Saclay/SPhT–T03/085
Karlsruhe/TTP03-16

Evolution Kernels from Splitting Amplitudes

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Abstract

We recalculate the next-to-leading order Altarelli–Parisi kernel using a method which relates it to the splitting amplitudes describing the collinear factorization properties of scattering amplitudes. The method breaks up the calculation of the kernel into individual pieces which have an independent physical interpretation.

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1. Introduction

The last quarter-century of experimental studies at colliders, complemented by theoretical investigations, have taught us that perturbative quantum chromodynamics (QCD) gives an excellent description of the strong interaction probed at short distances. Indeed, the theory has reached a sufficient maturity that it is no longer the target of experimental studies, but rather a tool in the search for new physics beyond the standard model.

Upcoming collider experiments at the Tevatron and the LHC will require continuing refinements and progress in our ability to make precise predictions of QCD and QCD-associated processes, along with an ability to give credible estimates of the associated uncertainties. Recent years have seen great progress in computing two-loop amplitudes [1, 2, 3] in QCD, which is one of the ingredients essential to a next-to-next-to-leading order (NNLO) description of collider processes.

The calculational framework for collider processes is based on our ability to compute short-distance matrix elements to increasing perturbative order in nonabelian gauge theories. It complements these process-dependent matrix elements with a general understanding of factorization, which separates out the process-independent long-distance aspects. The long-distance aspects of scattering processes are captured in the parton distribution functions of the scattering nucleon(s), and in fragmentation functions for identified outgoing hadrons. Up to subleading power corrections, such functions along with the strong coupling α_s are the only ingredients needed from outside perturbation theory for a description of collider scattering processes.

The parton distribution and fragmentation functions are functions of a momentum fraction, and of the scale Q^2 at which the hadron is probed. As is well-known, their values at different Q^2 are not independent, but are governed by the Altarelli–Parisi equation, whose kernel is computable in perturbation theory [4, 5, 6, 7, 8, 9]. It allows these functions to be evolved from a fixed scale Q_0^2 to other values of Q^2. The evolution kernels are known up to second order [10, 11, 12, 13, 14, 15, 16]. Curci, Furmanski, and Petronzio [10, 11] used the light-cone gauge, whereas Floratos et al. obtained their results [12, 13, 14, 15, 16] using a covariant gauge. Subtleties related to the proper renormalization of the gluon operator in covariant gauges were later clarified by Hamberg and van Neerven [17]. Based on this work the second-order computation was rechecked more recently by Mertig and van Neerven, who also computed the polarized kernels at this order [18]. The calculation of the polarized kernels was checked by Vogelsang [19] using the light-cone gauge. The use of light-cone gauge was also investigated in refs. [20, 21, 22]. Its use beyond next-to-leading order is subtle [23] and not fully tested. An NNLO calculation is being undertaken, using the operator approach, by Moch, Vermaseren and Vogt [24]. Furthermore fits to the first moments of the evolution kernels [25, 26] have already been used to determine the NNLO parton distribution functions [27, 28]. We believe it is useful to develop other methods applicable to an NNLO calculation, and that is our purpose here.

Accordingly, we wish here to introduce an alternative approach to computing the kernel. We will do so using gauge-independent quantities describing the collinear behavior of various amplitudes. This has the advantage of breaking down the computation into pieces which themselves already have a meaningful interpretation. It also allows us to avoid complications associated with prescriptions for light-cone gauge. In this paper, we focus our attention on flavor-independent contributions to the time-like gluon kernel.

At present there exist two different methods to calculate the Altarelli–Parisi kernels. To contrast previous methods with the approach presented in the present work, let us consider the basic ingredients of factorization. The key feature of factorization in the QCD-improved parton model is the following: a hadronic cross section (more precisely, the leading-twist contribution) can be factorized into a hard scattering coefficient and parton distribution or fragmentation functions (so-called PDFs). All information about the hard scattering process is contained in the hard-scattering coefficient which is in the realm of short-distance physics and hence can be calculated entirely in perturbative QCD. Long-distance effects, including the matching of partonic states to hadronic ones and vice versa, are captured by the PDFs. These functions can be expressed as expectation values of composite operators between hadronic states. The matrix elements of the composite operators cannot be calculated in perturbative QCD, although they are (in principle) amenable to non-perturbative techniques such as lattice QCD. On the other hand, these non-perturbative matrix elements are *universal*, that is process-independent, and hence can be measured in one process and then used to obtain predictions for all other processes of interest. While the matrix elements themselves cannot be calculated perturbatively, their scaling behavior *can* be, and it is this scaling behavior which is captured by the Altarelli–Parisi equation. Their scaling behavior, or equivalently their anomalous dimensions, are determined by the ultraviolet singularities of QCD corrections to the matrix elements.

The ultraviolet singularities, and hence the anomalous dimensions, are independent of the choice of external state in the composite-operator matrix element. For calculational purposes, we can therefore replace a hadronic state with a partonic one. This yields one method (the 'OPE approach') of calculating the evolution kernels.

The other method used in the literature is the so-called 'infrared approach'. Here one starts with unrenormalized quantities. The calculation yields singularities, in particular in the partonic, short-distance, cross section. (The singularities are typically regulated using a dimensional regulator.) One can distinguish between ultraviolet divergences and mass singularities, related to the collinear emission from initial-state partons (or from final-state partons fragmenting into identified hadrons). Soft singularities cancel between virtual and real corrections. Ultraviolet singularities are removed through the usual renormalization procedure, that is are absorbed into the definition of the physical coupling in terms of the 'bare' coupling. For mass singularities, the situation is more complicated. In the infrared approach, they are canceled by corresponding singularities in the 'bare' PDFs. Equivalently, they are absorbed into the definition of the physical PDFs in terms of the 'bare' PDFs. For this to be possible, the singularities must of course possess a universal, process-independent form. (They do.)

The mass singularities in the PDFs determine their evolution with respect to changes in the reference scale, so the consistency of this approach means that the singularities determine the evolution kernels. In turn, a determination of the remaining singularities in the ultraviolet-subtracted hard-scattering coefficient thus also yields a calculation of the evolution kernels. To determine these singularities, one could in principle calculate the complete partonic cross section for a specific process, and extract its singularities. In general, this is a formidable task. For example, in order to derive just the leading-order (LO) kernels one already needs to know the singularities of the next-to-leading order (NLO) hard scattering coefficient.

Fortunately, this task can be simplified. One way is to use the observation of Ellis, Georgi, Machacek, Politzer, and Ross [29, 30], that in axial gauge only a certain class of diagrams contribute to the singularities. It is on this observation that the derivation of Curci, Furmanski, and Petronzio [10, 11] is based. Their derivation is closely connected to the use of light-cone gauge whose treatment beyond next-to-leading order, as mentioned earlier, is not fully tested.

The method we shall use in the present paper is related but distinct. We use the knowledge of factorization at the amplitude level, more specifically the *splitting amplitudes* which describe the collinear limits of scattering amplitudes. While it is intuitively clear that the factorization and universality of the 'mass' singularities in a hard-scattering cross section follow from this factorization at the amplitude level, up to now no explicit demonstration has been given in the literature. We give such a demonstration, using the phase-space slicing method, and use the connection to derive the NLO evolution kernels. In the context of our derivation, the splitting amplitudes represent smaller parts of the calculation that can be verified independently. Furthermore, since they describe properties of on-shell amplitudes, they are manifestly gauge-invariant; the method we describe is accordingly independent of the use of light-cone gauge.

Splitting amplitudes have been used extensively as a check on new calculations of amplitudes, since the constraint of obeying the correct behavior in all collinear limits is a strong one. Indeed, the one-loop splitting amplitudes have also been used, via such constraints, to give conjectured forms [31] for certain classes of one-loop amplitude with an arbitrary number of external legs (later proven by Mahlon [32]), and as an aid in the initial derivation of another all-n class of amplitudes obtainable using the unitarity-based method [33].

In this work we will focus on the timelike $g \to gg$-kernel, with g denoting a gluon. To simplify the derivation we will consider a gauge theory with no light fermions, and with an additional massive colorless scalar coupled to the gluons via an effective vertex.

At leading order, the splitting amplitude can be regarded as the amplitude for finding a parton of given momentum fraction inside a parent parton. It is therefore natural to expect the Altarelli–Parisi kernel, which describes the probability of finding a parton of given momentum fraction inside a parent parton, to be the square of the splitting amplitude. We make this correspondence more precise in section 4. Beyond leading order, we expect that there will be virtual corrections to this picture. Indeed, they are given by the one-loop splitting amplitude, more precisely by its interference with the leading-order splitting amplitude. As usual in gauge theories, however, there are additional singularities due to the emission of soft or collinear partons, and so we must also integrate over corresponding real-emission contributions. Heuristically, these two contributions, less the iterated leading-order kernel, give the next-to-leading order Altarelli–Parisi kernel.

The outline of the paper is as follows. In section 2 we review briefly the factorization of color ordered amplitudes. In section 3 we setup the general framework. In the following section we illustrate the new approach by the rederivation of the leading-order kernel. We will discuss the leading-order derivation in great detail because parts of it will be reused in the derivation of the NLO kernel which we present in section 5. We give our conclusions in section 6.

2. Collinear Factorization

The properties of gauge theories are easiest to discuss in the context of a color decomposition [34, 35, 36, 37, 38]. At tree level, for all-gluon amplitudes such a decomposition takes the form,

$$\mathcal{A}_n^{(0)}(\{k_i,\lambda_i,a_i\}) = \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\, A_n^{(0)}(\sigma(1^{\lambda_1},\ldots,n^{\lambda_n}))\,, \tag{2.1}$$

where S_n/Z_n is the group of non-cyclic permutations on n symbols, and j^{λ_j} denotes the j-th momentum and helicity. We use the normalization

$$\mathrm{Tr}(T^a T^b) = \delta^{ab} \tag{2.2}$$

for the generators of SU(N). The same color decomposition as shown in eqn. (2.1) holds for the amplitudes we shall consider, for the process $\phi \to g \cdots g$, where ϕ denotes a colorless heavy scalar. One can write analogous formulæ for amplitudes with quark-antiquark pairs. The color-ordered or partial amplitude A_n is gauge invariant. In the collinear limit, $k_a \parallel k_b$ of two adjacent legs, the color-

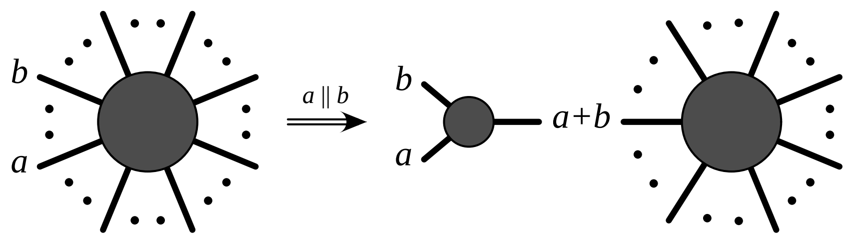

Figure 1: A schematic depiction of the collinear factorization of tree-level amplitudes, with the amplitudes labeled clockwise.

ordered amplitude A_n is singular. (It is finite when the two collinear legs are not adjacent arguments, that is when they are not color-connected.) This singular behavior has a universal form expressed by the tree-level factorization equation,

$$A_n^{(0)}(1,\ldots,a^{\lambda_a},b^{\lambda_b},\ldots,n) \quad \xrightarrow{k_a \parallel k_b} \\ g_s \sum_{\text{ph. pol. } \sigma} \mathrm{Split}^{\mathrm{tree}}_{-\sigma}(a^{\lambda_a},b^{\lambda_b}) A_{n-1}^{(0)}(1,\ldots,(a+b)^{\sigma},\ldots,n) + \cdots, \tag{2.3}$$

where the dots represent terms which are finite in the limit. In this equation, $\mathrm{Split}^{\mathrm{tree}}$ is the usual tree splitting amplitude, and the notation '$a+b$' means $k_a + k_b$. The QCD coupling is denoted by g_s. The notation 'ph. pol.' indicates a sum over physical polarizations only. ('Physical' here is in the sense of 'transverse', and their number may depend on the number of dimensions and on the variant of dimensional regularization employed.) This factorization is depicted schematically in fig. 1. At tree level, one may derive the splitting amplitudes from a string representation [39] or from the Berends–Giele recurrence relations [40]. It is characteristic of gauge theories that the splitting amplitude has a square-root singularity, $\mathrm{Split} \sim 1/\sqrt{s_{ab}}$, rather than a full inverse power of the two-particle invariant s_{ab}. Similar formulæ hold in the triple-collinear case [41, 42, 43, 44].

At one loop, the color decomposition analogous to eqn. (2.1) is

$$\mathcal{A}_n(\{k_i,\lambda_i,a_i\}) = \sum_J n_J \sum_{c=1}^{\lfloor n/2 \rfloor+1} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Gr}_{n;c}(\sigma)\, A_{n;c}^{[J]}(\sigma), \tag{2.4}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x and n_J is the number of particles of spin J. The leading color-structure factor,

$$\mathrm{Gr}_{n;1}(1) = N\ \mathrm{Tr}(T^{a_1} \cdots T^{a_n}), \tag{2.5}$$

is just N times the tree color factor, and the subleading color structures are given by

$$\mathrm{Gr}_{n;c}(1) = \mathrm{Tr}(T^{a_1} \cdots T^{a_{c-1}})\, \mathrm{Tr}(T^{a_c} \cdots T^{a_n}). \tag{2.6}$$

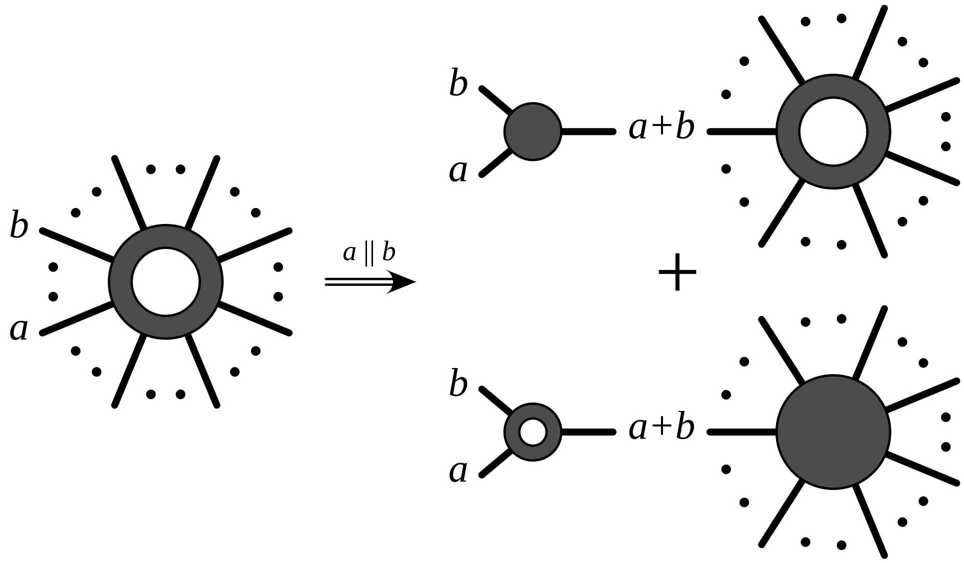

Figure 2: A schematic depiction of the collinear factorization of one-loop amplitudes.

S_n is the set of all permutations of n objects, and $S_{n;c}$ is the subset leaving $\mathrm{Gr}_{n;c}$ invariant. The decomposition eqn. (2.4) holds separately for different spins circulating around the loop. The usual normalization conventions take each massless spin-J particle to have two helicity states: gauge bosons, Weyl fermions, and complex scalars. (For internal particles in the fundamental $(N+\overline{N})$ representation, only the single-trace color structure ($c=1$) would be present, and the corresponding color factor would be smaller by a factor of N.)

The subleading color amplitudes $A_{n;c>1}$ are in fact not independent of the leading color amplitude $A_{n;1} \equiv A_n^{(1)}$. Rather, they can be expressed as sums over permutations of the arguments of the latter [45]. (For amplitudes with external fermions, the basic objects are primitive amplitudes [46] rather than the leading color one, but a similar dependence of the subleading color amplitudes on the leading-color ones holds.) As a result, it suffices to examine the collinear limits of leading color amplitudes. The collinear limits of the subleading color then follow using this relation. The leading color one-loop amplitudes obey the following factorization [45, 47],

$$\begin{aligned} &A_n^{(1)}(1,\ldots,a^{\lambda_a},b^{\lambda_b},\ldots,n) \xrightarrow{a\|b} \\ &\sum_{\text{ph. pol. }\sigma}\Big(g_s\,\mathrm{Split}^{\text{tree}}_{-\sigma}(a^{\lambda_a},b^{\lambda_b})\,A_{n-1}^{(1)}(1,\ldots,(a+b)^{\sigma},\ldots,n) \\ &\qquad\qquad +g_s^3\,\mathrm{Split}^{\text{1-loop}}_{-\sigma}(a^{\lambda_a},b^{\lambda_b})\,A_{n-1}^{(0)}(1,\ldots,(a+b)^{\sigma},\ldots,n)\Big)\,. \end{aligned} \tag{2.7}$$

This factorization is depicted schematically in fig. 2.

This form was originally deduced from explicit calculations of higher-point amplitudes [45, 46], but can also be proven more generally using the unitarity-based method [47]. The latter proof also provides an explicit formula for the one-loop splitting amplitude. We used it [48] to calculate all the one-loop splitting amplitudes relevant in QCD to all orders in the dimensional regularization parameter

$$\varepsilon = (4-d)/2 \tag{2.8}$$

with d being the dimensions of the spacetime. A subset of the terms higher order in ε are needed for singular phase-space integrations in NNLO jet calculations [49]. Bern et al. [50, 51, 52] derived the one-loop splitting amplitudes from an analysis of one-loop integrals.

3. Framework

In this section we derive a relation between the singularities in a 'partonic' (unsubtracted) cross section and the evolution kernels. We use this relation in following sections to compute the leading-order and the next-to-leading order kernels. Our derivation follows closely the one given in ref. [53]. As mentioned earlier we restrict attention to pure gluonic QCD together with an uncolored massive scalar ϕ to which the gluons couple via a higher-dimension operator. Such a coupling could, for example, be induced via a heavy quark loop which is integrated out. A similar derivation would of course hold for a gauge theory with fermions as well.

Consider the production of a glueball G in the decay of the massive scalar ϕ. In order to calculate the corresponding decay rate or differential distributions thereof, we need as ingredients the subtracted decay rate $\Gamma^R(\phi \to g+X)$ for the production of gluons in the decay of the massive scalar, and the gluon-to-glueball fragmentation function $\mathcal{D}^R_{g\to G}$. Using these quantities the energy distribution of the daughter glueball is given by the following expression,

$$\frac{d\Gamma(\phi \to G+X)}{dx_G} = \frac{d\Gamma^R(\phi \to g+X)}{dx_g} \otimes \mathcal{D}^R_{g\to G} \tag{3.1}$$

where the convolution '$\otimes$' is defined by

$$[f \otimes g](x) = \int_0^1 \int_0^1 dydz\, f(y)g(z)\delta(x-yz). \tag{3.2}$$

The energy fractions x_G and x_g are normalized to the mass m_ϕ of the scalar particle:

$$x_{G,g} = \frac{2E_{G,g}}{m_\phi}, \tag{3.3}$$

where E_G (E_g) denotes the energy of the glueball (gluon). We use the subscript R to indicate that both the fragmentation function as well as the subtracted decay rate depend on the subtraction method used to define them; we have not put in an explicit argument to show the dependence on the factorization and the renormalization scale. The universality of factorization allows us to write down a formula for the 'bare' decay rate $\Gamma^B(\phi \to g+X)$ of the scalar ϕ into an identified gluon in terms of the subtracted decay rate:

$$\frac{d\Gamma^B(\phi \to g+X)}{dx_g} = \frac{d\Gamma^R(\phi \to g+X)}{dx_g} \otimes \mathcal{D}_{g\to g}. \tag{3.4}$$

The 'bare' decay rate is unphysical, as it describes the 'probability' of finding a gluon with a given energy fraction inside a jet, while of course a lone gluon is not a colorless physical state. It will thus contain collinear or 'mass' singularities, as will $\mathcal{D}_{g\to g}$. Our purpose in considering $\Gamma^B(\phi \to g+X)$ is precisely to extract these singularities. In the following we will assume that all singularities (ultraviolet (UV), soft, and mass singularities) are regulated via a dimensional regulator. The advantage of the bare decay rate $\Gamma^B(\phi \to g+X)$ is that it can be calculated purely in perturbative QCD. As suggested by eqn. (3.4) the mass singularities which are present in $\mathcal{D}_{g\to g}$ must exactly match those in the (unphysical) partonic decay width $\Gamma^B(\phi \to g+X)$. To see this, expand $\mathcal{D}_{g\to g}$ in α_s,

$$\mathcal{D}_{g\to g}(z) = \delta(1-z) + \frac{\alpha_s}{2\pi}\mathcal{D}^{(1)}_{g\to g}(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathcal{D}^{(2)}_{g\to g}(z) + O(\alpha_s^3), \tag{3.5}$$

and then invert eqn. (3.4) to obtain,

$$\begin{aligned} \frac{d\Gamma^R(\phi \to g+X)}{dx_g} &= \frac{d\Gamma^B(\phi \to g+X)}{dx_g} \otimes \mathcal{D}^{-1}_{g\to g} \\ &= \frac{d\Gamma^B(\phi \to g+X)}{dx_g} \otimes \left[\delta(1-z) - \frac{\alpha_s}{2\pi}\mathcal{D}^{(1)}_{g\to g}(z) \right. \\ &\qquad \left. + \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\mathcal{D}^{(1)}_{g\to g} \otimes \mathcal{D}^{(1)}_{g\to g} - \mathcal{D}^{(2)}_{g\to g}(z)\right) + O(\alpha_s^3)\right], \end{aligned} \tag{3.6}$$

where we have used the identity

$$\begin{aligned} \delta(1-z) &= \left[\delta(1-z) + \frac{\alpha_s}{2\pi}\mathcal{D}^{(1)}_{g\to g}(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathcal{D}^{(2)}_{g\to g}(z)\right] \\ &\otimes \left[\delta(1-z) - \frac{\alpha_s}{2\pi}\mathcal{D}^{(1)}_{g\to g}(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 (\mathcal{D}^{(1)}_{g\to g} \otimes \mathcal{D}^{(1)}_{g\to g} - \mathcal{D}^{(2)}_{g\to g}(z))\right] + O(\alpha_s^3) \end{aligned} \tag{3.7}$$

to invert $\mathcal{D}_{g\to g}$. The left hand side of eqn. (3.6) is a finite quantity, and thus the right hand side must be so as well. If we now expand the 'bare' partonic decay width in α_s, we see that order by order the singularities in the partonic decay width must be canceled by those which appear in $\mathcal{D}^{(i)}_{g\to g}$. In particular, using

$$\frac{d\Gamma^B(\phi \to g+X)}{dx_g} = h^{(0)}(x_g) + \frac{\alpha_s}{2\pi}h^{(1)}(x_g) + \left(\frac{\alpha_s}{2\pi}\right)^2 h^{(2)}(x_g) + O(\alpha_s^3), \tag{3.8}$$

we obtain

$$\begin{aligned} \frac{d\Gamma^R(\phi \to g+X)}{dx_g} &= h^{(0)} + \frac{\alpha_s}{2\pi}\left[h^{(1)} - h^{(0)} \otimes \mathcal{D}^{(1)}_{g\to g}\right] \\ &+ \left(\frac{\alpha_s}{2\pi}\right)^2 \left[h^{(2)} - h^{(1)} \otimes \mathcal{D}^{(1)}_{g\to g} + h^{(0)} \otimes \left(\mathcal{D}^{(1)}_{g\to g} \otimes \mathcal{D}^{(1)}_{g\to g} - \mathcal{D}^{(2)}_{g\to g}(z)\right)\right] + O(\alpha_s^3). \end{aligned} \tag{3.9}$$

(The reader may worry about powers of α_s implicit in the higher-dimension operator coupling gluons to the heavy scalar. Such factors are frozen at the scale where the operator is generated, and in any event do not enter into the following arguments.) The usual application of this expansion is to the calculation of the differential decay width $d\Gamma^R(\phi \to g+X)/dx_g$: start with the 'bare' partonic decay width $h^{(i)}$ and use eqn. (3.9) (after ultraviolet subtractions as well) to obtain the finite subtracted decay width, which predicts the decay of the scalar ϕ into a glueball or more generally into hadrons. Here, we will use our knowledge of the collinear divergences in the 'bare' partonic decay width $\Gamma^B(\phi \to g+X)$, along with eqn. (3.9), to determine $\mathcal{D}_{g\to g}$.

Through $O(\alpha_s^3)$, we can express $\mathcal{D}_{g\to g}$ in terms of the evolution kernels $P^{(0)}$ and $P^{(1)}$,

$$\begin{aligned}\mathcal{D}_{g\to g} &= \delta(1-z) - \frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}S_\varepsilon P^{(0)}(z) \\ &+ \left(\frac{\alpha_s}{2\pi}\right)^2 {S_\varepsilon}^2\left[\frac{1}{2\varepsilon^2}P^{(0)}\otimes P^{(0)} + \frac{1}{4\varepsilon^2}\beta_0 P^{(0)} - \frac{1}{2}\frac{1}{\varepsilon}P^{(1)}\right] + O(\alpha_s^3).\end{aligned} \tag{3.10}$$

As usual β_0 denotes the first coefficient of the QCD β-function,

$$\beta_0 = \frac{1}{3}\left(11N - 2n_f\right) \overset{n_f=0}{=} \frac{11}{3}N, \tag{3.11}$$

and S_ε is the usual factor appearing in the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme:

$$S_\varepsilon = (4\pi)^\varepsilon e^{-\varepsilon\gamma} \tag{3.12}$$

with γ the Euler constant. The general structure of this expansion follows from the renormalization-group equation in the $\overline{\mathrm{MS}}$ scheme, which we choose. In particular, using eqn. (3.10) we obtain for the μ dependence of $\mathcal{D}^R_{g\to G}$:

$$\begin{aligned}\mu^2\frac{d}{d\mu^2}\mathcal{D}^R_{g\to G} &= -\mathcal{D}_{g\to g}\otimes\left(\beta(\alpha_s,\varepsilon)\frac{d}{d\alpha_s}\mathcal{D}^{-1}_{g\to g}\right)\otimes\mathcal{D}^R_{g\to G} \\ &= \left[\frac{\alpha_s}{2\pi}P^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)^2 P^{(1)} + \dots\right]\otimes\mathcal{D}^R_{g\to G}\end{aligned} \tag{3.13}$$

where we have used the β-function in d dimensions $\beta(\alpha_s,\varepsilon)$ in the $\overline{\mathrm{MS}}$ scheme:

$$\beta(\alpha_s,\varepsilon) = \mu^2\frac{d}{d\mu^2}\alpha_s = -\alpha_s\left(\varepsilon + \frac{\alpha_s}{4\pi}S_\varepsilon\beta_0\right) + O(\alpha_s^3). \tag{3.14}$$

Inserting eqn. (3.10) in eqn. (3.9) we obtain for the subtracted differential decay rate,

$$\begin{aligned}\frac{d\Gamma^R(\phi\to g+X)}{dx_g} &= h^{(0)} + \frac{\alpha_s}{2\pi}\left[h^{(1)} + h^{(0)}\otimes\frac{1}{\varepsilon}S_\varepsilon P^{(0)}(z)\right] + \left(\frac{\alpha_s}{2\pi}\right)^2\left[h^{(2)} + h^{(1)}\otimes\frac{1}{\varepsilon}S_\varepsilon P^{(0)}(z)\right. \\ &\left. + h^{(0)}\otimes\left(\frac{1}{2\varepsilon^2}{S_\varepsilon}^2 P^{(0)}\otimes P^{(0)} - \frac{1}{4\varepsilon^2}{S_\varepsilon}^2\beta_0 P^{(0)} + \frac{1}{2}\frac{1}{\varepsilon}{S_\varepsilon}^2 P^{(1)}\right)\right].\end{aligned} \tag{3.15}$$

The subtracted decay rate $\Gamma^R(\phi\to g+X)$ calculated via eqn. (3.15) is the $\overline{\mathrm{MS}}$-subtracted decay rate. In the following two sections we illustrate how to calculate the divergences in $h^{(1)}$ and $h^{(2)}$, and thereby determine the kernels $P^{(0)}$ and $P^{(1)}$.

4. The Leading-Order Kernel

In order to compute the leading-order kernel, $P^{(0)}$, we must isolate the collinear singularities in $h^{(1)}$. In terms of the matrix elements, the leading-order partonic decay rate is given by the following formula,

$$h^{(0)}(x) = \int dR^d(k_1,k_2)\,|\mathcal{A}^{(0)}(\phi\to g(k_1)g(k_2))|^2[\delta(x_g-x_1)+\delta(x_g-x_2)] \tag{4.1}$$

where $dR^d(k_1,\ldots,k_n)$ denotes the phase-space measure in d dimensions (including the symmetry factor $1/n!$) for n gluons with momenta $k_1,\ldots,k_n$,

$$dR^d(k_1,\ldots,k_n) = \frac{1}{n!}(2\pi)^d\delta(K-\sum_i k_i)\prod_{i=1}^{n}\frac{d^{d-1}k_i}{(2\pi)^{d-1}2k_i^0}. \tag{4.2}$$

The leading-order amplitude for the decay $\phi\to gg$ is given by $\mathcal{A}^{(0)}$. The energy fraction x_i of the identified gluon i is defined as in eqn. (3.3) but with $E_{G,g}$ replaced by its energy E_i. At next-to-leading order the partonic decay rate gets two additional contributions, from virtual and real-emission corrections,

$$\begin{aligned}
\frac{\alpha_s}{2\pi}h^{(1)}(x_g) &= \frac{\alpha_s}{2\pi}(h_v^{(1)}(x_g)+h_r^{(1)}(x_g)),\\
\frac{\alpha_s}{2\pi}h_v^{(1)}(x_g) &= \int dR^d(k_1,k_2)\; 2\mathrm{Re}[\mathcal{A}^{(1)}(\phi\to g(k_1)g(k_2))\mathcal{A}^{(0)*}(\phi\to g(k_1)g(k_2))]\\
&\qquad\times[\delta(x_g-x_1)+\delta(x_g-x_2)],\\
\frac{\alpha_s}{2\pi}h_r^{(1)}(x_g) &= \int dR^d(k_1,k_2,k_3)\;|\mathcal{A}^{(0)}(\phi\to g(k_1)g(k_2)g(k_3))|^2\\
&\qquad\times[\delta(x_g-x_1)+\delta(x_g-x_2)+\delta(x_g-x_3)],
\end{aligned} \tag{4.3}$$

where $\mathcal{A}^{(1)}(\phi\to gg)$ is the one-loop amplitude for the decay $\phi\to gg$ while $\mathcal{A}^{(0)}(\phi\to ggg)$ is the tree-level amplitude for the real-emission process $\phi\to ggg$.

Both contributions in eqn. (4.3) contain collinear singularities associated with the identified gluon and will thus contribute to the kernel $P^{(0)}(z)$. (The individual contributions also contain soft and other collinear singularities.) In the virtual corrections $h_v^{(1)}$ the singularities arise from the loop integration. The general structure of infrared divergences of one-loop amplitudes is known, we may extract them without further knowledge of the specific process at hand. This contribution will only contribute to the term proportional to $\delta(1-z)$ in $P^{(0)}(z)$.

Let us then turn to the contribution from real emission which will determine the 'non-trivial' z dependence of $P^{(0)}(z)$. (The $\delta(1-z)$ contribution can always be determined from the z dependent part through appeal to various sum rules, for example that associated with momentum conservation.) In the real emission contribution $h_r^{(1)}$ the singularities arise from the phase-space integration of singular terms in the matrix elements. These may be extracted by the phase space-slicing method [54, 55]. We will not need the whole apparatus developed for cross-section calculations, but only the following basic elements. The basic idea in the phase-space slicing method is to split phase space into *resolved* and *unresolved* regions. In resolved regions, all outgoing partons are 'resolved', which is to say none become soft or collinear. The unresolved regions are the remaining regions of the phase space; in these regions one or more final-state partons may be soft, or one or more pairs may become collinear.

For the three-gluon final state, we may use the following partition of unity to separate the various regions,

$$\begin{aligned}
1 &= [\Theta(s_{12}-s_{\min})+\Theta(s_{\min}-s_{12})][\Theta(s_{23}-s_{\min})+\Theta(s_{\min}-s_{23})]\\
&\quad [\Theta(s_{13}-s_{\min})+\Theta(s_{\min}-s_{13})]\\
&= \Theta(s_{12}-s_{\min})\Theta(s_{23}-s_{\min})\Theta(s_{13}-s_{\min}) \quad \text{(1, 2, 3 hard)}\\
&+ \Theta(s_{\min}-s_{12})\Theta(s_{23}-s_{\min})\Theta(s_{13}-s_{\min}) \quad \text{(1, 2 coll.)}\\
&+ \Theta(s_{12}-s_{\min})\Theta(s_{\min}-s_{23})\Theta(s_{13}-s_{\min}) \quad \text{(2, 3 coll.)}
\end{aligned}$$

$$
\begin{aligned}
&+\quad \Theta(s_{12}-s_{\min})\Theta(s_{23}-s_{\min})\Theta(s_{\min}-s_{13}) \quad \text{(1, 3 coll.)}\\
&+\quad \Theta(s_{\min}-s_{12})\Theta(s_{23}-s_{\min})\Theta(s_{\min}-s_{13}) \quad \text{(1 soft)}\\
&+\quad \Theta(s_{\min}-s_{12})\Theta(s_{\min}-s_{23})\Theta(s_{13}-s_{\min}) \quad \text{(2 soft)}\\
&+\quad \Theta(s_{12}-s_{\min})\Theta(s_{\min}-s_{23})\Theta(s_{\min}-s_{13}) \quad \text{(3 soft)}\\
&+\quad \Theta(s_{\min}-s_{12})\Theta(s_{\min}-s_{23})\Theta(s_{\min}-s_{13}) \quad \text{(`double unresolved')}, \qquad (4.4)
\end{aligned}
$$

with $s_{ij} = 2k_i \cdot k_j$. The arbitrary parameter $s_{\min}$ introduced in eqn. (4.4) controls the boundary between resolved and unresolved regions. In parentheses we have classified the different contributions into resolved, soft and collinear contributions. The 'double unresolved' contribution does not contribute because it is kinematically forbidden. Thus only the contributions classified as soft or collinear will yield singularities.

As we shall see, in fact only the collinear contributions survive at the end after combining virtual and real-emission contributions. As one would expect, the singularities arising from soft regions cancel between these two types of contributions. The evolution kernels are thus determined solely by the collinear or 'mass' singularities. From a practical point of view, such cancellations are an important consistency check on the calculation.

The cancellation of soft singularities can be seen heuristically from the form of the partonic observable we are 'measuring'. For an n-parton final state, it takes the form,

$$O_n = \sum_{i=1}^{n} \delta(x_g - x_i).$$

We see that $O_n \to O_{n-1}$ when one gluon becomes soft, but this does not happen when a pair becomes collinear. That is, the observable is soft-finite but not free of collinear singularities. Were the δ-functions not present in our integral, the sum of the virtual and real would of course be finite because it would just be the NLO correction to the total decay rate. With the δ-functions present, but one of the gluons soft, the corresponding term does not contribute, and all other terms are insensitive to the soft momentum. We can thus integrate the real-emission contribution over the soft region, generating poles in ε that cancels the corresponding singularity in the virtual corrections. In the case of collinear gluons, however, the two δ-functions depending on the momentum fractions of the collinear gluons will prevent us from integrating over the appropriate region of phase space. The collinear divergence in the virtual corrections will therefore not be fully canceled. The left-over singularity is precisely the term we are trying to extract.

We can use the symmetry of the matrix elements and of the phase-space measure to restrict attention to the contribution where gluons 1 and 2 are collinear. The other two collinear contributions follow through symmetrization. Using the color decomposition introduced in section 2 we may write,

$$\mathcal{A}^{(0)}(\phi \to g_1 g_2 g_3) = \sum_{\sigma \in S_3/Z_3} \mathrm{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}}) A^{(0)}(\sigma(1), \sigma(2), \sigma(3)). \qquad (4.5)$$

Calculating the square of $\mathcal{A}^{(0)}(\phi \to g_1 g_2 g_3)$ yields different color structures. For the determination of $P^{(0)}(z)$, it is sufficient to consider the leading-color structure (denoted by the subscript 'lc'):

$$
\begin{aligned}
|\mathcal{A}^{(0)}(\phi \to g_1 g_2 g_3)|^2\Big|_{\mathrm{lc}} &= \mathrm{Tr}(T^a T^b T^c)\mathrm{Tr}(T^c T^b T^a) \sum_{(ijk)=(123),(132)} |A^{(0)}(i,j,k)|^2 \Big|_{\mathrm{lc}}\\
&= N^3(|A^{(0)}(1,2,3)|^2 + |A^{(0)}(1,3,2)|^2). \qquad (4.6)
\end{aligned}
$$

Within these terms, examine the contribution from the phase-space region where gluons 1 and 2 become collinear,

$$\begin{aligned}
& \int dR^d(k_1,k_2,k_3)\ |\mathcal{A}^{(0)}(\phi\to g_1g_2g_3)|^2\Big|_{\rm lc}[\delta(x_g-x_1)+\delta(x_g-x_2)+\delta(x_g-x_3)] \\
& \times\Theta(s_{\rm min}-s_{12})\Theta(s_{23}-s_{\rm min})\Theta(s_{13}-s_{\rm min}) \\
= \ & 2g_s^2N\frac{1}{3}\int dR^d(P,k_3)dR^d_{\rm coll.}(k_1,k_2)\ |\mathcal{A}^{(0)}(P,k_3)|^2|\,\mathrm{Split}^{\rm tree}(k_1,k_2)|^2 \\
& \times[\delta(x_g-zx_P)+\delta(x_g-(1-z)x_P)+\delta(x_g-x_3)]\Theta(s_{\rm min}-s_{12})\Theta(s_{23}-s_{\rm min})\Theta(s_{13}-s_{\rm min}). \quad (4.7)
\end{aligned}$$

The factor 2 in front accounts for the two different color orderings. In deriving eqn. (4.7) we have used the factorization of the amplitudes as discussed in section 2. Using these results the factorization for the squared matrix element is given by

$$|A^{(0)}(k_1,\ldots,k_i^{\lambda_i},k_j^{\lambda_j},\ldots,k_n)|^2\xrightarrow{k_i\|k_j}g_s^2|\,\mathrm{Split}^{\rm tree}_{-\lambda}(k_i^{\lambda_i},k_j^{\lambda_j})|^2|A^{(0)}(k_1,\ldots,(k_i+k_j)^\lambda,\ldots,k_n)|^2 \qquad (4.8)$$

where we have eliminated cross terms of the form $\mathrm{Split}^{\rm tree}_{-\lambda}(k_i^{\lambda_i},k_j^{\lambda_j})\times\left(\mathrm{Split}^{\rm tree}_{\lambda}(k_i^{\lambda_i},k_j^{\lambda_j})\right)^*$. These terms vanish upon azimuthal integration. We have also made use of the factorization of the phase-space measure in the collinear limit [54],

$$dR^d(k_1,k_2,k_3)\xrightarrow{1\|2}\frac{1}{3}dR^d(P,k_3)dR^d_{\rm coll.}(k_1,k_2) \qquad (4.9)$$

where

$$dR^d_{\rm coll.}(k_1,k_2)=\mathcal{N}_c\,(s_{\rm min})^\varepsilon ds_{12}dz[s_{12}z(1-z)]^{-\varepsilon} \qquad (4.10)$$

and

$$\mathcal{N}_c=\frac{1}{16\pi^2}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{\rm min}}\right)^\varepsilon. \qquad (4.11)$$

We have defined z to be the momentum fraction in the collinear limit,

$$k_1=z(k_1+k_2)=zP, \quad \text{and} \quad k_2=(1-z)(k_1+k_2)=(1-z)P. \qquad (4.12)$$

We work throughout in $d=4-2\varepsilon$ dimensions. The leading-order splitting amplitude is given by

$$\mathrm{Split}^{\rm tree}(1,2)=-\frac{\sqrt{2}}{s_{12}}\left(-\varepsilon_1\cdot\varepsilon_2\,k_2\cdot\varepsilon_P+k_2\cdot\varepsilon_1\,\varepsilon_P\cdot\varepsilon_2-k_1\cdot\varepsilon_2\,\varepsilon_1\cdot\varepsilon_P\right). \qquad (4.13)$$

If we are only interested in the unpolarized kernels we have to sum over the final gluon polarizations and average over the fused-leg polarizations,

$$\widehat{\sum_{\rm pol.}}|\,\mathrm{Split}^{\rm tree}(1,2)|^2\equiv\frac{1}{2}\sum_{\lambda,\lambda_1,\lambda_2}|\,\mathrm{Split}^{\rm tree}_\lambda(1^{\lambda_1},2^{\lambda_2})|^2=\frac{2}{s_{12}}\frac{(z^2-z+1)^2}{z(1-z)}\equiv\frac{2}{s_{12}}p(z). \qquad (4.14)$$

The averaging depends in general on the variant of dimensional regularization, this form holding for the conventional scheme [56]. Inserting eqn. (4.10) and eqn. (4.14) in eqn. (4.7) we obtain

$$\begin{aligned}
& -\frac{4\mathcal{N}_cg_s^2N}{3\varepsilon}\int dR^d(P,k_3)\int_{\tilde z(k_3,P)}^{1-\tilde z(k_3,P)}dz\,[z(1-z)]^{-\varepsilon}|\mathcal{A}^{(0)}(P,k_3)|^2p(z) \\
& \times[\delta(x_g-zx_P)+\delta(x_g-(1-z)x_P)+\delta(x_g-x_3)]\Theta(s_{3P}-s_{\rm min}), \qquad (4.15)
\end{aligned}$$

with

$$\tilde{z}(i,j) = \frac{s_{\min}}{s_{ij}}. \tag{4.16}$$

eqn. (4.15) has almost the form we want: if we extend the region of the z-integral by adding and subtracting

$$\begin{aligned}
&-\frac{4\mathcal{N}_c g_s^2 N}{3\varepsilon}\int dR^d(P,k_3)\left(\int_0^{\tilde{z}(k_3,P)} + \int_{1-\tilde{z}(k_3,P)}^1\right) dz\,[z(1-z)]^{-\varepsilon}|\mathcal{A}^{(0)}(P,k_3)|^2 p(z) \\
&\times[\delta(x_g - zx_P) + \delta(x_g-(1-z)x_P)]\Theta(s_{3P}-s_{\min}),
\end{aligned} \tag{4.17}$$

we can write the expression in eqn. (4.15) as a convolution plus an additional term:

$$\begin{aligned}
&\frac{1}{3}\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}[K^{(0)}\otimes h^{(0)}](x_g) \\
+\quad &\frac{2}{3}N\int dR_2^d(1,2)|\mathcal{A}^{(0)}(1,2)|^2[\delta(x_g-x_1)+\delta(x_g-x_2)]\mathcal{C}^{(0)}(1,2,1),
\end{aligned} \tag{4.18}$$

where

$$K^{(0)}(z) = 16\pi^2\mathcal{N}_c N\left(\delta(1-z)\mathcal{N} - 2(z(1-z))^{-\varepsilon}p(z)\right), \tag{4.19}$$

$$\begin{aligned}
\mathcal{N} &= \int_0^1 dz z^{-\varepsilon}(1-z)^{-\varepsilon}p(z) = -\frac{3(1-\varepsilon)(4-3\varepsilon)}{2\varepsilon(3-2\varepsilon)(1-2\varepsilon)}\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \\
&= -\frac{2}{\varepsilon} - \frac{11}{6} + (\frac{1}{3}\pi^2 - \frac{67}{18})\varepsilon + O(\varepsilon^2),
\end{aligned} \tag{4.20}$$

and

$$\mathcal{C}^{(0)}(i,j,k) = -2\frac{\mathcal{N}_c g_s^2}{\varepsilon}\int_{\tilde{z}(i,j)}^{1-\tilde{z}(k,j)} dz[z(1-z)]^{-\varepsilon}p(z). \tag{4.21}$$

The other collinear regions – in which $k_2 \parallel k_3$ or $k_1 \parallel k_3$ – yield the same result. In the sum of the three we get thus a factor of 3 which cancels the factor $1/3$ from the phase space measure.

Using the factorization of color-ordered amplitudes in the soft limit [39]

$$A^{(0)}(1,\ldots,i^{\lambda_i},s^{\lambda_s},j^{\lambda_j},\ldots,n) \xrightarrow{s\text{ soft}} g_s N\,\mathrm{Soft}^{\mathrm{tree}}(i,s^{\lambda_s},j)\times A^{(0)}(1,\ldots,i^{\lambda_i},j^{\lambda_j},\ldots,n) \tag{4.22}$$

together with the factorization of the phase space measure in the soft limit [54],

$$dR^d(i,s,j) \xrightarrow{s\text{ soft}} \frac{1}{3}dR^d_{\mathrm{soft}}(i,s,j)dR^d(i,j) \tag{4.23}$$

with

$$dR^d_{\mathrm{soft}}(i,s,j) = \mathcal{N}_c(s_{\min})^{\varepsilon}\left(\frac{s_{is}s_{js}}{s_{ij}}\right)^{-\varepsilon}\frac{ds_{is}ds_{js}}{s_{ij}} \tag{4.24}$$

we obtain (after relabeling)

$$\begin{aligned}
&\int dR^d(k_1,k_2,k_3)\;|\mathcal{A}^{(0)}(\phi\to g(k_1)g(k_2)g(k_3))|^2\Big|_{\mathrm{lc}}[\delta(x_g-x_1)+\delta(x_g-x_2)+\delta(x_g-x_3)] \\
&\times\Theta(s_{\min}-s_{12})\Theta(s_{23}-s_{\min})\Theta(s_{\min}-s_{13}) + (2\text{ soft}) + (3\text{ soft}) \\
=\quad &\int dR^d(1,2)[\delta(x_g-x_1)+\delta(x_g-x_2)]|\mathcal{A}^{(0)}(1,2)|^2 2N\mathcal{S}^{(0)}(1,2)
\end{aligned} \tag{4.25}$$

for the singular contribution from the soft regions. The soft factor $\mathcal{S}^{(0)}(1,2)$ is given by

$$\mathcal{S}^{(0)}(1,2) = \sum_{\lambda} \int dR^d_{\mathrm{soft}}(2,3,1) g_s^2 |\,\mathrm{Soft}^{\mathrm{tree}}(2,3^{\lambda},1)|^2 \Theta(s_{\min} - s_{23}) \Theta(s_{\min} - s_{13}). \tag{4.26}$$

As we shall see, we will not need the explicit result for $\mathrm{Soft}^{\mathrm{tree}}$. Combining the soft and collinear contributions we obtain

$$\begin{aligned} h_r^{(1)}(x_g)\Big|_{\mathrm{lc,\ sing.}} &= \frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}[K^{(0)} \otimes h^{(0)}](x_g) \\ &\quad + \int dR^d(1,2)[\delta(x_g - x_1) + \delta(x_g - x_2)]|\mathcal{A}^{(0)}(1,2)|^2 2N(\mathcal{S}^{(0)}(1,2) + \mathcal{C}^{(0)}(1,2,1)) \end{aligned} \tag{4.27}$$

for the singular part of the real emission contribution. To complete our analysis of the singular part in $h^{(1)}$ it remains only to add in the contribution from the virtual corrections. The color decomposition of the one-loop amplitudes in pure gluonic QCD is given by (c.f. section 2)

$$\mathcal{A}^{(1)}(\phi \to g_1 \cdots g_n) = \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Gr}_{n;c}(\sigma)\, A_{n;c}(\sigma(1), \ldots, \sigma(n)), \tag{4.28}$$

with $\mathrm{Gr}_{n;c}(1)$ defined in eqn. (2.6). Once again we are only interested in the leading color-structure. So it is sufficient to include $\mathrm{Gr}_{n;1}(1)A_{n;1}(1,\ldots,n) \equiv N\ \mathrm{Tr}(T^{a_1}\cdots T^{a_n})A^{(1)}_{\mathrm{lc}}(1,\ldots,n)$ in our analysis. The singularity structure of the one-loop color-ordered amplitude $A^{(1)}_{\mathrm{lc}}(1,2)$ is known [54, 57]. Alternatively, we may reason as follows: suppose we are calculating the real-emission contribution to the total ϕ decay rate. In this case we must replace the sum over δ-functions by 1 in the derivation above. We see immediately that the singular contribution of the soft and collinear regions to the total decay rate reduces to

$$\int dR^d(1,2)|\mathcal{A}^{(0)}(1,2)|^2 2N(\mathcal{S}^{(0)}(1,2) + \mathcal{C}^{(0)}(1,2,1)). \tag{4.29}$$

On the other hand, unitarity [58, 59] dictates that the total decay rate must be free of soft and collinear singularities, order by order in perturbation theory. (Note that the $s_{\min}$-dependent terms on the right-hand side are finite.) This implies that the singular contribution from the leading-color one-loop amplitude satisfies

$$N\, 2\mathrm{Re}(A^{(1)}_{\mathrm{lc}}(1,2)A^{(0)*}(1,2))\Big|_{\mathrm{sing.}} = -|A^{(0)}(1,2)|^2 2N(\mathcal{S}^{(0)}(1,2) + \mathcal{C}^{(0)}(1,2,1))\Big|_{\mathrm{sing.}}. \tag{4.30}$$

As a consequence we see that the singular contribution from the virtual correction cancels the second term in eqn. (4.18). We thus obtain

$$h^{(1)}(x_g)\Big|_{\mathrm{lc,\ sing.}} = \frac{1}{\varepsilon}[h^{(0)} \otimes K^{(0)}](x_g) \tag{4.31}$$

for the surviving singular term. Note that $h^{(0)}$ should not be expanded in ε. Comparing with eqn. (3.15) we find

$$P^{(0)}(z) = -\,K^{(0)}(z)\Big|_{d=4} = 2N\left(\frac{11}{12}\delta(1-z) + \frac{1}{z} + \left[\frac{1}{1-z}\right]_+ - z^2 + z - 2\right), \tag{4.32}$$

where we have used

$$(1-z)^{-1-\varepsilon} = -\frac{1}{\varepsilon}\delta(1-z) + \left[\frac{1}{1-z}\right]_+ - \varepsilon\left[\frac{\ln(1-z)}{1-z}\right]_+ + O(\varepsilon^2). \tag{4.33}$$

In these equations, the plus prescription defines distributions via,

$$[F(z)]_+ = \lim_{\eta\to 0}\left\{\Theta(1-z-\eta)F(z) - \delta(1-z-\eta)\int_0^{1-\eta} F(y)dy\right\}, \tag{4.34}$$

so that if $g(z)$ is well-behaved at $z=1$, then

$$\int_x^1 dz\,\frac{g(z)}{(1-z)_+} \quad = \quad \int_x^1 \frac{g(z)-g(1)}{1-z} + g(1)\ln(1-x), \tag{4.35}$$

$$\int_x^1 dz\, g(z)\Big[\frac{\ln(1-z)}{1-z}\Big]_+ \quad = \quad \int_x^1 \frac{(g(z)-g(1))\ln(1-z)}{1-z} + \frac{g(1)}{2}\ln^2(1-x). \tag{4.36}$$

The phase-space slicing used here shares many features with that used for the calculation of a typical infrared-finite observable. In contrast to those observables, which necessarily allow the recombination of collinear partons in the real-emission terms, and hence allow integrating over collinear phase space in a process-independent way [54], the partonic decay rate $\Gamma^B(\phi\to g+X)$ describes an unphysical ‘probability’ of finding a gluon with a given energy fraction inside the jet. This does not allow the recombination of two collinear gluons to a hard one, and a further convolution with a physical state distribution function is necessary to produce an infrared-finite observable. The uncanceled singularity does have a universal form, however, which is why this ‘unphysical’ object can be used to extract it. The singularities associated with soft-gluon emission do cancel, because too soft a gluon won’t contribute to the energy fraction of any final-state hadron, and hence will drop out of the calculation.

5. Next-to-Leading Order Kernel

In order to calculate the NLO kernel we must study the singularities in the NNLO decay rate. Again keeping only the contribution dominant in the number of colors (denoted by the subscript lc) the decay rate is,

$$\begin{aligned}
&\left.\frac{d\Gamma^B(\phi\to g+X)}{dx_g}\right|_{\text{NNLO, lc}} = \\
&N^4\int dR^d(1,2)\; 2\mathrm{Re}(A^{(2)}_{\mathrm{lc}}(1,2)A^{(0)*}(1,2))\sum_{i=1}^{2}\delta(x_g-x_i) \\
+\;\; &N^4\int dR^d(1,2)\; |A^{(1)}_{\mathrm{lc}}(1,2)|^2\sum_{i=1}^{2}\delta(x_g-x_i) \\
+\;\; &2N^4\int dR^d(1,2,3)\; 2\mathrm{Re}(A^{(1)}_{\mathrm{lc}}(1,2,3)A^{(0)*}(1,2,3))\sum_{i=1}^{3}\delta(x_g-x_i) \\
+\;\; &6N^4\int dR^d(1,2,3,4)\; |A^{(0)}(1,2,3,4)|^2\sum_{i=1}^{4}\delta(x_g-x_i).
\end{aligned} \tag{5.1}$$

The factors of 2 and 6 in front of the integrals in the last two terms are combinatorial. Once again we are interested only in the singular terms. The singularities in the first two terms will contribute only to the coefficient of $\delta(1-z)$ in the NLO kernel. The structure of the singularities, needed for the direct computation of this coefficient is now known in a general two-loop amplitude [60, 61, 62]. However, the coefficient of the δ-function can always be computed using the sum rule constraints on the kernels. We will therefore not compute it directly, and turn our attention to the remaining terms in the kernel. These are determined by the three- and four-parton final states, which we discuss in the following two subsections.

5.1. Three-Parton Final State

The treatment of this contribution is very similar to the treatment of the three-parton final state in the computation of the leading-order kernel. In particular, we may use the same phase-space slicing. Here, we must consider the collinear limits of one-loop amplitudes in addition to those of tree amplitudes. Thus in addition to $\mathrm{Split}^{\text{tree}}$, the one-loop splitting amplitude $\mathrm{Split}^{\text{1-loop}}$ also makes an appearance. Symmetry again allows us to focus on the configuration $k_1 \parallel k_2$; the other two collinear regions will give equal contributions. We are interested in the singular contribution arising from

$$
\begin{aligned}
2\int dR^d(1,2,3)\;\sum_{i=1}^{3}\delta(x_g-x_i)2\mathrm{Re}(A^{(1)}_{\text{lc}}(1,2,3)A^{(0)*}(1,2,3))\\
\times\Theta(s_{\min}-s_{12})\Theta(s_{23}-s_{\min})\Theta(s_{13}-s_{\min}).
\end{aligned}
\tag{5.2}
$$

Using the factorization of the phase-space measure eqn. (4.9), along with the factorization of the leading color one-loop amplitudes [45, 47] (cf. eqn. (2.7)),

$$
\begin{aligned}
A^{(1)}_{\text{lc}}(1,\ldots,i^{\lambda_i},j^{\lambda_j},\ldots,n) \quad \xrightarrow{i\parallel j} \quad & g_{\text{s}}\,\mathrm{Split}^{\text{tree}}_{-\lambda}(i^{\lambda_i},j^{\lambda_j})\times A^{(1)}_{\text{lc}}(1,\ldots,(i+j)^{\lambda},\ldots,n)\\
&+g_{\text{s}}^3\,\mathrm{Split}^{\text{1-loop}}_{-\lambda}(i^{\lambda_i},j^{\lambda_j})\times A^{(0)}(1,\ldots,(i+j)^{\lambda},\ldots,n),
\end{aligned}
\tag{5.3}
$$

we obtain

$$
\begin{aligned}
&-\frac{4\mathcal{N}_c g_{\text{s}}^2}{3\varepsilon}\int dR^d(P,3)\int_{\tilde z(3,P)}^{1-\tilde z(3,P)} dz\,[z(1-z)]^{-\varepsilon}p(z)[\delta(x_g-zz_P)+\delta(x_g-(1-z)z_P)]\\
&\qquad\times 2\mathrm{Re}[A^{(1)}_{\text{lc}}(P,3)A^{(0)*}(P,3)]\\
&-\frac{2\mathcal{N}_c g_{\text{s}}^4}{3\varepsilon}\,s_{\min}{}^{-\varepsilon}\int dR^d(P,3)\int_{\tilde z(3,P)}^{1-\tilde z(3,P)} dz\,[z(1-z)]^{-\varepsilon}[\delta(x_g-zz_P)+\delta(x_g-(1-z)z_P)]\\
&\qquad\times 2|A^{(0)}(P,3)|^2\widehat{\sum_{\text{pol.}}}\mathrm{Re}[(s_{12})^{1+\varepsilon}\,\mathrm{Split}^{\text{1-loop}}(1,2)\,\mathrm{Split}^{\text{tree}*}(1,2)]\\
&+\frac{2}{3}\int dR^d(P,3)\;\delta(x_g-x_3)\,\mathcal{C}^{(0)}(3,P,3)2\mathrm{Re}(A^{(1)}_{\text{lc}}(P,3)A^{(0)}(P,3)^*)\\
&+\frac{2}{3}\int dR^d(P,3)\;\delta(x_g-x_3)\,\mathcal{C}^{(1)}(3,P,3)|A^{(0)}(P,3)|^2,
\end{aligned}
\tag{5.4}
$$

where

$$
\mathcal{C}^{(1)}(i,j,k)=-\frac{\mathcal{N}_c g_{\text{s}}^4}{2\varepsilon}\,s_{\min}{}^{-\varepsilon}\int_{\tilde z(3,P)}^{1-\tilde z(3,P)} dz[z(1-z)]^{-\varepsilon}2\widehat{\sum_{\text{pol.}}}\mathrm{Re}[(s_{12})^{1+\varepsilon}\,\mathrm{Split}^{\text{1-loop}}(1,2)\,\mathrm{Split}^{\text{tree}*}(1,2)].
\tag{5.5}
$$

Note that

$$
\widehat{\sum_{\text{pol.}}}\mathrm{Re}[(s_{12})^{1+\varepsilon}\,\mathrm{Split}^{\text{1-loop}}(1,2)\,\mathrm{Split}^{\text{tree}*}(1,2)]
\tag{5.6}
$$

does not depend on s_{12}. Extending the region of the z-integral (by adding and subtracting the corresponding term), neglecting $\delta(1-z)$-type contributions, adding the contribution from $k_2 \parallel k_3$ and $k_1 \parallel k_3$, and relabeling, we obtain

$$\begin{aligned}
& \frac{\alpha_s}{2\pi} N^4 \int dz \int dR^d(1,2)\ [\delta(x_g - zx_1) + \delta(x_g - zx_2)] 2\mathrm{Re}[A^{(1)}(1,2) A^{(0)*}(1,2)] \frac{1}{\varepsilon} K^{(0)}(z) \\
& + \left(\frac{\alpha_s}{2\pi}\right)^2 N^4 \int dz \int dR^d(1,2)\ [\delta(x_g - zx_1) + \delta(x_g - zx_2)] |A^{(0)}(1,2)|^2 \frac{1}{\varepsilon} K_v^{(1)}(z) \\
= \;\; & \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{\varepsilon} [h_v^{(1)} \otimes K^{(0)}](x_g) + \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{\varepsilon} [h^{(0)} \otimes K_v^{(1)}](x_g) \qquad (5.7)
\end{aligned}$$

where

$$K_v^{(1)}(z) = -\frac{1}{2}(16\pi^2)^2 \mathcal{N}_c\, s_{\min}^{-\varepsilon} N^2 (z(1-z))^{-\varepsilon} \widehat{\sum_{\text{pol.}}} \mathrm{Re}((s_{12})^{1+\varepsilon}\, \mathrm{Split}^{\text{1-loop}}(1,2)\, \mathrm{Split}^{\text{tree}}(1,2)^*). \qquad (5.8)$$

The explicit results for $\mathrm{Split}^{\text{1-loop}}(1,2)$ can be found in refs. [45, 51, 52, 48]:

$$\mathrm{Split}^{\text{1-loop}}(1,2) = r_1(z)\, \mathrm{Split}^{\text{tree}}(1,2) + r_2(z) \frac{(k_1 - k_2) \cdot \varepsilon_P}{\sqrt{2} s_{12}^2} (s_{12} \varepsilon_1 \cdot \varepsilon_2 - 2k_2 \cdot \varepsilon_1\, k_1 \cdot \varepsilon_2) \qquad (5.9)$$

with

$$\begin{aligned}
r_1(z) &= \frac{1}{2} \left(\frac{\mu^2}{-s_{12}}\right)^{\varepsilon} [z f_1(z) + (1-z) f_1(1-z) - 2 f_2], \\
r_2(z) &= \frac{\varepsilon^2}{(1-2\varepsilon)(3-2\varepsilon)} \left(\frac{\mu^2}{-s_{12}}\right)^{\varepsilon} f_2, \qquad (5.10)
\end{aligned}$$

$$\begin{aligned}
f_1(z) &= \frac{2}{\varepsilon^2} c_\Gamma \left[-\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\, z^{-1-\varepsilon}(1-z)^{\varepsilon} - \frac{1}{z} + \frac{(1-z)^{\varepsilon}}{z}\, {}_2F_1(\varepsilon, \varepsilon; 1+\varepsilon; z)\right] \\
&= \frac{2}{\varepsilon^2} c_\Gamma \left[-\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\, z^{-1-\varepsilon}(1-z)^{\varepsilon} - \frac{1}{z} + \frac{(1-z)^{\varepsilon}}{z} + \frac{\varepsilon^2}{z} \mathrm{Li}_2(z)\right] + O(\varepsilon), \qquad (5.11) \\
f_2(z) &= -\frac{1}{\varepsilon^2} c_\Gamma, \qquad (5.12)
\end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function, Li_2 the dilogarithm, and the standard one-loop prefactor is,

$$c_\Gamma = \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{(4\pi)^{2-\varepsilon}\Gamma(1-2\varepsilon)} = \mathcal{N}_c \left(\frac{\mu^2}{s_{\min}}\right)^{-\varepsilon} + O(\varepsilon^3). \qquad (5.13)$$

Using the above results we find that

$$r_1(z) = \mathcal{N}_c \left(\frac{s_{\min}}{-s_{12}}\right)^{\varepsilon} \left(-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}(\ln(z) + \ln(1-z)) - \frac{1}{2}\ln(1-z)^2 + \ln(z)\ln(1-z) - \frac{1}{2}\ln(z)^2 - \frac{1}{6}\pi^2\right) \qquad (5.14)$$

and thus for the virtual contributions to $K^{(1)}$,

$$\begin{aligned}
K_v^{(1)}(z) \;=\; & -(16\pi^2 \mathcal{N}_c)^2 N^2 \Big\{ p(z) \Big[-\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon}(\ln(z) + \ln(1-z)\Big] \\
& -2\ln(1-z)^2 - 2\ln(z)\ln(1-z) - 2\ln(z)^2 + \frac{1}{3}\pi^2) + \frac{1}{6} \Big\} + O(\varepsilon). \qquad (5.15)
\end{aligned}$$

The above equations give the unrenormalized splitting amplitude; the renormalized one is, as given by eqn. (5.9) – eqn. (5.12)

$$\mathrm{Split}^{\text{1-loop,R}} = \mathrm{Split}^{\text{1-loop}} - \frac{1}{16\pi^2} S_\varepsilon \frac{1}{\varepsilon} \frac{11}{6}\, \mathrm{Split}^{\text{tree}}, \tag{5.16}$$

which adjusts eqn. (5.15) by,

$$\begin{aligned} \delta K_v^{(1)}(z) &= 16\pi^2 \mathcal{N}_c S_\varepsilon N^2 \frac{11}{3} \left(\frac{1}{\varepsilon} - \ln(z) - \ln(1-z) \right) p(z) + O(\varepsilon) \\ &= (16\pi^2 \mathcal{N}_c)^2 N^2 \left(\frac{\mu^2}{s_{\min}} \right)^{-\varepsilon} \frac{11}{3} \left(\frac{1}{\varepsilon} - \ln(z) - \ln(1-z) \right) p(z) + O(\varepsilon). \end{aligned} \tag{5.17}$$

5.2. Four-Parton Final State

We turn next to the four-gluon final state. The slicing of phase space is now more complicated than in the three-gluon case. A new feature arises: we can have *double unresolved* contributions. Such contributions can originate from two soft gluons, one soft gluon and a collinear pair, from two independent collinear pairs or from a triple collinear configuration. To derive a suitable slicing we start with the following partition of unity,

$$\begin{aligned} 1 &= [\Theta(s_{12} - s_{\min}) + \Theta(s_{\min} - s_{12})][\Theta(s_{23} - s_{\min}) + \Theta(s_{\min} - s_{23})] \\ &\times [\Theta(s_{34} - s_{\min}) + \Theta(s_{\min} - s_{34})][\Theta(s_{14} - s_{\min}) + \Theta(s_{\min} - s_{14})]. \end{aligned} \tag{5.18}$$

Each term in eqn. (5.1) is expressed in terms of a single color-ordered amplitude, and hence the vanishing of an invariant s_{ij} for non-adjacent gluons $i,\ j$ does not yield a singular contribution. Expanding the right hand side of eqn. (5.18) we get a decomposition of the four gluon phase space into sixteen different regions. We can classify these as chown in table 1.

In the first type of region, all invariants would be smaller than $s_{\min}$, but this is kinematically forbidden, and so it gives rise to no contribution. In the last type of region, all invariants are greater than $s_{\min}$, so that no singularities arise; we can set aside this region as well. In the penultimate type listed, one nearest-neighbor invariant is smaller than $s_{\min}$ while the other three are greater than it. These give rise to single-unresolved contributions for a collinear pair of gluons. These regions give the NLO corrections to processes with three distinct jets. Their treatment follows that of the single-unresolved regions in the three-parton final-state in section 4. As given, they contribute only to the leading-order kernel (and could be used to compute it had we not already used the NLO decay rate to do so).

It will nonetheless be convenient to add and subtract terms as done in section 4 for the three-parton final state; the subtracted terms will then enter into the calculation of the NLO kernel. (This amounts of course to shifting terms from other regions.) The contribution from the $1 \parallel 2$, for example, is given by

$$\begin{aligned} & 6N^4 \int dR^d(1,2,3,4)\, |A^{(0)}(1,2,3,4)|^2 \sum_{i=1}^{4} \delta(x_g - x_i) \\ & \qquad \times \Theta(s_{\min} - s_{12}) \Theta(s_{23} - s_{\min}) \Theta(s_{34} - s_{\min}) \Theta(s_{14} - s_{\min}) \\ = {} & \frac{N^3}{2} \int dR^d(1,2,3) \int_0^1 dz\, |A^{(0)}(1,2,3)|^2 \frac{\alpha_s}{2\pi} \frac{1}{\varepsilon} K^{(0)}(z) \end{aligned}$$

s_{12}	s_{23}	s_{34}	s_{41}	*Comments*
$<$	$<$	$<$	$<$	Triple-unresolved: kinematically forbidden.
$<$	$<$	$<$	$>$	1, 4 hard & separated. 1 $\parallel$ 2, 3 soft; or 3 $\parallel$ 4, 2 soft; or 2, 3 soft.
$<$	$<$	$>$	$<$	3, 4 hard & separated. 2 $\parallel$ 3, 1 soft; or 1 $\parallel$ 4, 2 soft; or 1, 2 soft.
$<$	$>$	$<$	$<$	2, 3 hard & separated. 1 $\parallel$ 2, 4 soft; or 3 $\parallel$ 4, 1 soft; or 1, 4 soft.
$>$	$<$	$<$	$<$	1, 2 hard & separated. 2 $\parallel$ 3, 4 soft; or 1 $\parallel$ 4, 3 soft; or 3, 4 soft.
$<$	$<$	$>$	$>$	1, 3, 4 hard. 1 $\parallel$ 2 $\parallel$ 3; or 1 $\parallel$ 3, 2 soft; or 1 $\not\parallel$ 3, 2 soft.
$<$	$>$	$>$	$<$	2, 3, 4 hard. 2 $\parallel$ 3 $\parallel$ 4; or 2 $\parallel$ 4, 3 soft; or 2 $\not\parallel$ 4, 1 soft.
$>$	$>$	$<$	$<$	1, 2, 3 hard. 1 $\parallel$ 3 $\parallel$ 4; or 1 $\parallel$ 3, 4 soft; or 1 $\not\parallel$ 3, 4 soft.
$>$	$<$	$<$	$>$	1, 2, 4 hard. 1 $\parallel$ 2 $\parallel$ 4; or 2 $\parallel$ 4, 3 soft; or 2 $\not\parallel$ 4, 3 soft.
$<$	$>$	$<$	$>$	All hard. 1 $\parallel$ 2, 3 $\parallel$ 4. Double-collinear region.
$>$	$<$	$>$	$<$	All hard. 2 $\parallel$ 3, 1 $\parallel$ 4. Double-collinear region.
$<$	$>$	$>$	$>$	All hard. 1 $\parallel$ 2. No contribution to NLO kernel.
$>$	$<$	$>$	$>$	All hard. 2 $\parallel$ 3. No contribution to NLO kernel.
$>$	$>$	$<$	$>$	All hard. 3 $\parallel$ 4. No contribution to NLO kernel.
$>$	$>$	$>$	$<$	All hard. 1 $\parallel$ 4. No contribution to NLO kernel.
$>$	$>$	$>$	$>$	All hard. No singularities. No contribution to NLO kernel.

Table 1: Classification of the four parton phase space. The notation $<$ ($>$) means that the corresponding invariant s_{ij} is smaller (greater) than $s_{\min}$.

$$
\begin{aligned}
& \times \sum_{i=1}^{3} \delta(x_g - z z_i)\Theta(s_{12}-s_{\min})\Theta(s_{23}-s_{\min})\Theta(s_{13}-s_{\min}) \\
+ \quad & \frac{3}{2}N^4 \int dR^d(k_1,k_2,k_3)\ \mathcal{C}^{(0)}(3,1,2)|A^{(0)}(1,2,3)|^2 \\
& \times[\delta(x_g-x_1)+\delta(x_g-x_2)+\delta(x_g-x_3)] \\
& \times\Theta(s_{21}-s_{\min})\Theta(s_{31}-s_{\min})\Theta(s_{23}-s_{\min}),
\end{aligned}
\tag{5.19}
$$

where we have used the cyclic invariance of the color-ordered amplitudes together with the freedom to relabel the momenta. We adjusted the boundaries of the z-integrals as described in section 4. The last term will contribute only to δ-type terms; dropping it and including the contribution from $2 \parallel 3$, $3 \parallel 4$, $4 \parallel 1$ we obtain,

$$
\begin{aligned}
& 2N^3 \int_0^1 dz \int dR^d(1,2,3)\ \ |A^{(0)}(1,2,3)|^2 \frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}K^{(0)}(z) \\
& \qquad \times \sum_{i=1}^{3} \delta(x_g - z z_i)\Theta(s_{12}-s_{\min})\Theta(s_{23}-s_{\min})\Theta(s_{13}-s_{\min}).
\end{aligned}
\tag{5.20}
$$

We can further rewrite it using eqn. (4.27),

$$
\begin{aligned}
& 2N^3 \int dz \int dR^d(1,2,3)\ |A^{(0)}(1,2,3)|^2 \sum_{i=1}^{3}\delta(x_g - x_i z)\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}K^{(0)}(z) \\
& \quad \times\Theta(s_{12}-s_{\min})\Theta(s_{23}-s_{\min})\Theta(s_{13}-s_{\min}) \\
= \quad & 2N^3 \int dz \int dR^d(1,2,3)\ |A^{(0)}(1,2,3)|^2 \sum_{i=1}^{3}\delta(x_g - x_i z)\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}K^{(0)}(z) \\
& -\left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{\varepsilon^2}[h^{(0)}\otimes K^{(0)}]\otimes K^{(0)} \\
& -N^2 \int dR^d(1,2)\ [\delta(y-x_1 z)+\delta(y-x_2 z)]|A^{(0)}(1,2)|^2 2(\mathcal{S}^{(0)}(1,2)+\mathcal{C}^{(0)}(1,2,1))\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}K^{(0)}(z) \\
= \quad & \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{\varepsilon} h_r^{(1)} \otimes K^{(0)} - \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{\varepsilon^2} h^{(0)}\otimes K^{(0)}\otimes K^{(0)} \\
& -N^2 \int dR^d(1,2)\ [\delta(y-x_1 z)+\delta(y-x_2 z)]|A^{(0)}(1,2)|^2 2(\mathcal{S}^{(0)}(1,2)+\mathcal{C}^{(0)}(1,2,1))\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}K^{(0)}(z).
\end{aligned}
\tag{5.21}
$$

The last term does not show a proper factorization because the combination $\mathcal{S}^{(0)}(1,2)+\mathcal{C}^{(0)}(1,2,1)$ is still process dependent, but it will eventually cancel against virtual corrections with similar structure.

We continue with contributions where two invariants are smaller than $s_{\min}$. There are two types, the third and fourth types in table 1. The latter,

$$
\begin{aligned}
& \Theta(s_{\min}-s_{12})\Theta(s_{23}-s_{\min})\Theta(s_{\min}-s_{34})\Theta(s_{14}-s_{\min}) \\
+ \quad & \Theta(s_{12}-s_{\min})\Theta(s_{\min}-s_{23})\Theta(s_{34}-s_{\min})\Theta(s_{\min}-s_{14}),
\end{aligned}
\tag{5.22}
$$

correspond to configurations with two independent pairs of collinear gluons. Here, the collinear integral over the variable not present in a δ-function can be done, while the other remains, so we obtain

$$
\int dz \int dR_2^d(1,2)\mathcal{C}^{(0)}(zk_1,k_2,(1-z)k_1)[\delta(x_g-x_1 z)+\delta(x_g-x_2 z)]|A^{(1)}(1,2)|^2\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}K^{(0)}(z), \tag{5.23}
$$

where we have adjusted the boundaries of the z-integral as usual, and have dropped terms which contribute only to the δ-function in the kernel. Note that the integral $\mathcal{C}^{(0)}(zk_1,2,(1-z)k_1)$ still depends on the momentum fraction z,

$$\mathcal{C}^{(0)}(zk_1,k_2,(1-z)k_1)-\mathcal{C}^{(0)}(k_1,k_2,k_1)=-\frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{s_{\min}}\right)^{\varepsilon}\left(\frac{s_{12}}{s_{\min}}\right)^{\varepsilon}\frac{1}{\varepsilon^2}\left(z^{\varepsilon}+(1-z)^{\varepsilon}-2\right). \tag{5.24}$$

In the third type of region in table 1, two invariants with a common momentum ('adjacent' invariants) are smaller than $s_{\min}$,

$$\begin{aligned}
&\Theta(s_{\min}-s_{12})\Theta(s_{23}-s_{\min})\Theta(s_{34}-s_{\min})\Theta(s_{\min}-s_{14})\\
+\ &\Theta(s_{\min}-s_{12})\Theta(s_{\min}-s_{23})\Theta(s_{34}-s_{\min})\Theta(s_{14}-s_{\min})\\
+\ &\Theta(s_{12}-s_{\min})\Theta(s_{\min}-s_{23})\Theta(s_{\min}-s_{34})\Theta(s_{14}-s_{\min})\\
+\ &\Theta(s_{12}-s_{\min})\Theta(s_{23}-s_{\min})\Theta(s_{\min}-s_{34})\Theta(s_{\min}-s_{14}).
\end{aligned} \tag{5.25}$$

These contain subregions with qualitatively different types of unresolved contributions, in which we must use different factorizations – as long as we are not using a limiting function unifying different limits (see for example ref. [63]). For example, consider the region defined by

$$\Theta(s_{\min}-s_{12})\Theta(s_{\min}-s_{23})\Theta(s_{34}-s_{\min})\Theta(s_{14}-s_{\min}). \tag{5.26}$$

Gluons 1, 3, 4 cannot be soft, otherwise the last two Θ functions would vanish. The constraint may be satisfied in two distinct ways: gluon 2 can be soft, or three momenta can be collinear, $1 \parallel 2 \parallel 3$ (with or without gluon being soft).

To distinguish the different subregions, we may introduce additional Θ functions. For example, multiply eqn. (5.26) by

$$1=\Theta(s_{123}-s_{\min})+\Theta(s_{\min}-s_{123})\left(\Theta(s_{24}-s_{\min})+\Theta(s_{\min}-s_{24})\right) \tag{5.27}$$

with $s_{ijk}=(k_i+k_j+k_k)^2$ to obtain

$$\begin{aligned}
&\Theta(s_{\min}-s_{12})\Theta(s_{\min}-s_{23})\Theta(s_{34}-s_{\min})\Theta(s_{14}-s_{\min})\Theta(s_{123}-s_{\min}) && \text{2 soft}\\
&+\Theta(s_{34}-s_{\min})\Theta(s_{14}-s_{\min})\Theta(s_{\min}-s_{24})\Theta(s_{\min}-s_{123}) && 1\parallel 3 \text{ and 2 soft}\\
&+\Theta(s_{34}-s_{\min})\Theta(s_{14}-s_{\min})\Theta(s_{24}-s_{\min})\Theta(s_{\min}-s_{123}). && 1\parallel 2\parallel 3
\end{aligned} \tag{5.28}$$

In the first term, gluons 1, 3, and 4 are resolved, while gluon 2 is soft, so this will contribute only to δ-function terms. In the second term, the matrix element will not be singular enough to produce a pole unsuppressed by $s_{\min}$ [41]. In the third term, we must use the triple-collinear factorization.

Before turning to the factorization and the computation of resulting integrals, let us consider the last type of region (the second type in table 1), where three nearest-neighbor invariants are smaller than $s_{\min}$,

$$\begin{aligned}
&\Theta(s_{\min}-s_{12})\Theta(s_{\min}-s_{23})\Theta(s_{\min}-s_{34})\Theta(s_{14}-s_{\min})\\
+\ &\Theta(s_{\min}-s_{12})\Theta(s_{\min}-s_{23})\Theta(s_{34}-s_{\min})\Theta(s_{\min}-s_{14})\\
+\ &\Theta(s_{\min}-s_{12})\Theta(s_{23}-s_{\min})\Theta(s_{\min}-s_{34})\Theta(s_{\min}-s_{14})\\
+\ &\Theta(s_{12}-s_{\min})\Theta(s_{\min}-s_{23})\Theta(s_{\min}-s_{34})\Theta(s_{\min}-s_{14}).
\end{aligned} \tag{5.29}$$

To match the term we selected above, examine

$$\Theta(s_{34}-s_{\min})\Theta(s_{\min}-s_{14})\Theta(s_{\min}-s_{12})\Theta(s_{\min}-s_{23}). \tag{5.30}$$

Once again we must introduce additional Θ-functions to distinguish between configurations in which different factorization formulæ apply. In the case at hand, we must distinguish between

$$1,2 \text{ soft}; \quad 1 \text{ soft}, 2 \parallel 3; \quad \text{or } 2 \text{ soft}, 1 \parallel 4. \tag{5.31}$$

Multiplying eqn. (5.30) by

$$1 = (\Theta(s_{13}-s_{\min})+\Theta(s_{\min}-s_{13}))(\Theta(s_{\min}-s_{24})+\Theta(s_{24}-s_{\min})) \tag{5.32}$$

we obtain

$$\begin{aligned}
&\Theta(s_{34}-s_{\min})\Theta(s_{\min}-s_{23})\Theta(s_{\min}-s_{12})\Theta(s_{\min}-s_{14}) \\
&\quad\times\Big(\Theta(s_{\min}-s_{24})\Theta(s_{\min}-s_{13}) \qquad 1,\text{ 2 soft} \\
&\qquad +\Theta(s_{24}-s_{\min})\Theta(s_{\min}-s_{13}) \quad 1 \text{ soft, } 2 \parallel 3 \\
&\qquad +\Theta(s_{13}-s_{\min})\Theta(s_{\min}-s_{24}) \quad 2 \text{ soft, } 1 \parallel 4\Big)
\end{aligned} \tag{5.33}$$

where we have dropped the contribution containing $\Theta(s_{13}-s_{\min})\Theta(s_{24}-s_{\min})$ because it is kinematically forbidden. The first term in eqn. (5.33) contributes only to δ-function terms in the kernel, and we will not consider it further.

We will want to attach the first of the remaining two terms to the region defined by eqn. (5.28), while the last term we will associate with a the similar product of theta functions with $(1,2,3,4) \to (4,1,2,3)$. To organize this, define functions aggregating various theta functions above,

$$\begin{aligned}
\Theta_C(a,b,c;d) &= \Theta(s_{ad}-s_{\min})\Theta(s_{bd}-s_{\min})\Theta(s_{cd}-s_{\min})\Theta(s_{\min}-s_{abc}), \\
\Theta_W(a,b,c) &= \Theta(s_{\min}-s_{ab})\Theta(s_{\min}-s_{bc})\Theta(s_{\min}-s_{ac}), \\
\Theta_S(a;b,c;d) &= \Theta(s_{\min}-s_{ad})\Theta(s_{bd}-s_{\min})\Theta(s_{cd}-s_{\min}).
\end{aligned} \tag{5.34}$$

The complete surviving contribution of terms with two or three small invariants, other than the double-collinear already accounted for in eqn. (5.22), is then given by inserting

$$\begin{aligned}
\sum_{\rho=(1234),(2341),(3412),(4123)} &[\Theta_C(\rho_1,\rho_2,\rho_3;\rho_4) \\
&\quad +\Theta_S(\rho_1;\rho_2,\rho_3;\rho_4)\Theta_W(\rho_1,\rho_2,\rho_3) \\
&\quad +\Theta_S(\rho_3;\rho_2,\rho_1;\rho_4)\Theta_W(\rho_1,\rho_2,\rho_3)]
\end{aligned} \tag{5.35}$$

into the integrand of the last term in eqn. (5.1).

Each of the different permutations ρ corresponds roughly to a triple-collinear region, with added regions where one of the 'outer' gluons becomes soft. That is, they correspond to regions where a three-particle invariant becomes small. Let us focus on one of these contributions, say as above the region where $s_{123} \to 0$.

Using the factorizations [41, 42, 43]

$$|A^{(0)}(1,2,3,4)|^2 \xrightarrow{1\|2\|3} |\,\mathrm{Split}^{\mathrm{tree}}(1,2,3)|^2 \times |A^{(0)}(1+2+3,4)|^2 \tag{5.36}$$

and

$$|A^{(0)}(1,2,3,4)|^2 \xrightarrow{1\,\mathrm{soft},2\|3} \mathrm{M}(1\,\mathrm{soft},2 \parallel 3) \times |A^{(0)}(1+2+3,4)|^2 \tag{5.37}$$

the integrand in the region where $s_{123} \to 0$ is given by

$$\begin{aligned} I_{123} \;=\;& |A^{(0)}(k_1+k_2+k_3,4)|^2 \\ &\times\Big(\Theta_S(1;2,3;4)\Theta_W(1,2,3)\,\mathrm{M}(1\,\mathrm{soft},2 \parallel 3) \\ &\quad + \Theta_S(3;2,1;4)\Theta_W(1,2,3)\,\mathrm{M}(3\,\mathrm{soft},1 \parallel 2) \\ &\quad + \Theta_C(1,2,3;4)|\,\mathrm{Split}^{\mathrm{tree}}(1,2,3)|^2\Big) + \text{ non-singular.} \end{aligned} \tag{5.38}$$

As noted in ref. [41] the function describing the mixed 'soft-collinear' limit, $\mathrm{M}(a\,\mathrm{soft},b \parallel c)$, can be derived from the triple-collinear splitting function. Consequently, the difference of the two limiting functions – the soft-collinear one and the triple collinear one – gives only a finite contribution when integrated over a region where both are valid. In particular, as far as extracting poles is concerned, contributions of the form

$$\Theta_S(1;2,3;4)\,(\mathrm{M}(1\,\mathrm{soft},2 \parallel 3) - |\,\mathrm{Split}^{\mathrm{tree}}(1,2,3)|^2) \tag{5.39}$$

can be dropped. We can use this to combine different contributions so as to simplify the structure of the resulting integrals we must compute. In general, the fewer constraints (theta functions), the better.

Using

$$\begin{aligned} &\Theta(s_{34}-s_{\min})\Theta(s_{14}-s_{\min})\Theta(s_{24}-s_{\min}) = \\ &1-\Theta(s_{\min}-s_{14})\Theta(s_{24}-s_{\min})\Theta(s_{34}-s_{\min}) - \Theta(s_{14}-s_{\min})\Theta(s_{24}-s_{\min})\Theta(s_{\min}-s_{34}) \\ &-\Theta(s_{\min}-s_{14})\Theta(s_{24}-s_{\min})\Theta(s_{\min}-s_{34}) - \Theta(s_{\min}-s_{24}) \end{aligned} \tag{5.40}$$

along with eqn. (5.39), we obtain

$$\begin{aligned} I_{123} \;=\;& \Big(\Theta(s_{\min}-s_{123})|\,\mathrm{Split}^{\mathrm{tree}}(1,2,3)|^2 \\ &+ \Theta_S(1;2,3;4)(\Theta_W(1,2,3)-\Theta(s_{\min}-s_{123}))\,\mathrm{M}(1\,\mathrm{soft},2 \parallel 3) \\ &+ \Theta_S(3;2,1;4)(\Theta_W(1,2,3)-\Theta(s_{\min}-s_{123}))\,\mathrm{M}(3\,\mathrm{soft},1 \parallel 2) \\ &- \Theta(s_{\min}-s_{14})\Theta(s_{24}-s_{\min})\Theta(s_{\min}-s_{34})\Theta(s_{\min}-s_{123})|\,\mathrm{Split}^{\mathrm{tree}}(1,2,3)|^2 \\ &- \Theta(s_{\min}-s_{24})\Theta(s_{\min}-s_{123})||\,\mathrm{Split}^{\mathrm{tree}}(1,2,3)|^2\Big)|A^{(0)}(1+2+3,4)|^2. \end{aligned} \tag{5.41}$$

The theta functions in the penultimate term require both gluons 1 and 3 to be soft, and hence that term can contribute only to δ-function terms in the kernel. The theta function in the last term forces gluon 2 to be soft, and hence as discussed above, the term will not contribute any singular terms.

One may also be tempted to drop the second and third terms. This temptation should be resisted, because although the region is small, the soft-collinear factorization function is nonetheless sufficiently singular to produce a contribution, as shown by a careful analysis in a different context [64].

Let us first evaluate the primary contribution, given by the first term in eqn. (5.41). The other regions related by cyclic invariance (where respectively s_{234}, s_{134}, and s_{124} vanish) give equal contributions; adding all four, we obtain

$$24N^4 \int dR^d(1,2,3,4)\, |A^{(0)}((1+2+3),4)|^2 \sum_{i=1}^{3} \delta(x_g - x_i) |\,\mathrm{Split}^{\mathrm{tree}}(1,2,3)|^2 \Theta(s_{\min} - s_{123}). \quad (5.42)$$

Factorizing the phase-space measure in the triple collinear region [64],

$$dR^d(1,2,3,4) \xrightarrow{1\|2\|3} \frac{1}{12} dR^d(P,4) dR^d_{\mathrm{coll.}}(1,2,3), \quad (5.43)$$

with

$$\begin{aligned} dR^d_{\mathrm{coll.}}(1,2,3) \;&=\; \frac{1}{2^8\pi^5} \frac{1}{\Gamma(1-2\varepsilon)} (4\pi)^{2\varepsilon} (-\Delta)^{-\frac{1}{2}-\varepsilon} dz_1 dz_2 dz_3 ds_{123} ds_{12} ds_{13} ds_{23} \\ &\quad \delta(1 - z_1 - z_2 - z_3) \delta(s_{123} - s_{12} - s_{13} - s_{23}), \end{aligned} \quad (5.44)$$

$$\Delta = (z_3 s_{12} - z_1 s_{23} - z_2 s_{13})^2 - 4 z_1 z_2 s_{23} s_{13}, \quad (5.45)$$

and

$$k_i = z_i(k_1 + k_2 + k_3) = z_i P \quad (5.46)$$

the contribution of eqn. (5.42) takes the form,

$$\begin{aligned} & 2N^4 \int dR^d(P,4) dR^d_{\mathrm{coll.}}(1,2,3)\, |A^{(0)}((1+2+3),4)|^2 \\ & \quad \sum_{i=1}^{3} \delta(x_g - x_i) |\,\mathrm{Split}^{\mathrm{tree}}(1,2,3)|^2 \Theta(s_{\min} - s_{123}) \\ = \;& N^2 \int dz \int dR^d(1,2) |A^{(0)}(1,2)|^2 (\delta(x_g - z_1 z) + \delta(x_g - z_2 z)) \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{\varepsilon} K_r^{(1)}(z) \\ = \;& \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{\varepsilon} [h^{(0)} \otimes K_r^{(1)}](x_g) \end{aligned} \quad (5.47)$$

where

$$\left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{\varepsilon} K_r^{(1)}(z) = N^2 \int dR^d_{\mathrm{coll.}}(1,2,3) \sum_{i=1}^{3} \delta(z - z_i) |\,\mathrm{Split}^{\mathrm{tree}}(1,2,3)|^2 \Theta(s_{\min} - s_{123}). \quad (5.48)$$

Note that the factorization of the phase space given in eqn. (5.44) is exact up to an additional factor $(1 - \frac{s_{123}}{s_{1234}})^{d-3}$ if one defines the 'momentum fractions' outside the collinear region by

$$z_i = \frac{2k_i \cdot \tilde{k}_4}{2\tilde{k}_{ijk} \cdot \tilde{k}_4} = \frac{2k_i \cdot \tilde{k}_4}{s_{1234}}, \quad s_{ijkl} = (k_i + k_j + k_k + k_l)^2 \quad (5.49)$$

with $y = s_{123}/s_{1234}$ and

$$\tilde{k}_4 = \frac{1}{1-y} k_4, \quad \tilde{k}_{ijk} = k_i + k_j + k_k - y\tilde{k}_4. \quad (5.50)$$

Using eqn. (5.44) together with [41],

$$\sum_{\text{ph. pol.}} |\,\text{Split}^{\text{tree}}(1,2,3)|^2 = 2\times\Bigg\{$$
$$\begin{aligned}
&\frac{(1-\varepsilon)}{s_{12}^2 s_{123}^2}\frac{(z_2 s_{123}-(1-z_3)s_{23})^2}{(1-z_3)^2}+\frac{2(1-\varepsilon)s_{23}}{s_{12}s_{123}^2}+\frac{3(1-\varepsilon)}{2s_{123}^2}\\
+\quad&\frac{1}{s_{12}s_{123}}\left(\frac{(1-z_3(1-z_3))^2}{z_3 z_1(1-z_1)}-2\frac{z_2^2+z_2z_3+z_3^2}{1-z_3}+\frac{z_2z_1-z_2^2z_3-2}{z_3(1-z_3)}+2\varepsilon\frac{z_2}{1-z_3}\right)\\
+\quad&\frac{1}{2s_{12}s_{23}}\left(3z_2^2-2\frac{(2-z_1+z_1^2)(z_2^2+z_1(1-z_1))}{z_3(1-z_3)}+\frac{1}{z_3z_1}+\frac{1}{(1-z_3)(1-z_1)}\right)\Bigg\}\\
+\quad&(s_{12}\leftrightarrow s_{23}, z_1\leftrightarrow z_3),
\end{aligned} \tag{5.51}$$

(there is a factor of $1/4$ included here compared to ref. [41] to account for our normalization of the color matrices) we obtain

$$\begin{aligned}
K_r^{(1)}(z) \;=\; & N^2\frac{1}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{s_{\min}}\right)^{2\varepsilon}\Bigg\{-\frac{5}{\varepsilon^2}p(z)+\frac{1}{\varepsilon}\Bigg(10p(z)\ln(1-z)-4(1+z-p(z))\ln(z)\\
&-\frac{1}{6}\frac{(-102z^3+55z^4+105z^2-102z+55)}{z(1-z)}\Bigg)-\frac{1}{2}(p(z)-12(1+z))\ln(z)^2\\
&-p(-z)S_2(z)-8(1+z)\,\text{Li}_2(z)-8p(z)\ln(z)\ln(1-z)\\
&-10p(z)\ln(1-z)^2+\frac{1}{6}\frac{73z^2+z+88}{z}\ln(z)\\
&+\frac{1}{3}\frac{-102z^3+55z^4+105z^2-102z+55}{z(1-z)}\ln(1-z)-\frac{67}{9}p(z)\\
&-\frac{1}{36}\frac{-330z+95+101z^2}{(1-z)}+\pi^2\left(p(z)+\frac{4}{3}(1+z)\right)\Bigg\},
\end{aligned} \tag{5.52}$$

with $p(z)$ defined in eqn. (4.14), and where [19]

$$S_2(z) = \int_{\frac{z}{1+z}}^{\frac{1}{1+z}}\frac{dw}{w}\ln\left[\frac{1-w}{w}\right] = -2\,\text{Li}_2(-z)-2\ln z\ln(1+z)+\frac{1}{2}\ln^2 z-\frac{\pi^2}{6}. \tag{5.53}$$

In the derivation of eqn. (5.52) we have used the integrals given in the appendix.

Next, we compute the two additional contributions from the second and third terms in eqn. (5.41). The limiting function which we need to integrate is given by [41]

$$\text{M}(i\,\text{soft}, j\parallel k) = 2\frac{(1-z_j+z_j^2)^2}{(1-z_j)z_js_{jk}}\frac{(z_js_{jk}+z_js_{ijk}+s_{ij})}{s_{ij}s_{ijk}}\frac{s_{jl}+s_{kl}}{s_{il}}, \tag{5.54}$$

with l being the momenta of the adjacent color connected hard parton in the antenna containing the soft gluon. (The normalization is again different from [41] on account of different normalization conventions for the color matrices.) The term

$$\begin{aligned}
&\Theta_S(1;2,3;4)(\Theta_W(1,2,3)-\Theta(s_{\min}-s_{123}))\,\text{M}(1\,\text{soft},2\parallel 3)\\
&=\Theta(s_{34}-s_{\min})\Theta(s_{24}-s_{\min})\Theta(s_{\min}-s_{14})(\Theta_W(1,2,3)-\Theta(s_{\min}-s_{123}))\,\text{M}(1\,\text{soft},2\parallel 3)
\end{aligned} \tag{5.55}$$

in eqn. (5.41) yields the following contribution,

$$
\begin{aligned}
& 6g_s^4 N^4 \int dR^d(1,2,3,4)|A^{(0)}(1,2,3,4)|^2 \sum_{i=1}^{4} \delta(x_g - x_i) \\
& \Theta(s_{34}-s_{\rm min})\Theta(s_{24}-s_{\rm min})\Theta(s_{\rm min}-s_{14})(\Theta_b(1,2,3)-\Theta(s_{\rm min}-s_{123}))\,\mathrm{M}(1\,\mathrm{soft},2\parallel 3) \\
& = \frac{1}{4}\left(\frac{\alpha_s}{2\pi}\right)^2 N^2 \int dz \int dR^d(P,4)\;|A^{(0)}(P,4)|^2 \delta(x_g - z x_P) K_\delta(z),
\end{aligned} \tag{5.56}
$$

where

$$
\begin{aligned}
K_\delta(z) = {} & 128\pi^4 N^2 \int dR^d_{\rm coll.}(1,2,3)(\delta(z-z_2)+\delta(z-(1-z_2)))\mathrm{M}(1\,\mathrm{soft},2\parallel 3) \\
& \times \Theta(s_{34}-s_{\rm min})\Theta(s_{24}-s_{\rm min})\Theta(s_{\rm min}-s_{14})(\Theta_W(1,2,3)-\Theta(s_{\rm min}-s_{123})).
\end{aligned} \tag{5.57}
$$

The easiest way to obtain $K_\delta(z)$ is to calculate the contributions from $\Theta_W(1,2,3)$ and $\Theta(s_{\rm min}-s_{123})$ separately and then take the difference. The integration over the region given by $\Theta(s_{\rm min}-s_{123})$ is similar to that which yields eqn. (5.48). For the integration over the region given by $\Theta_W(1,2,3)$ we can use the result given in [64]. Subtracting the two contributions the final result reads:

$$
\begin{aligned}
K_\delta(z) = \quad & -2N^2\left(\frac{4\pi\mu^2}{s_{\rm min}}\right)^{2\varepsilon}\left(\frac{s_{P4}}{s_{\rm min}}\right)^{\varepsilon}\frac{1}{\Gamma(1-2\varepsilon)}p(z) \\
& \times\frac{1}{\varepsilon^3}\left(\frac{\Gamma(1-2\varepsilon)}{\Gamma(1-\varepsilon)^2}(1-z)^{-\varepsilon}+\frac{\Gamma(1-2\varepsilon)}{\Gamma(1-\varepsilon)^2}(z)^{-\varepsilon}-2z^{-\varepsilon}(1-z)^{-\varepsilon}\right).
\end{aligned} \tag{5.58}
$$

One can also compute the integral directly, as a check, and we obtain the same result. Including all other terms with the same structure ($3\,\mathrm{soft},1\parallel 2$ and cyclic permutations), we finally obtain

$$
\left(\frac{\alpha_s^2}{2\pi}\right)^2 N^2 \int dz \int dR^d(P,4)\;|A^{(0)}(1,2)|^2[\delta(x_g - z x_1)+\delta(x_g - z x_2)]K_\delta(z). \tag{5.59}
$$

Combining the separate contributions from eqn. (5.21), eqn. (5.23), eqn. (5.47), and eqn. (5.59) our final result for the singular part of the four parton final state is,

$$
\left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{1}{\varepsilon}h_r^{(1)}\otimes K^{(0)}+\frac{1}{\varepsilon}h^{(0)}\otimes\left(K_r^{(1)}-\frac{1}{\varepsilon}K^{(0)}\otimes K^{(0)}\right)+\frac{1}{\varepsilon}\mathcal{V}(z)\right) \tag{5.60}
$$

in which,

$$
\begin{aligned}
& -2(\mathcal{S}^{(0)}(1,2)+\mathcal{C}^{(0)}(1,2,1))\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}K^{(0)}(z)+\mathcal{C}^{(0)}(z1,2,(1-z)1)\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}K^{(0)}(z)+\left(\frac{\alpha_s^2}{2\pi}\right)^2 K_\delta(z) \\
= \quad & -\left(\frac{\alpha_s}{2\pi}\right)^2 N^2\frac{1}{\Gamma(1-\varepsilon)^2}\left(\frac{4\pi\mu^2}{s_{\rm min}}\right)^{2\varepsilon}\frac{1}{\varepsilon^3}2(z(1-z))^{-\varepsilon}p(z)\left(-2-\frac{11}{6}\varepsilon+(\frac{2}{3}\pi^2-\frac{67}{18})\varepsilon^2+O(\varepsilon^3)\right) \\
\equiv \quad & \left(\frac{\alpha_s}{2\pi}\right)^2\frac{1}{\varepsilon^2}\mathcal{V}(z).
\end{aligned} \tag{5.61}
$$

5.3. Assembling the Kernel

Upon combining the results for the singular contribution of three- and four-parton final states, eqn. (5.7) and eqn. (5.60) respectively, we obtain

$$
\frac{1}{\varepsilon}(h_v^{(1)}+h_r^{(1)})\otimes K^{(0)}+h^{(0)}\otimes\left(\frac{1}{\varepsilon}K_r^{(1)}-\frac{1}{\varepsilon}K^{(0)}\otimes\frac{1}{\varepsilon}K^{(0)}+\frac{1}{\varepsilon^2}\mathcal{V}+\frac{1}{\varepsilon}K_v^{(1)}\right). \tag{5.62}
$$

Note that the equation above contains only $1/\varepsilon^2$ and $1/\varepsilon$ singularities, the $1/\varepsilon^3$ poles present in individual terms cancel in the sum. In addition, the singularity is independent of $s_{\min}$ as it ought to be. Inserting this result in eqn. (3.15) we obtain

$$\begin{aligned}
\frac{d\Gamma^R}{dx_g} &= \frac{1}{\varepsilon}h^{(1)}\otimes K^{(0)} + h^{(0)}\otimes\left(\frac{1}{\varepsilon}K_r^{(1)} - \frac{1}{\varepsilon}K^{(0)}\otimes\frac{1}{\varepsilon}K^{(0)} + \frac{1}{\varepsilon^2}\mathcal{V} + \frac{1}{\varepsilon}K_v^{(1)}\right) \\
&+ h^{(1)}\otimes\frac{1}{\varepsilon}P^{(0)} + h^{(0)}\otimes\left(\frac{1}{2\varepsilon^2}S_\varepsilon{}^2P^{(0)}\otimes P^{(0)} - \frac{1}{4\varepsilon^2}S_\varepsilon{}^2\beta_0P^{(0)} + \frac{1}{2}\frac{1}{\varepsilon}S_\varepsilon{}^2P^{(1)}\right) + O(\varepsilon^0) \\
&= h^{(0)}\otimes\left(\frac{1}{\varepsilon}K_r^{(1)} + \frac{1}{\varepsilon}K_v^{(1)} - \frac{1}{2}\frac{1}{\varepsilon}K^{(0)}\otimes\frac{1}{\varepsilon}K^{(0)} + \frac{1}{\varepsilon^2}\mathcal{V} - \frac{1}{4\varepsilon^2}S_\varepsilon{}^2\beta_0P^{(0)} + \frac{1}{2}\frac{1}{\varepsilon}S_\varepsilon{}^2P^{(1)}\right) \\
&+ h^{(1)}\otimes\frac{1}{\varepsilon}(S_\varepsilon P^{(0)} + K^{(0)}) \\
&+ h^{(0)}\otimes\left(\frac{1}{2\varepsilon^2}S_\varepsilon{}^2P^{(0)}\otimes P^{(0)} - \frac{1}{2\varepsilon^2}K^{(0)}\otimes K^{(0)}\right) + O(\varepsilon^0). \qquad (5.63)
\end{aligned}$$

Thanks to the leading order result in eqn. (4.32), we see that

$$h^{R,(1)} = h^{(1)} + \frac{1}{\varepsilon}h^{(0)}\otimes S_\varepsilon P^{(0)} \qquad (5.64)$$

is finite, so that we may write,

$$\begin{aligned}
\frac{d\Gamma^R}{dx_g} &= h^{(0)}\otimes\left(\frac{1}{\varepsilon}K_r^{(1)} + \frac{1}{\varepsilon}K_v^{(1)} - \frac{1}{2\varepsilon^2}K^{(0)}\otimes K^{(0)} + \frac{1}{\varepsilon^2}\mathcal{V} - \frac{1}{4\varepsilon^2}S_\varepsilon{}^2\beta_0P^{(0)} + \frac{1}{2}\frac{1}{\varepsilon}S_\varepsilon P^{(1)}\right) \\
&+ h^{R,(1)}\otimes\frac{1}{\varepsilon}(S_\varepsilon P^{(0)} + K^{(0)}) \\
&- h^{(0)}\otimes\frac{1}{2\varepsilon^2}\left(S_\varepsilon{}^2P^{(0)}\otimes P^{(0)} + 2S_\varepsilon P^{(0)}\otimes K^{(0)} + K^{(0)}\otimes K^{(0)}\right) + O(\varepsilon^0). \qquad (5.65)
\end{aligned}$$

The term involving $h^{R,(1)}$ is finite as it should be. The last term can be written in a more suggestive form,

$$-h^{(0)}\otimes\frac{1}{2\varepsilon^2}(S_\varepsilon P^{(0)} + K^{(0)})\otimes(S_\varepsilon P^{(0)} + K^{(0)}). \qquad (5.66)$$

The leading-order result implies that

$$S_\varepsilon P^{(0)} + K^{(0)} \qquad (5.67)$$

is of order ε, and the convolution will not produce additional singularities. This has been checked via an explicit calculation. We may therefore drop the term in eqn. (5.66). The requirement that the remaining, first, term on the right-hand side of eqn. (5.65) be finite then allows us to extract the next-to-leading order kernel $P^{(1)}$,

$$\frac{1}{2}S_\varepsilon{}^2P^{(1)} = \left(-K_r^{(1)} - K_v^{(1)} + \frac{1}{2\varepsilon}K^{(0)}\otimes K^{(0)} - \frac{1}{\varepsilon}\mathcal{V}(z) + \frac{1}{4\varepsilon}S_\varepsilon{}^2\beta_0P^{(0)}\right). \qquad (5.68)$$

Intuitively, the first two terms are in some sense the radiative corrections to the leading-order splitting amplitude, while the following two terms,

$$\frac{1}{2\varepsilon}K^{(0)}\otimes K^{(0)} - \frac{1}{\varepsilon}\mathcal{V}(z) \qquad (5.69)$$

just remove the iteration of the leading-order kernel. The last term can be thought of as an ultraviolet renormalization. Plugging in the explicit results for $K_r^{(1)}$, $K_v^{(1)}$, $\mathcal{V}$ [eqn. (5.15), eqn. (5.17), eqn. (5.52), eqn. (5.61)], together with

$$\begin{aligned}
[K^{(0)}\otimes K^{(0)}](z) &= (16\pi^2\mathcal{N}_c N)^2\Bigg(\delta(1-z)\mathcal{N}^2+\frac{22}{3}p(z)+12-\frac{44}{3}\frac{1}{z}-12z+\frac{44}{3}z^2 \\
&\quad -8\ln(z)p(z)+4\frac{(1-4z+3z^2+z^4)}{z(1-z)}\ln(z)+8\ln(1-z)p(z) \\
&+\ \varepsilon\Bigg[-\frac{2}{9}\frac{(1-z)}{z}(67-2z+67z^2)+\frac{4}{3}\pi^2\frac{1+3z^2-4z^3+z^4}{z(1-z)}-16(1+z)\,\mathrm{Li}_2(z) \\
&\quad +\frac{8}{3}\frac{(1-z)}{z}(11+2z+11z^2)\ln(1-z)-12p(z)\ln(1-z)^2 \\
&\quad +\frac{4}{3}\frac{1}{z}(11+3z+12z^2)\ln(z)+8(1+z)\ln(z)^2 \\
&\quad +2(2\ln(z)^2-\frac{11}{3}\ln(z)-\frac{11}{3}\ln(1-z)-\frac{2}{3}\pi^2+\frac{67}{9})p(z)\Bigg]\Bigg)+O(\varepsilon^2), \qquad (5.70)
\end{aligned}$$

we finally obtain,

$$\begin{aligned}
P^{(1)}(z) &= N^2\Bigg(\frac{27}{2}(1-z)+\frac{67}{9}(z^2-\frac{1}{z})+(\frac{11}{3}-\frac{25}{3}z-\frac{44}{3}\frac{1}{z})\ln(z)-4(1+z)\ln(z)^2 \\
&+\ (4\ln(z)\ln(1-z)-3\ln(z)^2+\frac{22}{3}\ln(z)-\frac{1}{3}\pi^2+\frac{67}{9})p(z)+2p(-z)S_2(z)\Bigg) \qquad (5.71)
\end{aligned}$$

in agreement with known results for the time-like kernel [10, 12, 13].

6. Conclusion

Intuitively, the Altarelli–Parisi kernel summarizes that part of collinearly unresolved radiation from a short-distance process which must be absorbed into the scaling evolution of descriptions of initial- or final-state hadrons. The computation described above makes this direct connection precise, and shows how to use it in order to compute the kernel. The approach described in the present paper effectively breaks down the NLO computation into smaller and simpler parts, whose intermediate terms have an independent meaning and are subject to consistency checks on their own.

As described in section 5.3, the approach effectively computes the the next-to-leading order kernel as a radiative correction to the leading-order kernel (after cancellation of soft singularities), less an iteration of the leading-order kernel. As usual, the radiative corrections arise from a virtual correction and a real-emission correction. Both are computed from higher-order splitting amplitudes, quantities which govern the collinear behavior of gauge-theory amplitudes. These intermediate quantities are useful elsewhere in their own right, and can be checked independently. For example, the virtual corrections are basically given by the one-loop splitting amplitude, which should satisfy certain supersymmetry identities [48]. The real-emission correction arises from integrating the splitting amplitude describing the behavior of tree-level amplitudes in a gauge theory as three color-connected partons become collinear.

The complete computation of the NNLO corrections to the Altarelli–Parisi kernel remains an important goal for particle theorists. These corrections, and parton distribution functions relying on them, are needed for programs evaluating jet production to NNLO at hadron colliders. In the approach described in this paper, the required ingredients would be the two-loop $1 \to 2$ splitting amplitudes; the one-loop $1 \to 3$ splitting amplitudes; and the $1 \to 4$ splitting amplitudes. The second ingredient has been calculated by Catani and de Florian [65] and the third by Del Duca, Frizzo and Maltoni [66].

Acknowledgments: One of us (P.U.) would like to thank Arnd Brandenburg, Aude Gehrmann-de Ridder, Gudrun Heinrich, Sven Moch, and Stefan Weinzierl for useful discussions. We would also like to thank Zvi Bern, Sven Moch, and Stefan Weinzierl for a careful reading of the manuscript.

A. Double unresolved phase space integration

The calculation of contributions from the four-parton final state in particular eqn. (5.48) leads to the consideration of integrals of the following form

$$C_{n,m} = s_{ijk}^{-(2-n-m)} \int\limits_{-\Delta>0} ds_{ij} ds_{jk} \frac{1}{s_{ij}^n s_{jk}^m} (-\Delta)^{-\frac{1}{2}-\varepsilon} \tag{A.1}$$

with

$$\Delta = (s_{ik}(1-z_i-z_k) - z_i s_{jk} - s_{ij} z_k)^2 - 4 z_k z_i s_{jk} s_{ij}. \tag{A.2}$$

The integration can be done in $d = 4-2\varepsilon$ dimensions yielding

$$\begin{aligned}
C_{1,1} &= -2\pi \frac{1}{\varepsilon} s_{ijk}^{-1-2\varepsilon} z_i^{-\varepsilon} z_k^{-\varepsilon} z_j^{-1-2\varepsilon} (1-z_i)^{\varepsilon} (1-z_k)^{\varepsilon} {}_2F_1(-\varepsilon, -\varepsilon, 1-\varepsilon, \frac{z_i}{1-z_i} \frac{z_k}{1-z_k}), \\
C_{1,0} &= -\pi \frac{1}{\varepsilon} s_{ijk}^{-1-2\varepsilon} z_i^{-\varepsilon} z_k^{-\varepsilon} z_j^{-\varepsilon} \frac{1}{1-z_k}, \\
C_{1,-1} &= -\pi \frac{1}{\varepsilon} s_{ijk}^{-1-2\varepsilon} \frac{1}{1-2\varepsilon} z_i^{-\varepsilon} z_k^{-\varepsilon} z_j^{1-\varepsilon} \frac{1}{(1-z_k)^2} (1-\varepsilon-\varepsilon \frac{z_i z_k}{z_j}), \\
C_{0,1} &= -\pi \frac{1}{\varepsilon} s_{ijk}^{-1-2\varepsilon} z_i^{-\varepsilon} z_j^{-\varepsilon} z_k^{-\varepsilon} (1-z_i)^{-1}, \\
C_{0,0} &= \pi \frac{1}{\varepsilon} s_{ijk}^{-1-2\varepsilon} \frac{\varepsilon}{1-2\varepsilon} z_i^{-\varepsilon} z_j^{-\varepsilon} z_k^{-\varepsilon},
\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}
& z_b^2 C_{2,0} - 2 z_b (1-z_c) C_{2,-1} + (1-z_c)^2 C_{2,-2} \\
&= \pi \frac{1}{\varepsilon} \frac{1}{1-2\varepsilon} s_{abc}^{-1-2\varepsilon} z_a^{-\varepsilon} z_b^{-\varepsilon} z_c^{-\varepsilon} (1-z_a)^2 \left(\varepsilon - \frac{z_b(2+4\varepsilon)}{z_b + z_a z_c} + \frac{z_b^2 (2+4\varepsilon)}{(z_b + z_a z_c)^2} \right).
\end{aligned} \tag{A.4}$$

Note that the $C_{2,m}$ integrals can not appear in arbitrary combinations in a gauge theory amplitude. The individual integrals appearing in eqn. (A.4) are not regularized in dimensional regularization, in contrast the specific combination is regularized. The combination given above is exactly the combination appearing in eqn. (5.48).

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