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The criterion of irreducibility of multi–loop Feynman integrals.

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The integration by parts recurrence relations allow to reduce some Feynman integrals to more simple ones (with some lines missing). Nevertheless the possibility of such reduction for the given particular integral was unclear. The recently proposed technique for studying the recurrence relations as by-product provides with simple criterion of the irreducibility.

1 Introduction.

The growing accuracy of the experimental tests of the Standard Model of particle physics makes neseccary calculation of higher order quantum corrections. The latter, in turn, are expressed through so called multi-loop Feynman integrals.

For example, the calculation of the $O(\alpha_s^2)$ correction to $R(s)$ (cross-section $e^+e^- \to (hadrons)$) demands calculation of the massive 3–loop propagator diagrams. At present such diagrams cannot be evaluated explicitly, but it is possible to construct the reliable approximation, considering their expansions in various kinematical regions [1]. After some algebraic manipulations the coefficient in these expansions can be related to a huge number (of oder of millions) of 3-loop propagator massless and vacuum massive diagrams, which differ from each other by degrees of their denominators.

Fortunately, one need not calculate all these integrals separately, because it is possible (using the integration by part algorithm [2]) find algebraical relations between these integrals. Indeed, suppose we need to evaluate the L-loop

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Feynman integral in the dimensional regularization:

$$B(\underline{n}) \equiv \int \cdots \int \frac{d^D p_1 \dots d^D p_L}{D_1^{n_1} \dots D_N^{n_N}}, \tag{1}$$

where p_i $(i = 1, \dots, L)$ are loop momenta and $D_a = A_a^{ij} p_i \cdot p_j - m_a^2$ (a=1,...,N). According to general rules of d-dimensional integration we can integrate by part in this integral or, in other words, the integral of total derivative according to loop momenta should vanish. From other side we can evaluate this derivative as linear combination of the $B(\underline{n})$ with n_i shifted by ± 1:

$$0 = \int \cdots \int (\partial / \partial p_i) \cdot p_k \frac{d^D p_1 \dots d^D p_L}{D_1^{n_1} \dots D_N^{n_N}} = R(I^+, I^-) B(\underline{n}), \tag{2}$$

where $\mathbf{I}_c^- B(.., n_c, ..) \equiv B(.., n_c - 1, ..)$ and $\mathbf{I}_c^+ B(.., n_c, ..) \equiv n_c B(.., n_c + 1, ..)$.

One can use these relations to relate an integral with some values of n_i to more simple integrals. In particular, the millions of integrals appeared in the $O(\alpha_s^2)$ calculations mentioned above can be related with very few integrals (6 massless propagator and 3 massive vacuum integrals), which should be calculated explicitly.

The most important step in this method is to find the proper combination of the relations of (2) type, which allow to reduce the given integral to the desirable set of integrals. In other words, one should represent the relations (2) in recursive form, which allows to reduce each index n_i to basic value (usually 0 or 1). This problem was solved for some important cases [3], but there is no systematic way for the general case. It is even unknown, is it possible to reduce the given integral to more simple integrals.

The answer to the last question is the main subject of the present paper. In the next section the general form of the irreducibility criterion is suggested. Then its application is illustrated by a 3-loop example. In the last section some general remarks are presented.

2 Irreducibility criterion.

Suppose we want to check the possibility to represent, using the relations (2), the given integral $B(\underline{n}_0)$ as linear combination of integrals from the set $\{B(\underline{n}_i), i \in (1, ..., k)\}$:

$$0 = B(\underline{n}_0) - \sum_{i=1}^{k} f_i B(\underline{n}_i). \tag{3}$$

The relations (3) (if exist) should be the sequence of the relations (2) (in fact, their linear combinations).

Let us assume that we found the other solution of (2) in form of some function $s(\underline{n})$. Any such function should also fit the (3) as the sequence of the relations (2). Suppose we are able to construct some specific $s(\underline{n})$ with the properties

$$s(\underline{n}_0) \neq 0, \quad s(\underline{n}_i) = 0 \text{ for } i \in (1, ..., k). \tag{4}$$

Such $s(\underline{n})$ evidently cannot fit (3) and hence (3) cannot be the sequence of (2). So we got the sufficient criterion of irreducibility: the $B(\underline{n}_0)$ cannot be represented as linear combination of integrals from the set $\{B(\underline{n}_i),\ i \in (1, ..., k)\}$ if exist the partial solution of (2) with properties (4).

In practice, one can construct such solutions using the method of finding the explicit solutions of the recurrence relations for Feynman integrals, proposed in [4]. The key idea is to represent these solution in the form of auxlary integral

$$s(\underline{n}) = \int \frac{dx_1...dx_N}{x_1^{n_1}...x_N^{n_N}} g(x_i). \tag{5}$$

One can check that action of the $R(I^+, I^-)$ on (5) leads to

$$R(I^-, I^+)s(\underline{n}) = \int \frac{dx_1...dx_N}{x_1^{n_1}...x_N^{n_N}} R(x_i, \partial/\partial x_i)g(x_i) + (\text{surface terms}). \tag{6}$$

So, if one choose the integrand as solution of the

$$R(\partial_i, x_i)g(x_i) = 0 \tag{7}$$

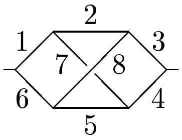
and cancel the surface terms by proper choose of integration contour, one can fit $R(I^+, I^-)s(\underline{n}) = 0$.

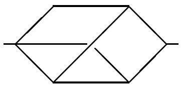
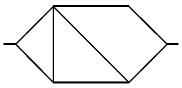
As it shown in [4], the (7) can be solved for the general case of multi-loop Feynman integral with arbitrary number of legs and with arbitrary masses, and the corresponding $g(x_i)$ can be represented as product of two polinoms in x_i, each polinom raised to non-integer degree (see [4] for details).

3 Example.

As example, let us consider the master 3-loop massless non-planar integral. This integral is supposed to be irreducible because of practical failure to simplify it. To the best of author's knowledge, no proof has been found.

So, let us check the possibility to reduce this integral to the linear combination of the simpler integrals (with at least one line missing):



$= c_1(d)$  $+ c_2(d)$  $+ \ldots$

where numbers numerate the lines. Let us define denominators as

$$D_1 = (l+p+q)^2,\ D_2 = (l+k+p+q)^2,\ D_3 = (k+p+q)^2,\ D_4 = (k+p)^2,$$

$$D_5 = p^2,\ D_6 = (l+p)^2,\ D_7 = k^2,\ D_8 = l^2,\ D_9 = 2\ l \cdot k.$$

The scalar products can be expressed through denominators in the following way:

$$k^2 = D_7,\ k \cdot l = \frac{1}{2} D_9,\ k \cdot p = \frac{1}{2}(D_4 - D_5 - D_7),\ l^2 = D_8,\ p^2 = D_5,$$

$$k \cdot q = \frac{1}{2}(D_2 - D_1 - D_4 + D_5 - D_9),\ l \cdot q = \frac{1}{2}(D_2 - D_3 + D_5 - D_6 - D_9),$$

$$l \cdot p = \frac{1}{2}(D_6 - D_5 - D_8),\ p \cdot q = \frac{1}{2}(D_1 - D_2 + D_3 - D_5 + D_9 - q^2). \quad (8)$$

According to [4], for this particular massless propagator-type case the function $g(\underline{x})$ will read

$$g(x_i) = (q^2)^{1-d/2} \det_{kl} \Big((p_k \cdot p_l)(x_i)\Big)^{d/2-5/2}, \quad (9)$$

where $p_k = (p, k, l, q)$ and $(p_k \cdot p_l)(x_i)$ means (8) with substitution $D_i = x_i$.

Let us construct $s(\underline{n})$ with property $s(\underline{n}) = 0$ if some of $(n_1, ..., n_8)$ less or equal to 0. The natural way to fit this condition is to chose the integration contours for $(x_1, ..., x_8)$ as small circles around zero. In this case, according to Cauchi theorem, the integrations will lead to calculation of the $(n_i - 1)$ coefficient in the Taylor expansion of the integrand (we omit for simplicity the overall factor):

$$s(\underline{n}) \propto \int \frac{dx_9}{x_9^{n_9}} \left[\frac{\partial_1^{n_1-1} \ldots \partial_8^{n_8-1}}{(n_1-1)! \ldots (n_8-1)!} P(x_i)^{d/2-5/2} \right] \Bigg|_{x_1,\ldots,x_8=0}. \quad (10)$$

Note that according to (8,9) $P(x_i)|_{x_1,\ldots,x_8=0} \propto x_9^2(q^2 - x_9)^2$. It means that remaining x_9 integral can be expressed through Pochhammer's symbols $(a)_n \equiv$

$\Gamma(a+n)/\Gamma(a)$:

$$\int dx_9 x_9^k (x_9^2(q^2-x_9)^2)^{d/2-5/2} \propto (d-4)_k/(2d-8)_k. \qquad (11)$$

Formulas (10,11) define $s(\underline{n})$ with desired properties (in particular, $s(1,..,1,0) = 1$). It means, according to general statement, that integration by part relations can not reduce master 3-loop massless non-planar integral $B(1,..,1,0)$ to linear combination of more simple integrals.

4 Final remarks and Conclusion.

In this paper the sufficient criterion of the irreducibility was suggested: the given integral is irreducible if the specific solution of the recurrence relations (with properties (4)) exists. Formally one can prove the reverse statement with the following restriction: suppose the given integral $B(\underline{n}_0)$ is irreducible and we are able to construct the procedure which reduce any integral to the given integral and some others:

$$B(\underline{n}) = c_0(\underline{n})B(\underline{n}_0) + c_1(\underline{n})B(\underline{n}_1) + .. \qquad (12)$$

Then the coefficient $c_0(\underline{n})$ near the given integral will serve as the separating function for this integral. Proof: acting by recurrence relations on both sides of (12) we got zero on the left side, and linear combination of basic integrals on the right side. The coefficient near the given integral $R_{ik}c_0(\underline{n})$ should be zero because otherwise this integral will be linear combination of others. It means that $c(\underline{n})$ is the solution of (2). Its "separating" initial conditions are fitted by construction.

So formally the existence of a separating solution is equivalent to irreducibility. But from practical point of view to prove irreducibility one should explicitly construct this solution. Our experience (up to 4-loop) shows that representation (5) provides with such solutions, although at 4-loop level it demands some efforts to find the proper integration contours.

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