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# Two-loop QCD corrections to the heavy quark pair production cross section in $e^+e^-$ annihilation near the threshold

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## Abstract

We present the $\mathcal{O}(\alpha_s^2)$ corrections to the cross section for the reaction $e^+e^- \to \gamma^* \to Q\bar{Q}$ in the energy region close to the threshold. We assume that the energy of the reaction is such that both the perturbative expansion in the strong coupling constant $\alpha_s$ and expansion in the velocity $\beta$ of the heavy quarks can be used. We explicitly obtain terms $\mathcal{O}(\alpha_s^2/\beta^2, \alpha_s^2/\beta, \alpha_s^2)$ in the relative correction to the threshold cross section. Using the ideas of asymptotic expansions, we demonstrate how an expansion of Feynman diagrams in the threshold region is constructed. From this analysis we obtain a matching relation between the vector current in full QCD and the quark–antiquark current in NRQCD at leading order in $1/m$ and the second order in the strong coupling constant.

Theoretical predictions for the cross section of the reaction $e^+e^- \to Q\bar{Q}$ in the energy region close to the $Q\bar{Q}$ threshold are of considerable interest for various phenomena. They are important for determinations of the $b$ and $c$ quark masses, as well as the value of the QCD coupling constant $\alpha_s(\mu)$, with $\mu \sim 1-2$ GeV, if one uses the sum rule approach for $\Upsilon$ and $J/\psi$ hadrons proposed in [1]; achievement of the $\mathcal{O}(\alpha_s^2)$ accuracy is considered very important for these quantities. One should also mention the ongoing efforts to determine the decay rates of the heavy quarkonia to leptons with the $\mathcal{O}(\alpha_s^2)$ accuracy [2]. Also, for the future $e^+e^-$ or $\mu^+\mu^-$ colliders one is considering precision measurements of the top quark properties by studying its threshold production region. It is well known that for these and other applications, where the threshold region is of interest, the fixed order perturbative calculations break down and a resummation of the terms singular at threshold is mandatory. In the leading order, such resummation yields the well known Sommerfeld–Sakharov factor for the threshold production cross section. Aiming at the $\mathcal{O}(\alpha_s^2)$ accuracy for the threshold cross section, the first thing to be calculated is the perturbative expression for the cross section at the order $\mathcal{O}(\alpha_s^2)$ in the energy region where $\alpha_s \ll \beta \ll 1$, where $\beta$ denotes the quark velocity, $\beta = \sqrt{1-4m^2/s}$, and $s$ is the total energy squared [3,4].

From the technical point of view, the problem of perturbative calculations in the threshold region has never received much attention. On the one hand, it is clear that a small parameter in which a useful expansion could in principle be constructed — $\beta$ — is there; on the other hand, it has not been quite clear how to use this parameter systematically in order to obtain a significant simplification of the Feynman integrals which should be calculated. On the conceptual level, the recognition of the existence of small and large scales in the threshold problems was formulated as the Non–Relativistic QCD (NRQCD), an effective field theory which should be used to describe physics in the threshold region [2]. In this framework the corrections which originate from the scales $k \sim m$ can be incorporated through the so–called matching procedure. This requires calculations in the full QCD, in order to provide the matching conditions. It is expected that since the infrared behaviors of both QCD and NRQCD are the same, the matching calculations could be considerably simplified. To the best of our knowledge, however, it has not yet been demonstrated, how the matching calculations in QCD/NRQCD should be organized in order to achieve such simplification in practice.

For practical reasons it is useful to be able to use the smallness of the relative velocity of the quarks on the level of individual Feynman diagrams, i.e. to formulate a prescription which operates with diagrams and subgraphs, rather than with composite operators and effective field theories. The advantages of such approach are its transparency and better control over the calculation. Indeed, similar approaches in other kinematic situations have recently permitted to complete many previously impossible calculations, both in QCD and in the electroweak theory (see for example [5–7]).

In this paper we consider a variant of the asymptotic expansions which can be used in the threshold region. It allows to construct an expansion in $\beta$ of a given Feynman diagram. For the resulting Feynman integrals one can construct algorithms which permit their calculation in any order in $\beta$, and can be encoded in a symbolic manipulation language.

An approach of this type has recently been discussed in [8]. Although its correctness has not been proven, its construction is analogous to the well established asymptotic expansions [9–11].

Using this approach we calculate the $\mathcal{O}(\alpha_s^2)$ correction to the cross section of the reaction $e^+e^- \to Q\bar{Q}$, up to terms of order $\mathcal{O}(\beta^0)$ relative to the Born cross section. In the result the terms with a color factor $C_F^2$, as well as the corrections induced by vacuum polarization insertions due to heavy and light fermions are identical to the previously obtained results in the Abelian gauge theory [4,12]. This provides a non-trivial test of the asymptotic expansion method applied in the threshold region. The non-abelian terms presented below are a new result.

We begin with introducing some notations. The cross section of the reaction $e^+e^- \to \gamma^* \to \bar{Q}Q$ is written as

$$\sigma_{e^+e^-\to\bar{Q}Q} = \sigma^{(0)}\left[1 + C_F\left(\frac{\alpha_s}{\pi}\right)\Delta^{(1)} + C_F\left(\frac{\alpha_s}{\pi}\right)^2\Delta^{(2)}\right], \qquad (1)$$

where

$$\sigma^{(0)}(s) = \frac{4}{3}\frac{\alpha^2}{s}\, N_c\, e_Q^2 \frac{\beta(3-\beta^2)}{2}, \qquad \beta = \sqrt{1-\frac{4m^2}{s}}, \tag{2}$$

$s$ is the total energy squared of the reaction, $m$ is the mass of the heavy quark $Q$, $N_c = 3$ is the number of colors and $e_Q$ is the charge of the quark $Q$ in units of the electric charge. We also always use $\alpha_s \equiv \alpha_s(m)$ in the $\overline{MS}$ scheme and the on–shell renormalization for the heavy quark propagators.

The terms $\Delta^{(1)}$ and $\Delta^{(2)}$ represent, respectively, the one– and two–loop corrections. Vertex corrections which contribute at these orders are shown in Fig. 1. The term $\Delta^{(1)}$ is known in exact form as a function of $\beta$. Near the threshold we are interested in the expressions for both $\Delta^{(1,2)}$ up to and including terms $\mathcal{O}(\beta^0)$[1]. We note, that up to this order in $\beta$, there is no need to consider radiation of real gluons in the process of interest. With this restriction the expression for $\Delta^{(1)}$ reads

$$\Delta^{(1)} = \frac{\pi^2}{2\beta} - 4 + \mathcal{O}(\beta). \tag{3}$$

The two terms in this expression are known to be of rather different origins. The term proportional to the inverse power of $\beta$ is the so–called Coulomb correction, while the second term is known to be a hard correction. Let us demonstrate how these corrections could be calculated using asymptotic expansions.

In this approach for each diagram one has to identify those regions of the integration momenta which can give non-vanishing contributions at the given order in $\beta$. In the calculations close to threshold, one should distinguish four different regions (the following description applies to the center of mass frame of the $Q\bar{Q}$ pair) [8]:

1. hard region, where the momenta of the quanta are of the order of $k \sim m$,
2. soft region, where $k_0 \sim |\mathbf{k}| \sim m\beta$,
3. potential region, where $k_0 \sim m\beta^2$ and $|\mathbf{k}| \sim m\beta$,
4. ultrasoft region $k \sim m\beta^2$.

One should construct all possible subgraphs of a given graph assigning the above labels to the lines in all possible combinations. After such assignment, the routing of the momenta should satisfy the "scale conservation." Namely, two ultrasoft lines, for instance, can not produce a potential line. On the contrary, two potential lines can produce an ultrasoft line and so on.

To calculate $\Delta^{(1)}$ to necessary order in $\beta$, one has to consider the real part of the one–loop correction to the vertex $\gamma^* Q\bar{Q}$. In this diagram only potential and hard regions contribute. The other two regions generate massless tadpoles which vanish in dimensional regularization,

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[1]The term $\mathcal{O}(\beta)$ in $\Delta^{(1)}$ can also be used for the matching on non–relativistic resummed cross–section. This term represents a kinematical correction and leads, essentially, to the replacement $\pi^2/(2\beta) \to \pi^2(1+\beta^2)/(2\beta)$, see [3,4].

which we use throughout this calculation. The potential subgraph gives rise to a finite contribution $\pi^2/(2\beta)$. The hard subgraph (which to the order $\mathcal{O}(\beta^0)$ is obtained by simply considering the vertex correction at the point $s = 4m^2$ in dimensional regularization) gives the constant term $-4$, if combined with the ultraviolet renormalization of the Born cross section.

To see the importance of this classification we recall the following fact. The above result for $\Delta^{(1)}$ is valid in the region $\beta \gg \alpha_s$, where perturbative calculations are still justified. When one approaches the region of small $\beta$, one should perform a resummation of all terms of the form $\alpha_s^n/\beta^n$. Such resummation results in the well known Sommerfeld–Sakharov factor, which is the modulus square of the fermion wave function at the origin, when the Coulomb interaction between nonrelativistic $Q$ and $\bar{Q}$ is taken into account exactly. It is also believed that part of the subleading terms of the form $\alpha_s^{n+1}/\beta^n$ can be resummed by multiplying the Sommerfeld–Sakharov factor by the so–called hard correction:

$$\sigma = \sigma^{(0)} \left(1 - 4C_F\frac{\alpha_s}{\pi}\right) |\Psi(0)|^2,$$
$$|\Psi(0)|^2 = \frac{z}{1-\exp(-z)}, \qquad z = \frac{C_F \alpha_s \pi}{\beta}. \tag{4}$$

It is interesting that this hard renormalization constant for the Sommerfeld–Sakharov factor, which has been known already for a long time, is the same as the contribution of the hard subgraph of the one–loop vertex correction and the ultraviolet renormalization. This is of course not accidental. If the approach based on asymptotic expansions in the threshold calculations is to be trusted, it is fairly clear, that the above formula is correct and the one–loop hard correction does indeed provide the renormalization of the Coulomb ladder to all orders in the coupling constant. A simplest way to see this is to say that the contribution of the hard subgraph provides a renormalization of the NRQCD vector current in order to match it on the complete vector current in QCD (see a more detailed discussion below).

One can hope that the knowledge of the contributions coming from hard subgraphs at order $\mathcal{O}(\alpha_s^2)$ would permit a determination of the two–loop renormalization of the Coulomb ladder.

We now turn to the consideration of the second order correction, described by the term $\Delta^{(2)}$. It is convenient to decompose it into terms proportional to various SU(3) color factors (in SU(3) $C_F = 4/3$, $C_A = 3$, $T_R = 1/2$, and $N_{L,H}$ are the numbers of quark flavors of mass 0 and $m$, respectively):

$$\Delta^{(2)} = C_F\ \Delta_A^{(2)} + C_A\ \Delta_{NA}^{(2)} + N_L T_R\ \Delta_L^{(2)} + N_H T_R\ \Delta_H^{(2)}. \tag{5}$$

The only unknown term in this expression is $\Delta_{NA}^{(2)}$, which arises in the non–abelian theory. $\Delta_{A,H,L}^{(2)}$ are the same in the abelian and non–abelian theory. In the framework of QED $\Delta_A^{(2)}$ were obtained in [4] in the threshold region. The terms which describe the contribution of both massless and massive fermions $\Delta_{L,H}^{(2)}$ were calculated in an analytical form for arbitrary $\beta$ in [12]. Also, we would like to mention, that the leading $\mathcal{O}(\alpha_s^2/\beta)$ terms in $\Delta_{NA}^{(2)}$ were predicted in [13] using the observation that similar terms in $\Delta_L^{(2)}$ are directly related to the contribution of light fermions to the Coulomb–like QCD interaction potential between heavy quark and anti–quark. These terms can be also extracted from the expression for the cross section for $e^+e^- \to Q\bar{Q}$, which includes the resummation of the terms $\alpha_s^{(n+1)}/\beta^n$. Our calculation confirms these results and allows a determination of the $\mathcal{O}(\beta^0)$ terms in $\Delta_{NA}^{(2)}$, which is the main new result of the present paper.

Various contributions to $\Delta^{(2)}$ are:

$$\Delta_A^{(2)} = \frac{\pi^4}{12\beta^2} - 2\frac{\pi^2}{\beta} + \frac{\pi^4}{6} + \pi^2\left(-\frac{35}{18} - \frac{2}{3}\log\beta + \frac{4}{3}\log 2\right) + \frac{39}{4} - \zeta_3, \tag{6}$$

$$\Delta_{NA}^{(2)} = \frac{\pi^2}{\beta}\left(\frac{31}{72} - \frac{11}{12}\log 2\beta\right) + \pi^2\left(\frac{179}{72} - \log\beta - \frac{8}{3}\log 2\right) - \frac{151}{36} - \frac{13}{2}\zeta_3, \tag{7}$$

$$\Delta_L^{(2)} = \frac{\pi^2}{\beta}\left(\frac{1}{3}\log 2\beta - \frac{5}{18}\right) + \frac{11}{9}, \tag{8}$$

$$\Delta_H^{(2)} = \frac{44}{9} - \frac{4\pi^2}{9}. \tag{9}$$

The terms $\Delta_{A,L,H}^{(2)}$ coincide with the results obtained in QED. The non-abelian piece $\Delta_{NA}^{(2)}$ was studied numerically in [13], where Padé approximation was used to obtain $\Delta_{NA}^{(2)}$ as a function of $\beta$. A comparison of our result for $\Delta_{NA}^{(2)}$ with the results for the similar quantity presented in [13] shows, that for $\beta \ll 1$ there is a reasonable agreement.

The details of the derivation of the above results cannot be presented here for lack of space. We only briefly explain how they are obtained using the asymptotic expansion in the threshold region. The contributions of all momenta regions, relevant for this calculation, are separated according to a classification given above and the integrands of the loop integrations are expanded in the respective small parameters (which vary from one momentum region to another). All divergences which arise in the course of this procedure are regulated using dimensional regularization. The calculation of the hard contribution is done by solving a system of recurrence relations which allows to reduce any hard integral to a limited set of master integrals. All integrals, where only potential lines are involved, are similar in their form to the integrals which one encounters in the non–relativistic perturbation theory and can be done easily. When some lines of a given diagram are either soft or ultrasoft, the calculation can be performed loop-by–loop. For this, the results for the one–loop eikonal integrals [15] are useful as well as some results obtained in [16].

Let us elucidate the origin of various terms in the above expressions for $\Delta_A^{(2)}$ and $\Delta_{NA}^{(2)}$. The most interesting contributions come from the hard subgraphs, because these contributions could be used to discuss the renormalization of the Sommerfeld–Sakharov factor at the next–to–leading order as well as a matching of the NRQCD quark–antiquark vector current on the full QCD vector current at order $\mathcal{O}(\alpha_s^2)$ and the leading order in $1/m$.

We note the following: in both $\Delta_{A,NA}^{(2)}$ the terms which are not accompanied by powers of $\pi$ come from hard subgraphs. These are $39/4 - \zeta_3$ and $-151/36 - 13\zeta_3/2$, respectively, for $\Delta_A^{(2)}$ and $\Delta_{NA}^{(2)}$. The terms of the order $\mathcal{O}(\beta^0)$, which are multiplied by a single power of

$\pi^2$, are more tricky. First, one sees that there is a $\log\beta$ contribution in these pieces. This implies that these terms get contributions from both hard and (all possible) soft scales, and these contributions are not finite separately. This in turn means that at the $\mathcal{O}(\alpha_s^2)$ order the matching coefficient of the NRQCD vector quark–antiquark current is not finite any longer. Removing this divergence by using the $\overline{MS}$ renormalization of the low energy effective field theory, we arrive at finite (but scale dependent) matching coefficients.

We write

$$\bar{\psi}\gamma^i\psi = \left(1 - 2C_F\frac{\alpha_s}{\pi} + c_2(\mu)C_F\left(\frac{\alpha_s}{\pi}\right)^2\right)\left[\psi_h^+\sigma^i\chi_h\right]_\mu, \tag{10}$$

where the operator at the LHS of the above equation is the quark–antiquark current in the full QCD, while the RHS represents the quark–antiquark current in the effective field theory (with $\psi_h$ and $\chi_h$ being two-component spinors), multiplied by a Wilson (matching) coefficient. We note that to the order $\mathcal{O}(\alpha_s, 1/m^2)$ the matching condition for the quark–antiquark vector current has been obtained in [14].

The knowledge of contributions coming from hard subgraphs allows us to obtain this matching coefficient directly:

$$c_2(\mu) = C_F c_A + C_A c_{NA} + N_L T_R c_L + N_H T_R c_H, \tag{11}$$

$$\begin{aligned} c_A(\mu) = {} & \pi^2\left(-\frac{79}{36} - \frac{1}{3}\log\left(\frac{\mu}{m}\right) + \log 2\right) \\ & + \frac{23}{8} - \frac{1}{2}\zeta_3, \end{aligned} \tag{12}$$

$$\begin{aligned} c_{NA}(\mu) = {} & \pi^2\left(\frac{89}{144} - \frac{1}{2}\log\left(\frac{\mu}{m}\right) - \frac{5}{6}\log 2\right) \\ & - \frac{151}{72} - \frac{13}{4}\zeta_3, \end{aligned} \tag{13}$$

$$c_L(\mu) = \frac{11}{18}, \tag{14}$$

$$c_H(\mu) = -\frac{2}{9}\pi^2 + \frac{22}{9}. \tag{15}$$

We note finally that within the present approach it is possible to calculate also the higher order terms in the expansion in $\beta$. However, already for the terms of order $\beta^1$ in $\Delta^{(2)}$ one should incorporate the real gluon radiation. The expansion of the real radiation and the phase space in this case can be obtained as a slight generalization of a similar expansion, discussed in [15] in the context of heavy quark decays.

In conclusion, we have calculated the $\mathcal{O}(\alpha_s^2)$ correction to the heavy quark production cross section in $e^+e^-$ annihilation in the threshold region, assuming $\alpha_s \ll \beta \ll 1$. A method of systematic expansion of the Feynman diagrams near thresholds in powers and logarithms of the quark velocity $\beta$ enabled for the first time an evaluation of the non-abelian terms. At the same time, comparison of the abelian terms with previously obtained results gives us confidence in the correctness of the method. The results of the present paper can be further used for matching-like calculations, to arrive at the predictions for the threshold cross section at the energy region, where $\beta \sim \alpha_s$. We have also given a matching relation between full QCD and NRQCD effective currents to leading order in $1/m$ and second order in the strong coupling constant.

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FIGURES

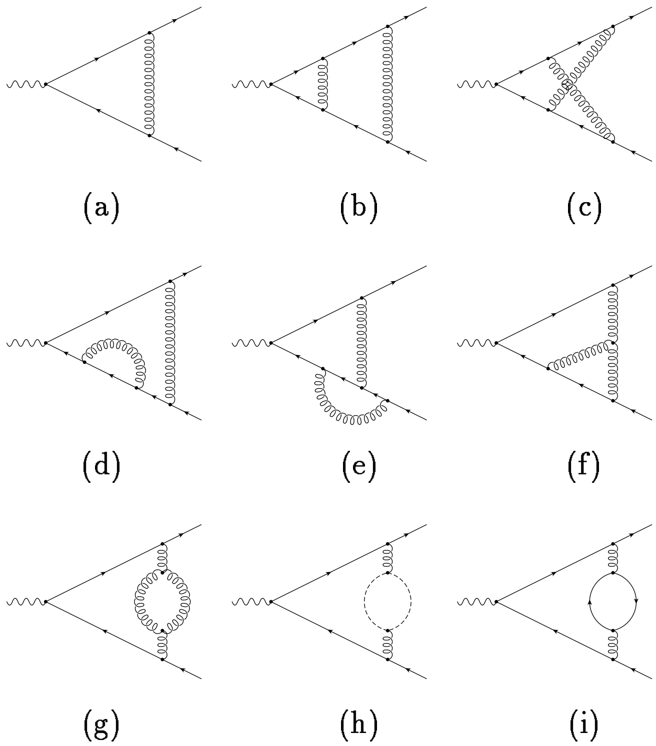


FIG. 1. Two-loop QCD corrections to the quark pair production; (a) is the one loop vertex correction; (b-i) are the two loop gluonic and fermionic corrections.