestly free of them to all orders. It is natural to conjecture that the same property holds in the DRED scheme. For discussion of the relationship between β_s^{NSVZ} and $\beta_s^{\overline{\mathrm{DR}}}$ see Ref. [7].

6 Discussion

In this paper we have applied DRED to gauge theories with gauge groups $SU(N)$, $SO(N)$ and $Sp(N)$, and calculated both the gauge β-function and the mass anomalous dimension to the four-loop level. These calculations required careful treatment of the evanescent Yukawa and quartic couplings of the ε-scalar. In the supersymmetric limit we explicitly verified that the β-function for the evanescent Yukawa coupling reproduces the gauge β-function through three loops.

The results for $\beta_s^{\overline{\mathrm{DR}}}$ and $\gamma_m^{\overline{\mathrm{DR}}}$ in the special case of QCD as described in Ref. [1] and [2] are easily obtained from the results of this paper by specialising to the $SU(N)$ case, and setting $N = 3$ and $v_4 = 0$, with the fermions in the fundamental representation.

Predictions based on theories with low energy supersymmetry require careful consideration of the transition between the $\overline{\mathrm{MS}}$ and $\overline{\mathrm{DR}}$ renormalisation schemes. If, for example, the decoupling of supersymmetric particles is carried out in several steps (as in split supersymmetry, for example [22]) then it is essential to take into account the evanescent couplings (for a recent discussion and treatment of the running of α_s and m_b in the MSSM, see Ref. [23].

A Group Theory

We consider a gauge group $\mathcal{G}$ with generators R^a satisfying[4]

$$\left[R^a, R^b\right] = i f^{abc} R^c. \tag{75}$$

We work throughout with a fermion representation consisting of n_f sets of Dirac fermions or $2n_f$ sets of two-component fermions, in irreducible representations with identical Casimirs, using R^a to denote the generators in one such representation. Thus $R^a R^a$ is proportional to the unit matrix:

$$R^a R^a = C_R.I \tag{76}$$

For the adjoint representation we have

$$C_A \delta_{ab} = f_{acd} f_{bcd}. \tag{77}$$

$I_2(R)$ is defined by

$$\mathrm{Tr}[R^a R^b] = I_2(R)\delta^{ab}. \tag{78}$$

Thus we have

$$C_R d_R = I_2(R) N_A \tag{79}$$

where N_A is the number of generators and d_R is the dimensionality of the representation R. Evidently $I_2(A) = C_A$. The fully symmetric tensors d_R^{abcd} and d_A^{abcd} are defined by

$$\begin{aligned} d_R^{abcd} &= \frac{1}{6}\mathrm{Tr}[R^{(a} R^b R^c R^{d)}], \\ d_A^{abcd} &= \frac{1}{6}\mathrm{Tr}[F^{(a} F^b F^c F^{d)}], \end{aligned} \tag{80}$$

where

$$(F^a)^{bc} = i f^{bac} \tag{81}$$

and

$$\begin{aligned} R^{(a} R^b R^c R^{d)} &= R^a R^b R^c R^d + R^a R^b R^d R^c + R^a R^c R^b R^d \\ &+ R^a R^c R^d R^b + R^a R^d R^b R^c + R^a R^d R^c R^b, \end{aligned} \tag{82}$$

(similarly for $F^{(a} F^b F^c F^{d)}$).

The additional tensor invariants occurring in our results for β_s and γ_m are defined as

$$\begin{aligned} D_2(A) &= d_A^{abcd} d_A^{abcd} \\ D_2(RA) &= d_R^{abcd} d_A^{abcd} \\ D_3(A) &= d_A^{abcd} d_A^{cdef} d_A^{abef} \\ D_3(RAA) &= d_R^{abcd} d_A^{cdef} d_A^{abef}. \end{aligned} \tag{83}$$

In table 2-4 we present results for the various tensor invariants for the groups $SU(N)$, $SO(N)$ and $Sp(N)$, when the fermion representation R is the fundamental representation.

[4] Useful sources for some of the material in this section have included Refs. [11, 16, 24].

B The groups $SO(N)$ and $Sp(N)$

In this section we derive explicit expressions for the β-functions for the ε-scalar quartic interactions for the groups $SO(N)$ and $Sp(N)$. These may also be derived from Eqs. (54) using tables 3, 4.

B.1 The case $\mathcal{G} = SO(N)$

Let us consider $SO(N)$. The defining representation of the generators of $SO(N)$ is given by the set of matrices

$$(M_{[ij]})_{kl} = -i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj}), \tag{84}$$

satisfying the algebra

$$\left[M_{[ij]}, M_{[kl]}\right] = -i\left(\delta_{jk}M_{[il]} - \delta_{ik}M_{[jl]} - \delta_{jl}M_{[ik]} + \delta_{il}M_{[jk]}\right) \tag{85}$$

or

$$\left[M_{[ij]}, M_{[kl]}\right] = if_{[ij][kl][mn]}M_{[mn]} \tag{86}$$

where the structure constants are given by

$$f_{[ij][kl][mn]} = \tfrac{1}{2}\left[\delta_{ik}\delta_{jm}\delta_{ln} + \ldots(7\text{terms})\right] \tag{87}$$

such that they are antisymmetric in ij, kl, and mn exchange. With this normalisation of the generators it is straightforward to show that the adjoint quadratic Casimir C_A is given by

$$f_{[ij][kl][mn]}f_{[ij][kl][pq]} = C_A(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}) \tag{88}$$

with

$$C_A = 2(N-2). \tag{89}$$

We will, however, present results for an arbitrary normalisation of the generators such that

$$C_A = b(N-2), \tag{90}$$

where b is a constant. Useful checks on our calculations will be provided by the isomorphisms

$$SO(3) \equiv \frac{SU(2)}{Z_2} \tag{91}$$

and

$$SO(6) \equiv \frac{SU(4)}{Z_2} \tag{92}$$

which mean that the Lie algebras of $SO(3)$ and $SU(2)$, and of $SO(6)$ and $SU(4)$ are identical. Note that to compare our result for $SO(3)$ with the corresponding result for $SU(2)$ (where with the conventional normalisation we have $C_A = 2$) we will need to set $b = 2$, while to compare $SO(6)$ with $SU(4)$ we will similarly need to set $b = 1$.

The basis for 4-tensors for $SO(N)$ for $N \geq 4$ has $\gamma = 6$ and can be chosen to be[5](we adopt a shorthand notation with $[i_1 i_2] \to i$ etc.):

$$
\begin{aligned}
P_1 &= \delta_{ij}\delta_{kl}, \\
P_2 &= \delta_{ik}\delta_{jl}, \\
P_3 &= \delta_{il}\delta_{kj}, \\
P_4 &= f_{ijm}f_{klm}, \\
P_5 &= f_{ikm}f_{jlm}, \\
P_6 &= \mathrm{tr}[F_iF_jF_kF_l],
\end{aligned}
\tag{93}
$$

where $(F_i)_{mn} = f_{min}$.

Some useful identities for reduction of various 4-tensors to the basis are as follows :

$$
\begin{aligned}
\mathrm{Tr}[F_iF_jF_pF_kF_lF_p] &= \frac{b^3}{2}(N-4)(-2P_1+2P_2+P_3) \\
&+ \frac{b^2}{4}(N-8)(P_4-2P_5) - \frac{b}{2}(N-2)P_6
\end{aligned}
\tag{94}
$$

$$
\begin{aligned}
\mathrm{Tr}[F_iF_mF_jF_n](F_kF_l)_{mn} &= \frac{b^3}{2}(N-4)(-2P_1+P_2+P_3) \\
&- \frac{b^2}{4}(N-8)P_4 + \frac{b^2}{2}(N-8)P_5.
\end{aligned}
\tag{95}
$$

It is interesting that this doesn't involve the basis element P_6.

$$
\begin{aligned}
\mathrm{Tr}[F_iF_jF_mF_n]\mathrm{Tr}[F_kF_lF_mF_n] &= \frac{b^4}{4}(N-2)(N-4)(6P_1-P_2-P_3) \\
&+ \frac{b^3}{8}(N^3-6N^2+16N-24)P_4 \\
&- \frac{b^3}{4}N(N-6)P_5 \\
&+ \frac{b^2}{4}(N^2-6N+20)P_6
\end{aligned}
\tag{96}
$$

$$
\begin{aligned}
\mathrm{Tr}[F_iF_mF_jF_nF_kF_mF_lF_n] &= \frac{b^4}{4}(N-2)(N-4)(P_1+P_3) \\
&+ \frac{b^3}{8}(N^2-14N+32)(-2P_4+P_5) \\
&- \frac{b^2}{4}(N-8)P_6
\end{aligned}
\tag{97}
$$

[5]As will become clear our results will not be applicable to the special case $N = 8$, which we will not consider further

The results for the one-loop β-functions in a basis as in Eq. (28) and with v_i defined as for $SU(N)$ are as follows:

$$\begin{aligned}
\beta_{v_1} &= 8v_1^2 + 2(N^2 - N + 2)v_1v_2 - 4b(N-2)v_1v_3 + 6b^2(N-2)^2v_1v_4 \\
&+ 12v_2^2 + 8b(N-2)v_2v_3 + 8b^2(N-2)^2v_2v_4 \\
&+ 20b^4(N-2)(N-4)v_4^2 + 16b^3(N-4)v_3v_4 - 12b(N-2)v_1\alpha_s, \\
\beta_{v_2} &= (N^2 - N + 8)v_2^2 + 12v_1v_2 - 4b(N-2)v_2v_3 + 6b^2(N-2)^2v_2v_4 \\
&- 8b^3(N-4)v_3v_4 + 6b^4(N-2)(N-4)v_4^2 - 12b(N-2)v_2\alpha_s, \\
\beta_{v_3} &= 4b(N-2)v_3^2 + 12v_1v_3 - 4v_2v_3 - 4b(N-2)v_2v_4 \\
&+ 2b^2(N^2 - 6N + 20)v_3v_4 - 4b^3(N-4)v_4^2 - 12b(N-2)v_3\alpha_s, \\
\beta_{v_4} &= \tfrac{1}{2}b^2(3N^2 - 28N + 140)v_4^2 + 12v_1v_4 + 20v_2v_4 - 2v_3^2 \\
&- 2b(N-2)v_3v_4 - 12b(N-2)v_4\alpha_s + 6\alpha_s^2. \qquad (98)
\end{aligned}$$

If we set $N = 6$ and $b = 1$ in Eq. (98), we reproduce precisely the results obtained by setting $N = 4$ in Eq. (32). Similarly, if we set $N = 3$ and $b = 2$ in Eq. (98), we reproduce precisely the results obtained by setting $N = 2$ in Eq. (32), setting $v_3 = v_4 = 0$ in both cases.

B.1.1 The fermion contribution

As in the $SU(N)$ case, the fermion loop contribution to the scalar anomalous dimension results in a contribution of $\Delta\beta_{v_i} = 8n_f I_2(R)v_i\alpha_e$ to each β-function in Eq. (98). The 1PI fermion box diagram makes a contribution to the β-functions (appropriately normalised) of the form as Eq. (39); for the adjoint representation this becomes:

$$\overline{H}_i\Delta\beta_{v_i} = \alpha_e^2(-2b(N-2)\overline{H}_3 - 4\overline{H}_4), \qquad (99)$$

and hence for the complete β-functions including a single two-component fermion in the adjoint representation we have from Eq. (98):

$$\begin{aligned}
\beta_{v_1} &= 8v_1^2 + 2(N^2 - N + 2)v_1v_2 - 4b(N-2)v_1v_3 + 6b^2(N-2)^2v_1v_4 \\
&+ 12v_2^2 + 8b(N-2)v_2v_3 + 8b^2(N-2)^2v_2v_4 + 16b^3(N-4)v_3v_4 \\
&+ 20b^4(N-2)(N-4)v_4^2 - 12b(N-2)v_1\alpha_s + 4b(N-2)v_1\alpha_e, \\
\beta_{v_2} &= (N^2 - N + 8)v_2^2 + 12v_1v_2 - 4b(N-2)v_2v_3 + 6b^2(N-2)^2v_2v_4 \\
&- 8b^3(N-4)v_3v_4 + 6b^4(N-2)(N-4)v_4^2 \\
&- 12b(N-2)v_2\alpha_s + 4b(N-2)v_2\alpha_e, \\
\beta_{v_3} &= 4b(N-2)v_3^2 + 12v_1v_3 - 4v_2v_3 - 4b(N-2)v_2v_4 \\
&+ 2b^2(N^2 - 6N + 20)v_3v_4 - 4b^3(N-4)v_4^2 - 12b(N-2)v_3\alpha_s \\
&+ 4b(N-2)v_3\alpha_e - 2b(N-2)\alpha_e^2, \\
\beta_{v_4} &= \tfrac{1}{2}b^2(3N^2 - 28N + 140)v_4^2 + 12v_1v_4 + 20v_2v_4 - 2v_3^2 \\
&- 2b(N-2)v_3v_4 - 12b(N-2)v_4\alpha_s + 6\alpha_s^2 \\
&+ 4b(N-2)v_4\alpha_e - 4\alpha_e^2, \qquad (100)
\end{aligned}$$

when it is once again easy to extract the supersymmetric result by setting $v_1 = v_2 = v_4 = 0$ and $v_3 = \alpha_e = \alpha_s$.

For a single two-component fermion in the fundamental representation, (and for $N \neq 8$) we find that

$$\mathrm{Tr}[M_i M_j M_k M_l] = \frac{1}{N-8}\Big[-b^2(P_1 + P_2 + P_3) + b(2P_4 - P_5) + P_6\Big] \tag{101}$$

and hence we find for fermions in the fundamental representation a contribution to the β-functions (for n_f flavours) of the form

$$\overline{H}_i \Delta\beta_{v_i} = \frac{2n_f\alpha_e^2}{N-8}\left[8b^2(\overline{H}_1 + \overline{H}_2) - 2b(N-10)\overline{H}_3 - 4\overline{H}_4\right]. \tag{102}$$

It is straightforward to incorporate these contributions into Eq. (98) in the same manner.

B.2 The case $\mathcal{G} = Sp(N)$

We will here be considering the unitary symplectic group. The generators of $Sp(N)$ satisfy

$$JR^a J = (R^a)^T \tag{103}$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{104}$$

and I is the unit matrix. Evidently N must be even. For the case $N = 2$ it is easy to check by explicitly constructing R^a to satisfy Eq. (103) that $Sp(2) \equiv SU(2)$. Another useful check on our calculations will be provided by the isomorphism

$$SO(5) \equiv \frac{Sp(4)}{Z_2}. \tag{105}$$

If we write $N = 2n$, the generators may be written as $L_{\alpha\beta}$, where an infinitesimal group element $\mathcal{S}$ may be written

$$\mathcal{S} = 1 + i\sum_{\alpha\beta} a_{\alpha\beta}L_{\alpha\beta} \tag{106}$$

where $a_{\alpha\beta} = a^*_{\beta\alpha}$ and

$$L_{\alpha\beta} = L_{-\beta-\alpha}, \quad \alpha,\beta = \pm1, \pm2, \cdots \pm n \tag{107}$$

(Thus the correspondence $Sp(2) \sim SU(2)$ is $L_{11} \sim J_3$, $L_{1-1}, L_{-1,1} \sim J_\pm = J_1 \pm iJ_2$.)

They obey the commutation relations

$$[L_{\alpha\beta}, L_{\gamma\delta}] = \left(\delta_{\beta\gamma}\delta_\beta L_{\alpha\delta} - \delta_{\alpha\delta}\delta_\alpha L_{\gamma\beta} + \delta_{\beta\overline{\delta}}\delta_\beta L_{\alpha\overline{\gamma}} - \delta_{\alpha\overline{\gamma}}\delta_\alpha L_{\overline{\beta}\delta}\right) \tag{108}$$

where

$$\delta_\alpha = -\delta_{-\alpha} = \begin{cases} 1 & \text{if } \alpha > 0, \\ -1 & \text{if } \alpha < 0. \end{cases} \tag{109}$$

and $\overline{\alpha} = -\alpha$ etc.

We find:

$$\begin{aligned}
\beta_{v_1} &= 8v_1^2 + 2(N^2+N+2)v_1v_2 - 2b(N+2)v_1v_3 + \tfrac{3}{2}b^2(N+2)^2v_1v_4 \\
&+ 12v_2^2 + 4b(N+2)v_2v_3 + 2b^2(N+2)^2v_2v_4 \\
&+ \tfrac{5}{4}b^4(N+2)(N+4)v_4^2 + 2b^3(N+4)v_3v_4 - 6b(N+2)v_1\alpha_s, \\
\beta_{v_2} &= (N^2+N+8)v_2^2 + 12v_1v_2 - 2b(N+2)v_2v_3 + \tfrac{3}{2}b^2(N+2)^2v_2v_4 \\
&- b^3(N+4)v_3v_4 + \tfrac{3}{8}b^4(N+2)(N+4)v_4^2 - 6b(N+2)v_2\alpha_s, \\
\beta_{v_3} &= 2b(N+2)v_3^2 + 12v_1v_3 - 4v_2v_3 - 2b(N+2)v_2v_4 \\
&+ \tfrac{1}{2}b^2(N^2+6N+20)v_3v_4 - \tfrac{1}{2}b^3(N+4)v_4^2 - 6b(N+2)v_3\alpha_s, \\
\beta_{v_4} &= \tfrac{1}{8}b^2(3N^2+28N+140)v_4^2 + 12v_1v_4 + 20v_2v_4 - 2v_3^2 \\
&- b(N+2)v_3v_4 - 6b(N+2)v_4\alpha_s + 6\alpha_s^2.
\end{aligned} \tag{110}$$

Using table 4 in Eq. (54) leads to the same results.

Setting $N = 2$, $b = 1$ and $v_3 = v_4 = 0$ in Eq. (110) above we indeed find agreement with the corresponding results for $SU(2)$, from Eq. (32).

B.2.1 The fermion contribution

The fermion loop contribution is similar to the the $SO(N)$ case. From the scalar anomalous dimension we get a contribution of $\Delta\beta_{v_i} = 8n_f I_2(R) v_i \alpha_e$ to each β-function in Eq. (110).

In the case of the adjoint representation, by similar algebra to that leading to Eq. (40), we obtain for the 1PI fermion box diagram a contribution:

$$\overline{H}_i \Delta\beta_{v_i} = \alpha_e^2(-b(N+2)\overline{H}_3 - 4\overline{H}_4), \tag{111}$$

and for the complete β-functions from Eq. (110):

$$\begin{aligned}
\beta_{v_1} &= 8v_1^2 + 2(N^2+N+2)v_1v_2 - 2b(N+2)v_1v_3 + \tfrac{3}{2}b^2(N+2)^2v_1v_4 \\
&+ 12v_2^2 + 4b(N+2)v_2v_3 + 2b^2(N+2)^2v_2v_4 \\
&+ \tfrac{5}{4}b^4(N+2)(N+4)v_4^2 + 2b^3(N+4)v_3v_4 - 6b(N+2)v_1\alpha_s \\
&+ 2b(N+2)v_1\alpha_e, \\
\beta_{v_2} &= (N^2+N+8)v_2^2 + 12v_1v_2 - 2b(N+2)v_2v_3 + \tfrac{3}{2}b^2(N+2)^2v_2v_4 \\
&- b^3(N+4)v_3v_4 + \tfrac{3}{8}b^4(N+2)(N+4)v_4^2 - 6b(N+2)v_2\alpha_s \\
&+ 2b(N+2)v_2\alpha_e, \\
\beta_{v_3} &= 2b(N+2)v_3^2 + 12v_1v_3 - 4v_2v_3 - 2b(N+2)v_2v_4 \\
&+ \tfrac{1}{2}b^2(N^2+6N+20)v_3v_4 - \tfrac{1}{2}b^3(N+4)v_4^2 - 6b(N+2)v_3\alpha_s, \\
&+ 2b(N+2)v_3\alpha_e - b(N+2)\alpha_e^2, \\
\beta_{v_4} &= \tfrac{1}{8}b^2(3N^2+28N+140)v_4^2 + 12v_1v_4 + 20v_2v_4 - 2v_3^2 \\
&- b(N+2)v_3v_4 - 6b(N+2)v_4\alpha_s + 6\alpha_s^2 \\
&+ 2b(N+2)v_4\alpha_e - 4\alpha_e^2,
\end{aligned} \tag{112}$$

when it is once again easy to extract the supersymmetric result by setting $v_1 = v_2 = v_4 = 0$ and $v_3 = \alpha_e = \alpha_s$. Setting $N = 4$ in Eq. (112) we reproduce precisely the results of setting $N = 5$ in Eq. (98), in accordance with Eq. (105); a good check of our calculation. Also, setting $N = 2$ and $v_3 = v_4 = 0$ in Eq. (112) we reproduce the results of setting $N = 2$ and $v_3 = v_4 = 0$ in Eq. (32).

For a single two-component fermion in the fundamental representation, we find (again reverting to a shorthand single index notation)

$$\mathrm{Tr}[L_\alpha L_\beta L_\gamma L_\delta] = \frac{1}{N+8}\Big[-\frac{b^2}{4}(P_1+P_2+P_3) - \frac{b}{2}(2P_4 - P_5) + P_6\Big], \tag{113}$$

(where the P-basis is defined in the same way as for $SO(N)$ in Eq. (93)) and hence a contribution to the β-functions (for n_f flavours) of the form

$$\overline{H}_i\Delta\beta_{v_i} = \frac{2n_f\alpha_e^2}{N+8}\left[2b^2(\overline{H}_1+\overline{H}_2) - b(N+10)\overline{H}_3 - 4\overline{H}_4\right]. \tag{114}$$

It is straightforward to incorporate these contributions into Eq. (110) in the same manner.

We can also check this result [6] using the identity $Sp(2) \equiv SU(2)$; setting $N = 2$ and $b = 1$ in Eq. (114) and using Eq. (46) we find agreement with the result of setting $N = 2$ and using Eq. (46) in Eq. (50).

Remarkably, the all β-functions for the $Sp(N)$ case (including β_s, β_e) together with γ_m

[6] Of course we cannot use Eq. (105) as a check here because the fundamental representation is different for the two groups.

can be derived from the corresponding $SO(N)$ versions by a series of simple substitutions:

$$\begin{aligned} b &\to \tfrac{1}{2}b \\ N &\to -N \\ \alpha_s &\to -\alpha_s \\ \alpha_e &\to -\alpha_e \\ v_3 &\to -v_3 \\ n_f &\to -n_f. \end{aligned} \tag{115}$$

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Group	$SU(N)$
C_A	bN
C_R	$b\frac{N^2-1}{2N}$
$I_2(A)$	bN
$I_2(R)$	$\frac{b}{2}$
N_A	N^2-1
$D_2(A)$	$\frac{b^4}{24}(N^2-1)(N^2+36)N^2$
$D_2(RA)$	$\frac{b^4}{48}N(N^2-1)(N^2+6)$
$D_2(R)$	$\frac{b^4}{96N^2}(N^2-1)(18-6N^2+N^4)$
$D_3(A)$	$\frac{b^6}{216}N^2(N^2-1)(324+135N^2+N^4)$
$D_3(RAA)$	$\frac{b^6}{432}N^3(N^2-1)(51+N^2)$

Table 2: $SU(N)$ Group invariants (here R is the fundamental representation).

Group	$SO(N)$
C_A	$b(N-2)$
C_R	$\frac{b}{2}(N-1)$
$I_2(A)$	$b(N-2)$
$I_2(R)$	b
N_A	$\frac{1}{2}N(N-1)$
$D_2(A)$	$\frac{b^4}{48}N(N-1)(N-2)(-296+138N-15N^2+N^3)$
$D_2(RA)$	$\frac{b^4}{48}N(N-1)(N-2)(22-7N+N^2)$
$D_2(R)$	$\frac{b^4}{48}N(N-1)(4-N+N^2)$
$D_3(A)$	$\frac{b^6}{864}(N-2)(N-1)N(-29440+23272N-7018N^2+971N^3-47N^4+2N^5)$
$D_3(RAA)$	$\frac{b^6}{864}N(N-2)(N-1)(2048-1582N+387N^2-31N^3+2N^4)$

Table 3: $SO(N)$ Group invariants (here R is the fundamental representation).

Group	$Sp(N)$
C_A	$b(N+2)$
C_R	$\frac{b}{4}(N+1)$
$I_2(A)$	$b(N+2)$
$I_2(R)$	$\frac{b}{2}$
N_A	$\frac{1}{2}N(N+1)$
$D_2(A)$	$\frac{b^4}{768}N(N+1)(N+2)(296+138N+15N^2+N^3)$
$D_2(RA)$	$\frac{b^4}{768}N(N+1)(N+2)(22+7N+N^2)$
$D_2(R)$	$\frac{b^4}{768}N(N+1)(4+N+N^2)$
$D_3(A)$	$\frac{b^6}{55296}(N+2)(N+1)N(29440+23272N+7018N^2+971N^3+47N^4+2N^5)$
$D_3(RAA)$	$\frac{b^6}{55296}N(N+2)(N+1)(2048+1582N+387N^2+31N^3+2N^4)$

Table 4: $Sp(N)$ Group invariants (here R is the fundamental representation).