One-loop computations in QFT: a modern perspective

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Lectures based on R.K. Ellis, Z. Kunszt, K. Melnikov, G. Zanderighi, ``One-loop computations in QFT: from Feynman diagrams to the unitarity cuts", hep-ph/1105.4319

Other useful references:
H. Ita, ``Susy theories and QCD: numerical approaches", hep-th/1109.6527
L. Dixon ``Calculating scattering amplitudes efficiently", hep-ph/9601359

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Outline

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Introduction

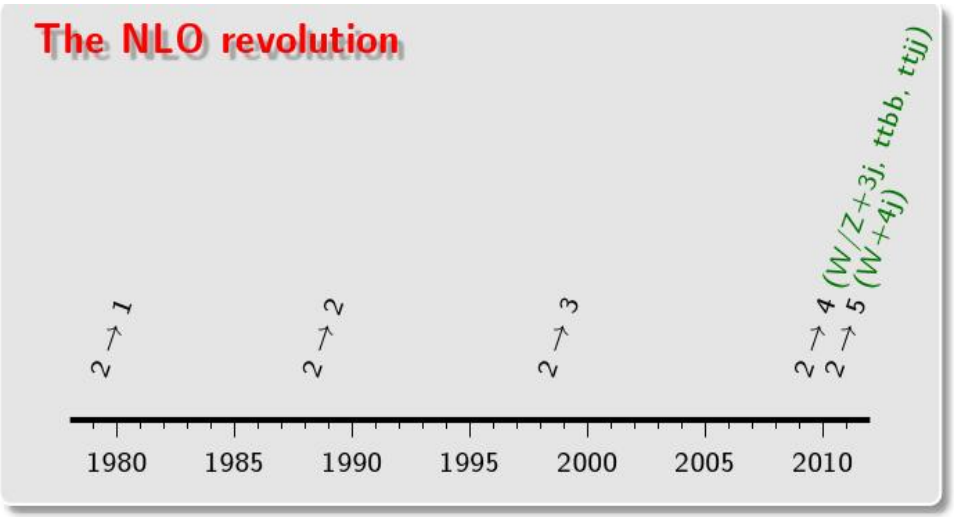
- One-loop (NLO) computations for hadron collider processes are very important for successful hadron collider phenomenology – they often provide first approximation in perturbative QCD which is reliable.
- In the past few years – remarkable progress with one-loop computations, right in time for the LHC start up.
- Complexity of NLO computations increases with partonic multiplicity of the process. The rule of thumb – increase of multiplicity by one unit requires decade of theoretical progress. This rule was true for about 30 years but, finally, it was broken.

An experimenter's wishlist

■ Hadron collider cross-sections one would like to know at NLO

Run II Monte Carlo Workshop, April 2001

Single boson	Diboson	Triboson	Heavy flavour
$W + \leq 5j$	$WW + \leq 5j$	$WWW + \leq 3j$	$t\bar{t} + \leq 3j$
$W + b\bar{b} + \leq 3j$	$WW + b\bar{b} + \leq 3j$	$WWW + b\bar{b} + \leq 3j$	$t\bar{t} + \gamma + \leq 2j$
$W + c\bar{c} + \leq 3j$	$WW + c\bar{c} + \leq 3j$	$WWW + \gamma\gamma + \leq 3j$	$t\bar{t} + W + \leq 2j$
$Z + \leq 5j$	$ZZ + \leq 5j$	$Z\gamma\gamma + \leq 3j$	$t\bar{t} + Z + \leq 2j$
$Z + b\bar{b} + \leq 3j$	$ZZ + b\bar{b} + \leq 3j$	$WZZ + \leq 3j$	$t\bar{t} + H + \leq 2j$
$Z + c\bar{c} + \leq 3j$	$ZZ + c\bar{c} + \leq 3j$	$ZZZ + \leq 3j$	$t\bar{b} + \leq 2j$
$\gamma + \leq 5j$	$\gamma\gamma + \leq 5j$		$b\bar{b} + \leq 3j$
$\gamma + b\bar{b} + \leq 3j$	$\gamma\gamma + b\bar{b} + \leq 3j$		
$\gamma + c\bar{c} + \leq 3j$	$\gamma\gamma + c\bar{c} + \leq 3j$		
	$WZ + \leq 5j$		
	$WZ + b\bar{b} + \leq 3j$		
	$WZ + c\bar{c} + \leq 3j$		
	$W\gamma + \leq 3j$		
	$Z\gamma + \leq 3j$		



Introduction

NLO predictions are currently available for major production channels:

1) multiple jets (up to 4 jets)

2) a gauge boson and up to 4 jets

3) multiple gauge bosons in association with jets (up to VV + 2j)

4) top quarks in association with jets (up to two) and gauge bosons (W,Z, photon)

5) Higgs and up to two jets

Process ($V \in \{Z, W, \gamma\}$)	Comments
Calculations completed since Les Houches 2005	
1. $pp \to VV$ jet	WW jet completed by Dittmaier/Kallweit/Uwer [27, 28]; Campbell/Ellis/Zanderighi [29]. ZZjet completed by Binoth/Gleisberg/Karg/Kauer/Sanguinetti [30]
2. $pp \to$ Higgs+2jets	NLO QCD to the gg channel completed by Campbell/Ellis/Zanderighi [31]; NLO QCD+EW to the VBF channel completed by Ciccolini/Denner/Dittmaier [32, 33] Interference QCD-EW in VBF channel [34, 35]
3. $pp \to VVV$	ZZZ completed by Lazopoulos/Melnikow/Petriello [36] and WWZ by Hankele/Zeppenfeld [37], see also Binoth/Ossola/Papadopoulos/Pittau [38] VBFNLO [39, 40] meanwhile also contains $WWW, ZZW, WW\gamma, ZZ\gamma, WZ\gamma, W\gamma\gamma, Z\gamma\gamma, \gamma\gamma\gamma$ $WZj, W\gamma j, \gamma jj, W\gamma\gamma j$
4. $pp \to t\bar{t}b\bar{b}$	relevant for $t\bar{t}H$, computed by Bredenstein/Denner/Dittmaier/Pozzorini [41, 42] and Bevilacqua/Czakon/Papadopoulos/Pittau/Worek [43]
5. $pp \to V$+3jets	W+3jets calculated by the Blackhat/Sherpa [44] and Rocket [45] collaborations Z+3jets by Blackhat/Sherpa [46]
Calculations remaining from Les Houches 2007	
6. $pp \to t\bar{t}$+2jets	relevant for $t\bar{t}H$, computed by Bevilacqua/Czakon/Papadopoulos/Worek [47, 48]
7. $pp \to VV\,b\bar{b}$, 8. $pp \to VV$+2jets	Pozzorini et al.[25],Bevilacqua et al.[23] W^+W^++2jets [49], W^+W^-+2jets [50], VBF contributions calculated by (Bozzi/)Jäger/Oleari/Zeppenfeld [51, 52, 53]
NLO calculations added to list in 2007	
9. $pp \to b\bar{b}b\bar{b}$	Binoth et al. [54, 55]
NLO calculations added to list in 2009	
10. $pp \to V + 4$ jets	top pair production, various new physics signatures Blackhat/Sherpa: W+4jets [22], Z+4jets [20] see also HEJ [56] for $W + n$jets
11. $pp \to Wb\bar{b}j$	top, new physics signatures, Reina/Schutzmeier [11]
12. $pp \to t\bar{t}t\bar{t}$	various new physics signatures
also: $pp \to 4$ jets	Blackhat/Sherpa [19]

CLOSED

In most cases, one can get the one-loop ingredients that are required for even higher-multiplicity processes in a relatively straightforward way. The real bottleneck now is the computation of real emission corrections for higher-multiplicity processes

Introduction

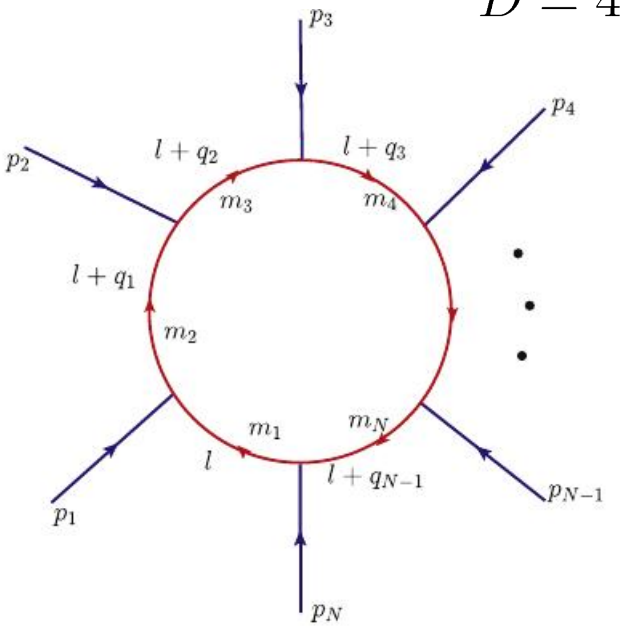
- The breakthrough that I just described is the consequence of a radically new technique for one-loop computations that has been developed in recent years. The goal of these lectures is to review this technique.
- We will start with a brief summary of traditional approaches for one-loop computations since it is important to understand what we attempt to improve upon; we will discuss the new framework after that.

Traditional approaches to one-loop computations

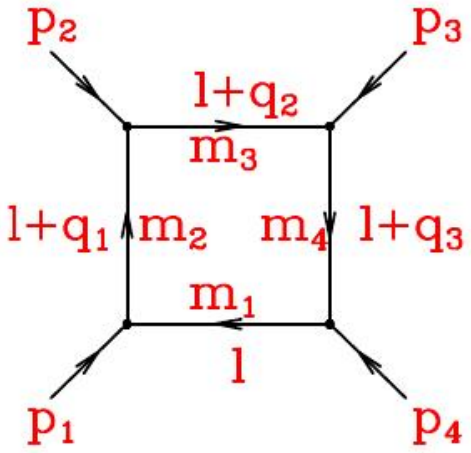
- One-loop integrals can be represented by a linear combination of four-, three-, two- and one-point master integrals and a remainder which is called the rational part. This distinction appears because the reduction coefficients are computed in four dimensions (D → 4 limit is taken)

$$I_N \sim \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{Num(l)}{((l+q_0)^2 - m_0^2)((l+q_1)^2 - m_1^2)...((l+q_{N-1})^2 - m_{N-1}^2)}$$

$$D = 4 - 2\epsilon \qquad q_j = \sum_{k=1}^{j} p_k \qquad p_1 + p_2 + ...p_N = 0$$



$$I_N = c_{4;j} I_{4;j} + c_{3;j} I_{3;j} + c_{2;j} I_{2;j} + c_{1;j} I_{1;j} + \mathcal{R} + \mathcal{O}(\epsilon)$$



Master integrals can be computed numerically using F77 program QCDloops (Ellis, Zanderighi)

Traditional approaches to one-loop computations

- The reason any integral can be reduced to a linear combination of scalar integrals is, essentially, Lorentz invariance. Let us consider an example of how such a reduction (Passarino-Veltman) can be performed

$$C^\mu = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l^\mu}{d_1 d_2 d_3} = p_1^\mu C_1 + p_2^\mu C_2 \qquad d_i = (l + \sum_{k=1}^{i-1} p_k)^2 - m_i^2$$

$$l \cdot p_1 = \frac{1}{2}\left(f_1 + d_2 - d_1\right), \quad f_1 = m_2^2 - m_1^2 - p_1^2$$

$$l.p_2 = \frac{1}{2}\left(f_2 + d_3 - d_2\right), \quad f_2 = m_3^2 - m_2^2 - p_2^2 - 2p_1 \cdot p_2$$

Contracting the integral with two vectors, p_1 and p_2, we obtain a system of linear equations

$$G_2 \begin{pmatrix} C_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} R_1^c \\ R_2^c \end{pmatrix} \qquad G_2 = \begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2 \cdot p_2 \end{pmatrix}$$

$$R_1^c = \frac{1}{2}\left(f_1 C_0(1,2,3) + B_0(1,3) - B_0(2,3)\right)$$

$$R_2^c = \frac{1}{2}\left(f_2 C_0(1,2,3) + B_0(1,2) - B_0(1,3)\right)$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = G_2^{-1} \begin{pmatrix} R_1^c \\ R_2^c \end{pmatrix}$$

Traditional approaches to one-loop computations

- If we consider higher-rank integrals, we can see that a reduction pattern emerges

$$C^{\mu\nu} = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l^\mu l^\nu}{d_1 d_2 d_3} = g^{\mu\nu} C_{00} + \sum_{i,j=1}^{2} p_i^\mu p_j^\nu C_{ij} \qquad C_{21} = C_{12}$$

By contracting with p_1 and p_2, canceling scalar products in numerators and denominators and equating terms with same momentum-dependence on the l.h.s and the r.h.s, we find

$$\begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} = G_2^{-1} \begin{pmatrix} R_1^{c1} \\ R_2^{c1} \end{pmatrix} \qquad \begin{pmatrix} C_{12} \\ C_{22} \end{pmatrix} = G_2^{-1} \begin{pmatrix} R_1^{c2} \\ R_2^{c2} \end{pmatrix}$$

$$R_1^{c1} = \frac{1}{2} \left(f_1 C_1(1,2,3) + B_1(1,3) + B_0(2,3) - 2C_{00}(1,2,3) \right)$$

$$R_2^{c1} = \frac{1}{2} \left(f_2 C_2(1,2,3) + B_1(1,2) - B_1(1,3) \right)$$

C_{00} is obtained by contracting the integral with the metric tensor

$$C_{00} = \frac{1}{2(D-2)} \left(2m_1^2 C_0(1,2,3) - f_2 C_2(1,2,3) - f_1 C_1(1,2,3) + B_0(2,3) \right)$$

Traditional approaches to one-loop computations

- The emergent structure of reduction ; the complexity decreases when going from top to bottom . Fully established procedure to reduce any one-loop diagram to scalar integrals

Table 2.1
Reduction chains for the Passarino–Veltman procedure; see Appendix A for a definition of all coefficients.

$D_{ijkl} \to D_{00ij}, D_{ijk}, C_{ijk}, C_{ij}, C_i, C_0$
$D_{00ij} \to D_{ijk}, D_{ij}, C_{ij}, C_i$
$D_{0000} \to D_{00i}, D_{00}, C_{00}$
$D_{ijk} \to D_{00i}, D_{ij}, C_{ij}, C_i$
$D_{00i} \to D_{ij}, D_i, C_i, C_0$
$D_{ij} \to D_{00}, D_i, C_i, C_0$
$D_{00} \to D_i, D_0, C_0$
$D_i \to D_0, C_0$
$C_{ijk} \to C_{00i}, C_{ij}, B_{ij}, B_i$
$C_{00i} \to C_{ii}, C_i, B_i, B_0$
$C_{ij} \to C_{00}, C_i, B_i, B_0$
$C_{00} \to C_i, C_0, B_0$
$C_i \to C_0, B_0$
$B_{ii} \to B_{00}, B_i, A_0$
$B_{00} \to B_i, B_0, A_0$
$B_i \to B_0, A_0$

$D_{ijkl}, C_{ijk}, B_{ij}, A$

are the 4-point, 3-point, 2-point and tadpole integrals, respectively

Why non-traditional approaches ?

- There are few issues with traditional approaches that require care
  - large number of diagrams (factorial growth with the number of external particles)
  - huge expressions that appear in the course of the reduction procedure
  - Gram determinants in high powers may lead to numerical instabilities
  - for multi-particle processes, hard to express the result in terms of truly independent structures which may be important for getting rid of spurious complexity
- It was thought for a while that these features will be a show-stopper for the Passarino-Veltman reduction but it did not happen; it turned out that all of these issues can be solved with a little bit of ingenuity and improved computing power
- However, in recent years an interesting alternative to the Passarino-Veltman reduction appeared. It was originally developed by Ossola, Papadopoulos and Pittau as a reduction procedure for one-loop integrals that operates at the level of an integrand.
- It was realized later on that this OPP procedure can be connected in an interesting way to deeper aspects of Quantum Field theory such as unitarity.
- Very often, the OPP procedure is presented using spinor-helicity formalism which is useful for certain things but leaves some other aspects of the construction (extension to D-dimensions, inclusion of masses etc.) somewhat unclear. We will start by preparting framework to discuss the OPP reduction procedure without spinor-helicity formalism.

Van Neerven – Vermaseren vector basis

- The important element of the construction of OPP procedure is the van Neerven – Vermaseren vector basis.
- Consider a linear vector space spanned by two non-orthogonal vectors q_1 and q_2 . While we can write $l^\mu = c_1 q_1^\mu + c_2 q_2^\mu$ it is not very convenient for finding c_1 and c_2 .
- A more convenient basis is $l^\mu = c_1 v_1^\mu + c_2 v_2^\mu$, where

$$v_1^\mu = \frac{\epsilon^{\mu q_2}}{\epsilon^{q_1 q_2}} \qquad v_2^\mu = \frac{\epsilon^{q_1 \mu}}{\epsilon^{q_1 q_2}} \qquad \epsilon^{\mu q} = \epsilon^{\mu\nu} q_\nu$$

Since $q_i \cdot v_j = \delta_{ij}$ we find $c_i = l \cdot q_i, \quad i = 1, 2.$

$$l^\mu = (l \cdot q_1) v_1^\mu + (l \cdot q_2) v_2^\mu$$

This representation may be useful for facilitating reduction if we think about momentum l as the loop momentum and momenta $q_{1,2}$ as the propagator momenta in a Feynman diagram

$$l \cdot q_i = \frac{1}{2}\left[\left((l+q_i)^2 - m_i^2\right) - (l^2 - m_0^2) + m_i^2 - m_0^2 - q_i^2\right].$$

Van Neerven – Vermaseren basis

- To extend this discussion to higher-dimensional spaces, we can re-write the basis vectors as

$$v_1^\mu = \frac{\epsilon^{\mu q_2}}{\epsilon^{q_1 q_2}} = \frac{\epsilon_{q_1 q_2}\epsilon^{\mu q_2}}{\epsilon_{q_1 q_2}\epsilon^{q_1 q_2}} = \frac{\delta^{\mu q_2}_{q_1 q_2}}{\delta^{q_1 q_2}_{q_1 q_2}} \qquad \epsilon_{\mu_1\mu_2}\epsilon^{\nu_1\nu_2} = \delta^{\nu_1\nu_2}_{\mu_1\mu_2} = \det|\delta^\nu_\mu| = \delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2} - \delta^{\nu_1}_{\mu_2}\delta^{\nu_2}_{\mu_1}$$

Consider now a Feynman integral that depends on N (independent) incoming momenta k_1, ...,k_N . Introduce the vNV basis by the straightforward generalization of the above

$$v_i^\mu = \frac{\delta^{k_1\ldots k_{i-1}\mu_i k_{i+1}..k_N}_{k_1\ldots.k_N}}{\delta^{k_1\ldots k_N}_{k_1\ldots k_N}} \qquad \delta^{\mu_1\ldots.\mu_N}_{\nu_1\ldots\nu_N} = \det|\delta^\mu_\nu| \qquad k_i\cdot v_j = \delta_{ij}$$

Beware that basis vectors are not orthonormal $v_i\cdot v_j \neq \delta_{ij}$

For N < D, need to introduce additional basis for part of the vector space that is not spanned by the external momenta (the transverse space)

$$l^\mu = \sum_{i=1}^{N} c_i v_i^\mu + \sum_{i=N+1}^{D} x_i n_i^\mu \qquad k_i\cdot n_j = 0 \qquad v_i\cdot n_j = 0 \qquad c_i = l\cdot k_i$$

$$g^{\mu\nu} = \sum_{i=1}^{N} k_i^\mu v_i^\nu + \sum_{i=N+1}^{D} n_i^\mu n_i^\nu = \sum_{i=1}^{N} k_i^\nu v_i^\mu + \sum_{i=N+1}^{D} n_i^\mu n_i^\nu$$

Completeness relation

Van Neerven – Vermaseren basis in two dimensions

- We will now use the vNV basis to investigate some properties of one-loop integrals in two dimensions. This is a useful exercise since
  - everything that we will learn is directly generalizable to four-dimensions
  - algebra is much more simple in two dimensions compared to a four-dimensional case

General statement: in D-dimensional space-time, the basis of master integrals consists of D-point, (D-1)-point, ...one-point integrals, so all integrals higher than D-point are completely reducible . We will try to understand why this statement is true for D=2

$$I_3 = \int \frac{\mathrm{d}^2 l}{(2\pi)^2}\ \mathcal{I}_3, \quad \mathcal{I}_3 = \frac{1}{d_0 d_1 d_2}, \quad d_i = (l+q_i)^2 - m_i^2, \quad q_0 = 0.$$

$$l^\mu = v_1^\mu (l\cdot q_1) + v_2^\mu (l\cdot q_2), \qquad v_1^\mu = \frac{\delta^{\mu q_2}_{q_1 q_2}}{\Delta_2}. \qquad v_2^\mu = \frac{\delta^{q_1 \mu}_{q_1 q_2}}{\Delta_2} \qquad \Delta_2 = q_1^2 q_2^2 - (q_1 q_2)^2$$

$$l\cdot q_i = \frac{1}{2}\left(d_i - d_0 - r_i\right), \quad r_i = q_i^2 - m_i^2 + m_0^2, \quad i = 1,2.$$

Van Neerven – Vermaseren basis in two dimensions

- Contract the loop momentum with itself

$$l^\mu = v_1^\mu (l\cdot q_1) + v_2^\mu (l\cdot q_2) \quad \Longrightarrow \quad 2(d_0+m_0^2) = \sum_{i=1}^{2} (l\cdot v_i)(d_i - d_0) - \sum (l\cdot v_i) r_i$$

Rewrite the last term in that equation using representation of l in terms of vNV basis vectors

$$\sum (l v_i) r_i = \frac{1}{2}\sum_j (w\cdot v_j)\,(d_j - d_0 - r_j)\,, \quad w^\mu = \sum_{i=1}^{2} v_i^\mu r_i$$

$$2(d_0+m_0^2) = \frac{1}{2}\left(\sum \left[2l\cdot v_i - w\cdot v_i\right] d_i - \sum (2l-w)\cdot v_i d_0 + \sum r_i (w\cdot v_i)\right).$$

Dividing by d_0, d_1, d_2 and collecting the terms, we find the reduced form of the integrand

$$\mathcal{I}_3 = \frac{1}{4m_0^2 - w^2}\left\{\frac{2l\cdot v_1 - w\cdot v_1}{d_0 d_2} + \frac{2l\cdot v_2 - w\cdot v_2}{d_0 d_1} - \frac{4 + (2l-w)\cdot(v_1+v_2)}{d_1 d_2}\right\}$$

Note that each reduction coefficient is linear in momentum and that the integration over l can be easily performed because this vector integral in the numerator is projected on the transverse space for particular integrals. This makes integration over loop momentum trivial

$$I_3 = \int \frac{\mathrm{d}^2 l}{(2\pi)^2}\mathcal{I}_3 = \frac{(-1)}{4m_0^2 - w^2}\left[(w\cdot v_1) I_{02} + (w\cdot v_2) I_{01} + (2 - w\cdot(v_1+v_2)) I_{12}\right).$$

Van Neerven – Vermaseren basis in two dimensions

- As the next example – consider the reduction of a (special) rank-two tensor integral to scalar integrals in two dimensions. We will, however, consider a d-dimensional integral, to account for a possible UV divergence

$$\mathcal{I}_2 = \frac{(\hat{n}\cdot l)^2}{d_1 d_2}, \quad d_1 = l^2 - m_1^2, \quad d_2 = (l+k)^2 - m_2^2, \quad \hat{n}\cdot k = 0, \quad k^2 \neq 0, \quad \hat{n}^2 = 1.$$

$$l^\mu = (l\cdot n)n^\mu + (l\cdot \hat{n})\hat{n}^\mu + (l\cdot n_\epsilon)\hat{n}_\epsilon^\mu \qquad n^\mu = \frac{k^\mu}{\sqrt{k^2}}$$

$$\frac{(\hat{n}\cdot l)^2}{d_1 d_2} = -\frac{\lambda^2+\mu^2}{d_1 d_2} + \frac{1}{4k^2}\left[\frac{r_1^2 - 2l\cdot k}{d_1} + \frac{r_2^2 + 2l\cdot k + 2k^2}{d_2}\right].$$

$$\mu^2 = (\hat{n}_\epsilon\cdot l)^2, \quad r_1^2 = k^2 + m_1^2 - m_2^2, \quad r_2^2 = k^2 + m_2^2 - m_1^2, \quad \lambda^2 = \frac{k^4 - 2k^2(m_1^2+m_2^2) + (m_1^2-m_2^2)^2}{4k^2}.$$

$$\frac{(\hat{n}\cdot l)^2}{d_1 d_2} = \frac{b_0 + b_1 \hat{n}\cdot l + b_2 (l\hat{n}_\epsilon)^2}{d_1 d_2} + \frac{a_{1,0} + a_{1,1}(n\cdot l) + a_{1,2}(\hat{n}\cdot l)}{d_1} + \frac{a_{2,0} + a_{2,1}(n\cdot l) + a_{2,2}(\hat{n}\cdot l)^2}{d_2}.$$

$$b_0 = -\lambda^2, \quad b_1 = 0, \quad b_2 = -1$$

We can find those coefficients in a different – and quite instructive – way

Van Neerven – Vermasseren basis in two dimensions

- Rewrite the previous reduction formula as

$$\frac{(\hat{n}\cdot l)^2}{d_1 d_2} = \frac{b_0 + b_1 \hat{n}\cdot l + b_2 (l\hat{n}_\epsilon)^2}{d_1 d_2} + \frac{a_{1,0} + a_{1,1}(n\cdot l) + a_{1,2}(\hat{n}\cdot l)}{d_1} + \frac{a_{2,0} + a_{2,1}(n\cdot l) + a_{2,2}(\hat{n}\cdot l)^2}{d_2}.$$

$$(\hat{n}\cdot l)^2 = b_0 + b_1 \hat{n}\cdot l + b_2 (l\hat{n}_\epsilon)^2 + \mathcal{O}(d_1, d_2)$$

Note that the right hand side simplifies if the above equation is studied for the loop momenta such that d_1 and/or d_2 vanish . In particular, by requiring that both $d_{1,2}$ vanish, we project the right hand side on b-coefficients only

$$d_1(l_c) = 0, \quad d_2(l_c) = 0 \quad l_c \cdot n_\epsilon = 0. \qquad l_c^{\pm} = -\frac{r_1^2}{2\sqrt{k^2}} n \pm i\lambda \hat{n}.$$

Note that the momentum for which both propagators vanish is, in general, complex

Compute the left-hand side and the right-hand side for two values of the l_c momentum

$$-\lambda^2 = b_0 + b_1 \hat{n}\cdot l_c^+, \quad -\lambda^2 = b_0 + b_1 \hat{n}\cdot l_c^-, \qquad \hat{n}\cdot l_c^{\pm} = \pm i\lambda \neq 0$$

$$b_0 = -\lambda^2, \quad b_1 = 0.$$

The reduction coefficients is found from the loop momenta that force the two propagators to be on the mass-shell. Clearly, these momenta are special. Also this feature of the reduction is not at all obvious in the regular Passarino-Veltman approach.

OPP reduction algorithm: parametrization of the integrand

- In a renormalizable quantum field theory, the rank of a tensor integral can not exceed the number of external legs. We will be concerned with this situation.
- We will prove the following equation for the integrand of a one-loop integral

$$I_N = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod_i d_i(l)} = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{\prod_i d_i(l)} \times \Bigg\{ \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}_{i_1,i_2,i_3,i_4,i_5}(l) \prod_{j\neq[i_1,i_2,i_3,i_4,i_5]} d_j(l)$$
$$+ \sum_{i_1,i_2,i_3,i_4} \tilde{d}_{i_1,i_2,i_3,i_4}(l) \prod_{j\neq[i_1,i_2,i_3,i_4]} d_j(l) + \sum_{i_1,i_2,i_3} \tilde{c}_{i_1,i_2,i_3}(l) \prod_{j\neq[i_1,i_2,i_3]} d_j(l)$$
$$+ \sum_{i_1,i_2} \tilde{b}_{i_1,i_2}(l) \prod_{j\neq[i_1,i_2]} d_j(l) + \sum_{i_1} \tilde{a}_{i_1}(l) \prod_{j\neq i_1} d_j(l) \Bigg\}.$$

The reduction formula is valid in D-dimensions assuming that external momenta are four-dimensional

The highest level master integral is a five-point function – this is a consequence of dealing with D-dimensional space

Each propagator appears on the right hand at most in first power

We will find that the reduction functions e,d,c,b,a have very particular dependence on the loop momentum that facilitates the reduction to master integrals

Parametrization of the intergand: the 5-point function

- We will begin with the discussion of the five-point function. The highest possible rank of a tensor integral is five. We assume that vectors $u_{i=1..5}$ are ``external" and therefore four-dimensional

$$\mathcal{I}_5 = \left(\prod_{i=1}^{5} u_i \cdot l\right) / \left(\prod_{i=0}^{4} d_i\right), \quad d_i = (l+q_i)^2 - m_i^2, \quad q_0 = 0.$$

$$l^\mu = \sum_{i=1}^{4} (l \cdot q_i) v_i^\mu + (l \cdot n_\epsilon) n_\epsilon^\mu. \qquad l \cdot q_i = \frac{1}{2}\left(d_i - d_0 - (q_i^2 - m_i^2 + m_0^2)\right).$$

$$N_5(l) = \left(\prod_i^4 u_i \cdot l\right)(u_5 \cdot l) = \frac{1}{2}\sum_{j=1}^{4}(u_5 \cdot v_j)\left(\prod_i^4 u_i \cdot l\right)(d_j - d_0) - \frac{1}{2}\sum_{j=1}^{4}(u_5 \cdot v_j)\left(\prod_i^4 u_i \cdot l\right)(q_j^2 - m_j^2 + m_0^2).$$

First term on the right-hand side is a collection of rank-four four-point functions; the last term is a rank-four five-point function. We can disregard the four-point function for now and continue with five-points.

Repeating this procedure over and over we get rid of the tensor structure from the five-point function and conclude that, in the OPP reduction formula, the coefficient of the 5-point function is a constant

$$\tilde{e}_{01234}(l) = \tilde{e}_{01234}.$$

Parametrization of the integrand: the four-point function

- For the four-point function, the highest rank of a tensor integral is four. We use the vVN basis and perform a computation similar to the five-point case

$$l^\mu = \sum_{i=1}^{3} (l \cdot q_i) v_i^\mu + (l \cdot n_4) n_4^\mu + (l \cdot n_\epsilon) n_\epsilon^\mu .$$

$$N_4(l) = \left(\prod_i^3 u_i \cdot l \right) (u_4 \cdot l) = \frac{1}{2} \sum_{j=1}^{3} u_4 \cdot v_j \left(\prod_i^3 u_i \cdot l \right) (d_j - d_0)$$

$$- \frac{1}{2} \sum_{j=1}^{3} u_4 \cdot v_j (q_j^2 - m_j^2 + m_0^2) \prod_i^3 u_i \cdot l + (l \cdot n_4)(u_4 \cdot n_4) \prod_i^3 u_i \cdot l.$$

The first two terms on the right hand side are ``reduced"; they are either lower-point or lower-rank integrals. The last term is the rank four again. To simplify it, repeat the same procedure for one of the scalar products

$$\left(\prod_{i=1}^{3} u_i \cdot l \right) (l \cdot n_4) \to \left(\prod_{i=1}^{2} u_i \cdot l \right) (l \cdot n_4)^2 .$$

$$2l \cdot q_i = d_i - d_0 - q_i^2 + m_i^2 - m_0^2, \quad i \in 1..3 \qquad\qquad l^2 = d_0 + m_0^2$$

$$(l \cdot n_4)^2 = -(l \cdot n_\epsilon)^2 + \text{const} + \mathcal{O}(d_0, d_1, d_2, d_3)$$

$$\tilde{d}_{0123}(l) = \tilde{d}_0 + \tilde{d}_1 (l \cdot n_4) + \tilde{d}_2 (l \cdot n_\epsilon)^2 + \tilde{d}_3 (l \cdot n_\epsilon)^2 (l \cdot n_4) + \tilde{d}_4 (l \cdot n_\epsilon)^4,$$

Parametrization of the intergand: the three-point function

- For the three-point function, the highest rank of the integral is three.

$$l^\mu = \sum_{i=1}^{2} (l \cdot q_i) v_i^\mu + +(l \cdot n_3) n_3^\mu + (l \cdot n_4) n_4^\mu + (l \cdot n_\epsilon) n_\epsilon^\mu.$$

$$\prod_{i=3}^{4} (l \cdot u_i) \to \sum_{i=3}^{4} c_{1i} (l \cdot n_i) + \sum_{i=3}^{4} c_{2i} (l \cdot n_i)^2 + \sum_{i=3}^{4} c_{3i} (l \cdot n_i)^3 + c_4 (l \cdot n_4)(l \cdot n_3) + c_5 (l \cdot n_3)^2 (l \cdot n_4) + c_6 (l \cdot n_3)(l \cdot n_4)^2.$$

We have one constraint on the projection of the loop momentum on the transverse space

$$(l \cdot n_3)^2 + (l \cdot n_4)^2 + (l \cdot n_\epsilon)^2 = \text{const} + \mathcal{O}(d_0, d_1, d_2)$$

Use this constraint to trade some scalar products for other scalar products

$$(l \cdot n_3)^2 l \cdot n_4, \;\; (l \cdot n_4)^2 l \cdot n_3 \;\Rightarrow\; (l \cdot n_\epsilon)^2 l \cdot n_4, \;\; (l \cdot n_\epsilon)^2 l \cdot n_3.$$

and choose $(l \cdot n_\epsilon)^2, \quad (l \cdot n_3)^2 - (l \cdot n_4)^2$ as independent scalar products.

$$\tilde{c}_{012}(l) = \tilde{c}_0 + \tilde{c}_1 (l \cdot n_3) + \tilde{c}_2 (l \cdot n_4) + \tilde{c}_3 ((l \cdot n_3)^2 - (l \cdot n_4)^2) + \tilde{c}_4 (l \cdot n_3)(l \cdot n_4) + \tilde{c}_5 (l \cdot n_3)^3 + \tilde{c}_6 (l \cdot n_4)^3 + \tilde{c}_7 (l \cdot n_\epsilon)^2 + \tilde{c}_8 (l \cdot n_\epsilon)^2 (l \cdot n_3) + \tilde{c}_9 (l \cdot n_\epsilon)^2 (l \cdot n_4).$$

Parametrization of the integrand: the two- and one-point functions

- We follow exactly the same way to derive the parametrization of two- and one-point functions. We find

$$l^\mu = (l\cdot q_1)v_1^\mu + \sum_{i=2}^{4}(l\cdot n_i)n_i^\mu + (l\cdot n_\epsilon)n_\epsilon^\mu, \qquad v_1^\mu = \frac{q_1^\mu}{\sqrt{q_1^2}}$$

$$\begin{aligned}\tilde{b}_{01}(l) = {} & \tilde{b}_0 + \tilde{b}_1(l\cdot n_2) + \tilde{b}_2(l\cdot n_3) + \tilde{b}_3(l\cdot n_4) \\ & + \tilde{b}_4((l\cdot n_2)^2 - (l\cdot n_4)^2) + \tilde{b}_5((l\cdot n_3)^2 - (l\cdot n_4)^2) + \tilde{b}_6(l\cdot n_2)(l\cdot n_3) \\ & + \tilde{b}_7(l\cdot n_3)(l\cdot n_4) + \tilde{b}_8(l\cdot n_2)(l\cdot n_4) + \tilde{b}_9(l\cdot n_\epsilon)^2,\end{aligned}$$

Note that the above parametrization is only valid if q is not a light-like vector. To deal with light-like vectors – external massless particles – we need to use a differentl parametrization

$$\tilde{a}_i(l) = \tilde{a}_0 + \tilde{a}_1(l\cdot n_1) + \tilde{a}_2(l\cdot n_2) + \tilde{a}_3(l\cdot n_3) + \tilde{a}_4(l\cdot n_4).$$

We have constructed the most general parametrization of possible numerators of various N-point functions using physical and transverse space. Why this is a useful thing to do?

To understand this, lets try to integrate one of the terms in the OPP reduction formula over the loop momentum

Usefulness of the OPP parametrization

- Recall how the OPP parametrization looks like:

$$I_N = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod_i d_i(l)} = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{\prod_i d_i(l)} \times \left\{ \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}_{i_1,i_2,i_3,i_4,i_5}(l) \prod_{j\neq[i_1,i_2,i_3,i_4,i_5]} d_j(l) \right.$$
$$+ \sum_{i_1,i_2,i_3,i_4} \tilde{d}_{i_1,i_2,i_3,i_4}(l) \prod_{j\neq[i_1,i_2,i_3,i_4]} d_j(l) + \sum_{i_1,i_2,i_3} \tilde{c}_{i_1,i_2,i_3}(l) \prod_{j\neq[i_1,i_2,i_3]} d_j(l)$$
$$\left. + \sum_{i_1,i_2} \tilde{b}_{i_1,i_2}(l) \prod_{j\neq[i_1,i_2]} d_j(l) + \sum_{i_1} \tilde{a}_{i_1}(l) \prod_{j\neq i_1} d_j(l) \right\}.$$

We take a few of the contributing terms and check to what extent the integration over the loop momentum can be performed. We begin with the 4-point function.

$$I_4 = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\tilde{d}_{0123}(l)}{d_0 d_1 d_2 d_3} \qquad d_i = (l+q_i)^2 - m_i^2, \quad q_0 = 0, \qquad q_{i=1..3} \cdot n_4 = 0$$

$$\tilde{d}_{0123} = \tilde{d}_0 + \tilde{d}_1 (l\cdot n_4) + \tilde{d}_2 (l\cdot n_\epsilon)^2 + \tilde{d}_3 (l\cdot n_\epsilon)^2 (l\cdot n_4) + \tilde{d}_4 (l\cdot n_\epsilon)^4 .$$

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l\cdot n_4}{d_0 d_1 d_2 d_3} = 0. \qquad I_4 = \tilde{d}_0 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{d_0 d_1 d_2 d_3} - \frac{i\pi^{D/2}\tilde{d}_4}{6(2\pi)^D} + \mathcal{O}(\epsilon)$$

The integration over the transverse space can be performed trivially; it removes a large number of the reduction coefficients

Usefulness of the OPP parametrization

- A similar simplification occurs for three-, two- and one-point integrals. The three-point example

$$I_3 = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\tilde{c}_{012}(l)}{d_0 d_1 d_2} \qquad d_i = (l+q_i)^2 - m_i^2, \quad q_0 = 0, \qquad q_{i=1,2} \cdot n_{3,4} = 0$$

$$\begin{aligned}\tilde{c}_{012}(l) = {} & \tilde{c}_0 + \tilde{c}_1(l\cdot n_3) + \tilde{c}_2(l\cdot n_4) + \tilde{c}_3((l\cdot n_3)^2 - (l\cdot n_4)^2) + \tilde{c}_4(l\cdot n_3)(l\cdot n_4) + \tilde{c}_5(l\cdot n_3)^3 + \tilde{c}_6(l\cdot n_4)^3 \\ & + \tilde{c}_7(l\cdot n_\epsilon)^2 + \tilde{c}_8(l\cdot n_\epsilon)^2(l\cdot n_3) + \tilde{c}_9(l\cdot n_\epsilon)^2(l\cdot n_4).\end{aligned}$$

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(l\cdot n_{3,4})^{i_{\mathrm{odd}}}}{d_0 d_1 d_2} = 0 \qquad \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(l\cdot n_3)^2 - (l\cdot n_4)^2}{d_0 d_1 d_2} = 0$$

$$I_3 = \tilde{c}_0 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{d_0 d_1 d_2} - \frac{i\pi^{D/2}\tilde{c}_7}{(2\pi)^D 2} + \mathcal{O}(\epsilon)$$

The c(l) reduction function is parametrized in terms of traceless tensors defined on the transverse space. The integration over directions of the transverse space can be done trivially

Usefulness of the OPP parametrization

- The OPP parametrization allows us to integrate over the transverse momentum spaces in the trivial way, so that we can write the result of the reduction in full generality.

$$I_N = \int \frac{\mathrm{d}^d l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod d_i(l)} = \sum_{i_1..i_5} \tilde{e}^{(0)}_{i_1..i_5} I_{i_1...i_5} + \sum_{i_1..i_4} \tilde{d}^{(0)}_{i_1..i_4} I_{i_1...i_4}$$

$$+ \sum_{i_1..3} \tilde{c}^{(0)}_{i_1 i_2 i_3} I_{i_1 i_2 i_3} + \sum_{i_1 i_2} \tilde{b}^{(0)}_{i_1 i_2} I_{i_1 i_2} + \sum_{i_1} \tilde{a}^{(0}_{i_1} I_{i_1} + \frac{i\pi^{D/2}}{(2\pi)^D} \mathcal{R} + \mathcal{O}(\epsilon)$$

$$\mathcal{R} = - \sum_{i_1 i_2 i_3 i_4} \frac{\tilde{d}^{(4)}_{i_1..i_4}}{6} - \sum_{i_1 i_2 i_3} \frac{\tilde{c}^{(7)}_{i_1 i_2 i_3}}{2} - \sum_{i_1 i_2} \left[\frac{m^2_{i_1} + m^2_{i_2}}{2} - \frac{(q_{i_1} - q_{i_2})^2}{6} \right] \tilde{b}^{(9)}_{i_1 i_2}$$

We see that, to compute any one-loop integral, we require several momentum-independent coefficients e_0, d_0, c_0, b_0, a_0 and a few terms that contribute to the rational part.

We will now turn to the discussion of how these coefficients can be computed

How to compute the reduction coefficients: five-point

- We have established that it is possible to write any relevant numerator function as

$$\mathrm{Num}(l) = \sum \tilde{e}_{i_1 i_2 i_3 i_4 i_5} \prod_{j \neq [i_1|i_5]} d_j + \tilde{d}_{i_1 i_2 i_3 i_4}(l) \prod_{j \neq [i_1|i_4]} d_j + ...$$

We now discuss how to compute the reduction functions in the above equation. We begin with the five-point coefficient e which, as we know, is momentum-independent constant

$$\tilde{e}_{01234} \Rightarrow d_i(l_c) = 0, \quad i \in [0..4]., \quad d_i = (l + q_i)^2 - m_i^2, \quad q_0 = 0.$$

$$l^\mu = \sum_{i=1}^{4} (l \cdot q_i) v_i^\mu + x n_\epsilon^\mu, \quad l \cdot q_i = \frac{1}{2}\left(d_i - d_0 - (q_i^2 - m_i^2 + m_0^2)\right)$$

$$d_i(l_c) = 0 \quad \Rightarrow \quad l \cdot q_i \to -\frac{1}{2}(q_i^2 - m_i^2 + m_0^2)$$

$$l^\mu \to l_c^\mu = V^\mu + x n_\epsilon^\mu, \quad V^\mu = -\frac{1}{2} \sum_{i=1}^{4} (q_i^2 - m_i^2 + m_0^2) v_i^\mu$$

$$l_c^2 = m_0^2 \quad \Rightarrow \quad x = \pm\sqrt{m_0^2 - V^2}.$$

$$\tilde{e}_{01234} = \frac{\mathrm{Num}(l_c)}{\prod_{i \neq 0,1,2,3,4} d_i(l_c)}$$

It is more reasonable to trade a constant, as a five-point reduction coefficient, for $(l \cdot n_\epsilon)^2$

Refining the five-point computation

- We get the constant, D-independent coefficient for the five-point because we compute regularized integrals rather than four-dimensional integrals.
- There is a price to pay: when we take the D → 4 limit, the computed five-point functions turn into a combination of the four-point functions, in accord with a general statement discussed earlier
- We have found, empirically, that this leads to numerical instabilities which can be qutie severe. To get rid of them, it is useful to change the normalization

$$\tilde{e}_{01234}(l) = \tilde{e}^{(0)}_{01234} \Rightarrow \tilde{e}^{(0)}_{01234}(l \cdot n_\epsilon)^2.$$

- With this change, the five-point integral decouples completely in D → 4 limit and does not destructively impact calculation of the four-point reduction functions

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(l \cdot n_\epsilon)^2}{d_0 d_1 d_2 d_3 d_4} = \mathcal{O}(\epsilon)$$

How to compute the reduction coefficients: four-point function

- Having computed the five-point reduction coefficient, we can re-write the numerator function in a way that is suitable for the direct extraction of the four-point coefficients

$$\mathrm{Num}(l) = \sum \tilde{e}_{i_1 i_2 i_3 i_4 i_5} \prod_{j \neq [i_1|i_5]} d_j + \tilde{d}_{i_1 i_2 i_3 i_4}(l) \prod_{j \neq [i_1|i_4]} d_j + ...$$

$$\mathrm{Num}_1(l) = \mathrm{Num}(l) - \sum \tilde{e}_{i_1 i_2 i_3 i_4 i_5} \prod_{j \neq [i_1|i_5]} d_j = \tilde{d}_{i_1 i_2 i_3 i_4}(l) \prod_{j \neq [i_1|i_4]} d_j + ...$$

The four-point function reduction coefficients can be computed using Num1 in which case there is no need to worry about leftovers of the five-point function coefficients.

Once the four-point reduction coefficients are found, they are also taken to the ``left hand side", offering the opportunity to find the three-point reduction coefficients in a simple way.

How to compute reduction coefficients : four-point function

- We need to design an algorithm that allows us to compute all the coefficients that enter the four-point reduction function . To a large extent, this is a repetition of what we just did for the five-point function.

$$\tilde{d}_{0123}(l) = \tilde{d}_0 + \tilde{d}_1 (l \cdot n_4) + \tilde{d}_2 (l \cdot n_\epsilon)^2 + \tilde{d}_3 (l \cdot n_\epsilon)^2 (l \cdot n_4) + \tilde{d}_4 (l \cdot n_\epsilon)^4 .$$

$$\tilde{d}_{0123}(l) \Rightarrow d_i(l_c) = 0, \quad i \in [0..3]., \quad d_i = (l + q_i)^2 - m_i^2, \quad q_0 = 0.$$

$$l^\mu = \sum_{i=1}^{3} (l \cdot q_i) v_i^\mu + x_\perp n_4^\mu + x n_\epsilon^\mu, \quad l \cdot q_i = \frac{1}{2}\left(d_i - d_0 - (q_i^2 - m_i^2 + m_0^2)\right)$$

$$d_i(l_c) = 0 \quad \Rightarrow \quad l \cdot q_i \to -\frac{1}{2}(q_i^2 - m_i^2 + m_0^2)$$

$$l^\mu \to l_c^\mu = V^\mu + x_\perp (\cos\phi n_4^\mu + \sin\phi n_\epsilon^\mu) \quad V^\mu = -\frac{1}{2}\sum_{i=1}^{4} (q_i^2 - m_i^2 + m_0^2) v_i^\mu$$

$$l_c^2 = m_0^2 \quad \Rightarrow \quad x_\perp = \sqrt{m_0^2 - V^2}.$$

Choose special values of ϕ to project on different reduction coefficients.

$$\tilde{d}_0 = \frac{1}{2}\left(\tilde{d}_{0123}(l_+) + \tilde{d}_{0123}(l_-)\right).$$

$$\tilde{d}_1 = \frac{1}{2\sqrt{m_0^2 - V^2}}\left(\tilde{d}_{0123}(l_+) - \tilde{d}_{0123}(l_-)\right)$$

How to compute reduction coefficients : three-point function

- For the three-point reduction coefficient we use a similar procedure

$$\tilde{c}_{012}(l) = \tilde{c}_0 + \tilde{c}_1(l\cdot n_3) + \tilde{c}_2(l\cdot n_4) + \tilde{c}_3((l\cdot n_3)^2 - (l\cdot n_4)^2) + \tilde{c}_4(l\cdot n_3)(l\cdot n_4) + \tilde{c}_5(l\cdot n_3)^3 + \tilde{c}_6(l\cdot n_4)^3$$
$$+ \tilde{c}_7(l\cdot n_\epsilon)^2 + \tilde{c}_8(l\cdot n_\epsilon)^2(l\cdot n_3) + \tilde{c}_9(l\cdot n_\epsilon)^2(l\cdot n_4).$$

$$l^\mu = \sum_{i=1}^{2}(l\cdot q_i)v_i^\mu + + x_\perp\,(\cos\phi n_3^\mu + \sin\phi n_4^\mu) + x_\epsilon n_\epsilon^\mu.$$

$$d_i(l_c) = 0 \quad \Rightarrow \quad l\cdot q_i \to -\frac{1}{2}(q_i^2 - m_i^2 + m_0^2)$$

$$l^\mu \to l_c^\mu = V^\mu + x_\perp\,(\cos\phi n_3^\mu + \sin\phi n_4^\mu) + x_\epsilon n_\epsilon^\mu \quad V^\mu = -\frac{1}{2}\sum_{i=1}^{4}(q_i^2 - m_i^2 + m_0^2)v_i^\mu$$

$$l_c^2 = m_0^2 \quad \Rightarrow \quad x_\perp^2 + x_\epsilon^2 = m_0^2 - V^2.$$

Choosing $x_\perp, x_\epsilon, \phi$ we determine all the relevant c-coefficients

Discrete Fourier transform is a useful way to solve the system of equations for ep-independent coefficients

A similar construction gives us also coefficients of the two-point and one-point functions

Discrete Fourier transform for the three-point function

- The discrete Fourier transform allows us to write simple formulas to find coefficients of the reduction function for three-points. Focus on the cut-constructible part.

$$\tilde{c}_{012}(l) = \tilde{c}_0 + \tilde{c}_1(l\cdot n_3) + \tilde{c}_2(l\cdot n_4) + \tilde{c}_3((l\cdot n_3)^2 - (l\cdot n_4)^2) + \tilde{c}_4(l\cdot n_3)(l\cdot n_4) + \tilde{c}_5(l\cdot n_3)^3 + \tilde{c}_6(l\cdot n_4)^3 + \tilde{c}_7(l\cdot n_\epsilon)^2 + \tilde{c}_8(l\cdot n_\epsilon)^2(l\cdot n_3) + \tilde{c}_9(l\cdot n_\epsilon)^2(l\cdot n_4).$$

$$l^\mu \sim +x_\perp\left(\cos\phi n_3^\mu + \sin\phi n_4^\mu\right), \quad t = e^{i\phi}$$

$$\tilde{c}_{012}(t) = \sum_{k=-3}^{3} c_k t^k$$

$$\sum_{n=0}^{k} e^{\frac{2\pi i r n}{k+1}} = \delta_{r0}(k+1), \quad r < k+1$$

$$c_m = \frac{1}{7}\sum_{n=0}^{6} \tilde{c}_{012}(t_n) t_n^{-m}$$

The discrete Fourier transform is neither unique nor superior way to solve a system of linear equations

A similar construction gives us also coefficients of the two-point and one-point functions

The OPP reduction: the re-cap

- We now summarize what we learned about one-loop reduction procedure using the OPP method

$$I_N = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod_i d_i(l)} = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{\prod_i d_i(l)} \times \left\{ \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}_{i_1,i_2,i_3,i_4,i_5}(l) \prod_{j\neq[i_1,i_2,i_3,i_4,i_5]} d_j(l) \right.$$
$$+ \sum_{i_1,i_2,i_3,i_4} \tilde{d}_{i_1,i_2,i_3,i_4}(l) \prod_{j\neq[i_1,i_2,i_3,i_4]} d_j(l) + \sum_{i_1,i_2,i_3} \tilde{c}_{i_1,i_2,i_3}(l) \prod_{j\neq[i_1,i_2,i_3]} d_j(l)$$
$$\left. + \sum_{i_1,i_2} \tilde{b}_{i_1,i_2}(l) \prod_{j\neq[i_1,i_2]} d_j(l) + \sum_{i_1} \tilde{a}_{i_1}(l) \prod_{j\neq i_1} d_j(l) \right\}.$$

$$I_N = \int \frac{\mathrm{d}^d l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod d_i(l)} = \sum_{i_1..i_5} \tilde{e}^{(0)}_{i_1..i_5} I_{i_1\ldots i_5} + \sum_{i_1..i_4} \tilde{d}^{(0)}_{i_1..i_4} I_{i_1\ldots i_4}$$
$$+ \sum_{i_1..3} \tilde{c}^{(0)}_{i_1 i_2 i_3} I_{i_1 i_2 i_3} + \sum_{i_1 i_2} \tilde{b}^{(0)}_{i_1 i_2} I_{i_1 i_2} + \sum_{i_1} \tilde{a}^{(0}_{i_1} I_{i_1} + \frac{i\pi^{D/2}}{(2\pi)^D} \mathcal{R} + \mathcal{O}(\epsilon)$$

$$\mathcal{R} = -\sum_{i_1 i_2 i_3 i_4} \frac{\tilde{d}^{(4)}_{i_1..i_4}}{6} - \sum_{i_1 i_2 i_3} \frac{\tilde{c}^{(7)}_{i_1 i_2 i_3}}{2} - \sum_{i_1 i_2} \left[\frac{m^2_{i_1} + m^2_{i_2}}{2} - \frac{(q_{i_1} - q_{i_2})^2}{6} \right] \tilde{b}^{(9)}_{i_1 i_2}$$

OPP reduction coefficients can be obtained from values of the loop momentum such that certain combinations of propagators vanish

Once the integrand-level reduction is performed, the integral-level reduction is obtained straightforwardly by means of parametric (rather than real) integration

Why the OPP procedure is important

- The OPP procedure allows us to reduce any one-loop integral to a basis set. It can be used in a calculation of Feynman diagrams as any other reduction procedure.
- It is not clear to me if the OPP applied to individual diagrams is ``better'' than say the Passarino-Veltman. OPP can be used to obtain numerical results for individual Feynman diagrams without much preparation. This is in contrast to PV reduction of high-rank high-point tensor integrals which requires quite a bit of analytic work. On the other hand, PV reduction is analytic while the OPP is numerical.
- The true strength of the OPP reduction procedure is different. Recall that the OPP-reduction coefficients can be computed using special values of the loop momenta. As we will explain now, this allows us to organize computations based on OPP in such a way that one deals directly with scattering amplitudes bypassing Feynman diagrams entirely.
- To accomplish this, scattering amplitudes need to be written in the ``color-ordered'' form. This is the next topic that we will discuss.

Color ordering for scattering amplitude

- While color algebra seems like a minor nuisance, compared to what we have to deal with anyhow, it is useful to organize calculations in QCD in such a way that color degrees of freedom factorize. Consider tree-level gluon scattering.

$$[T^a, T^b] = i\sqrt{2} f^{abc} T^c, \quad \mathrm{Tr}\left(T^a T^b\right) = \delta^{ab}$$

$$[F^a, F^b] = -F^a_{bc} F^c, \quad F^a_{bc} = -i\sqrt{2} f^{abc}, \quad \mathrm{Tr}(F^a F^b) = 2N_c \delta^{ab}.$$

Up to possible permutations, any N-gluon scattering amplitude is then given by the matrix element of product of (N-2) F-matrices $\left[F^{a_2} F^{a_3} .... F^{a_{n-1}}\right]_{a_1 a_n}$

$$\begin{aligned}\left(F^{a_2} F^{a_3} \ldots F^{a_{(n-2)}} F^{a_{(n-1)}}\right)_{a_1 a_n} &= \frac{1}{2N_c} \mathrm{Tr}\left(\left[\left[\ldots\left[\left[F^{a_1}, F^{a_2}\right], F^{a_3}\right], \ldots, F^{a_{n-2}}\right]\left[F^{a_{n-1}}, F^{a_n}\right]\right)\right. \\ &= \mathrm{Tr}\left(\left[\left[\ldots\left[\left[T^{a_1}, T^{a_2}\right], T^{a_3}\right], \ldots, T^{a_{n-2}}\right]\left[T^{a_{n-1}}, T^{a_n}\right]\right)\right.\end{aligned}.$$

$$\mathcal{A}_n^{\text{tree}} = \frac{g_s^{n-2}}{2N_c} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}\left(F^{a_{\sigma(1)}} F^{a_{\sigma(2)}} F^{a_{\sigma(3)}} \ldots F^{a_{\sigma(n)}}\right) A^{\text{tree}}_{n,\sigma},$$

$$\mathcal{A}_n^{\text{tree}} = g_s^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}\left(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} \ldots T^{a_{\sigma(n)}}\right) A^{\text{tree}}_{n,\sigma}.$$

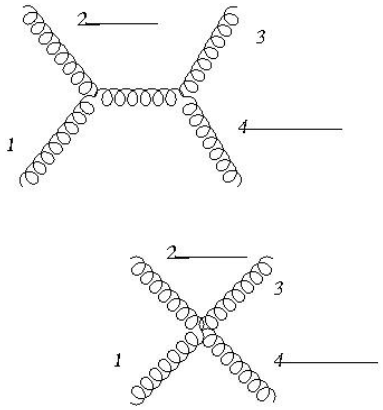
Amplitudes that multiply traces of products of F and T matrices are called color-ordered scattering amplitudes

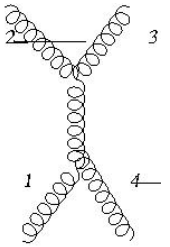
Properties of color ordered amplitudes

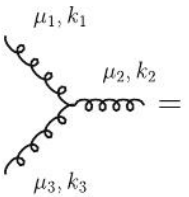
- For N-gluons, there is just one color-ordered amplitude (Bose symmetry)
- Color-ordered amplitudes are gauge-invariant
- Color-ordered amplitudes are cyclic-symmetric $m(g_1, g_2, ...g_n) = m(g_2, ..., g_n, g_1).$
- They satisfy reflection identity $m(g_1, g_2, ..g_n) = (-1)^n m(g_n, .., g_2, g_1)$
- and abelian identities, e.g

$$m_n(1, \underline{2}, \overline{3, \ldots, n}) \equiv m_n(g_1, g_2, g_3, \ldots, g_n) + m_n(g_1, g_3, g_2, \ldots, g_n) + \cdots m_n(g_1, g_3, \ldots, g_n, g_2) = 0.$$

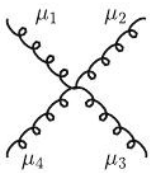
- Color-ordered amplitudes can be computed from the color-stripped Feynman rules
- Only such diagrams where physical ordering of gluons coincides with their ordering in a color-ordered amplitude $m(g_1, g_2, g_3, ...g_n)$ have to be considered
- These properties of color-ordered amplitudes are valid both at tree- and the one-loop level



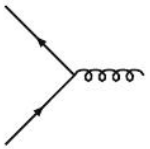




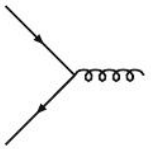
$$= i\frac{g_s}{\sqrt{2}}\left[(k_1 - k_2)_{\mu_3}\, g_{\mu_1\mu_2} + (k_2 - k_3)_{\mu_1}\, g_{\mu_2\mu_3} + (k_3 - k_1)_{\mu_2}\, g_{\mu_3\mu_1}\right]$$



$$i\frac{g_s^2}{2}\left[2g_{\mu_1\mu_3}\, g_{\mu_2\mu_4} - g_{\mu_1\mu_4}\, g_{\mu_2\mu_3} - g_{\mu_1\mu_2}\, g_{\mu_3\mu_4}\right]$$



$$= i\frac{g_s}{\sqrt{2}}\gamma_\mu$$



$$= -i\frac{g_s}{\sqrt{2}}\gamma_\mu$$

Proof of the abelian identities

- As a simple illustration of the color-ordering concept, we will prove the Abelian identities

$$m_n(1,\underline{2},\overline{3,\ldots,n}) \equiv m_n(g_1,g_2,g_3,\ldots,g_n) + m_n(g_1,g_3,g_2,\ldots,g_n) + \cdots m_n(g_1,g_3,\ldots,g_n,g_2) = 0.$$

Consider a gauge theory where the gauge group is the ``direct product'' of two groups $SU(N_1)$ and $SU(N_2)$ and consider the color-ordered amplitude in such a theory. The gluons from two different groups do not interact.

$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \qquad \left[T^a, T^b\right] = i\sqrt{2} f^{abc} T^c,$ where structure constants vanish if gluon indices belong to different groups

Take n-gluons from the first and m-gluons from the second group. The scattering amplitude is zero – no scattering – but we can write it in a sophisticated way.

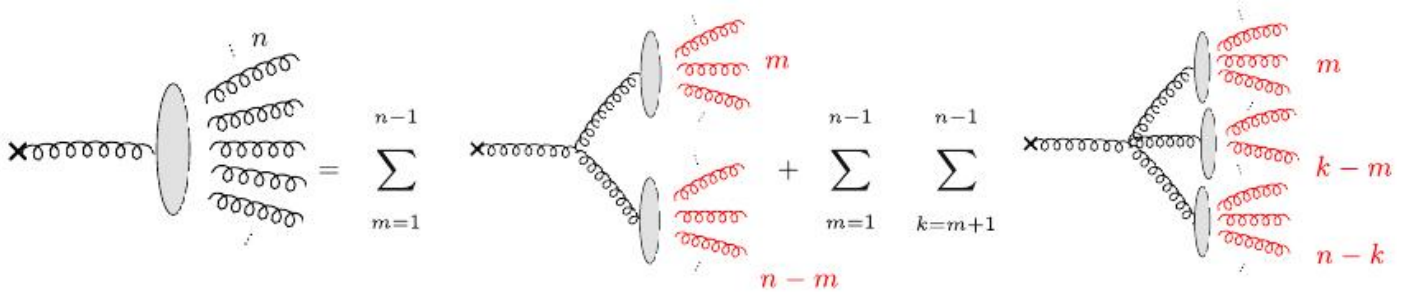
$$\mathcal{M} = \sum \mathrm{Tr}\left(T^{a_1}..T^{a_{n+m}}\right) m\left(1,....a_{n+m}\right) = 0.$$

Because generators of different SU(N)'s commute, traces are not independent. Collecting similar terms, we find that linear combinations of amplitudes that differ by relative placements of gluons that belong to two different gauge groups should vanish

$$m\left(1,\underline{2,...n},\overline{n+1,...n+m}\right) = 0$$

Calculation of color-ordered amplitudes

- Color-ordered amplitudes can be calculated both analytically and numerically. Analytic computations are based on spinor-helicity methods that are applicable in a most straightforward way for D=4 and for massless particles.
- We will see that for one-loop computations tree-level color-ordered amplitudes in higher-dimensional space-times and for massless and massive particle will play an important role. A robust framework to compute those is provided by Berends-Giele recurrence relations



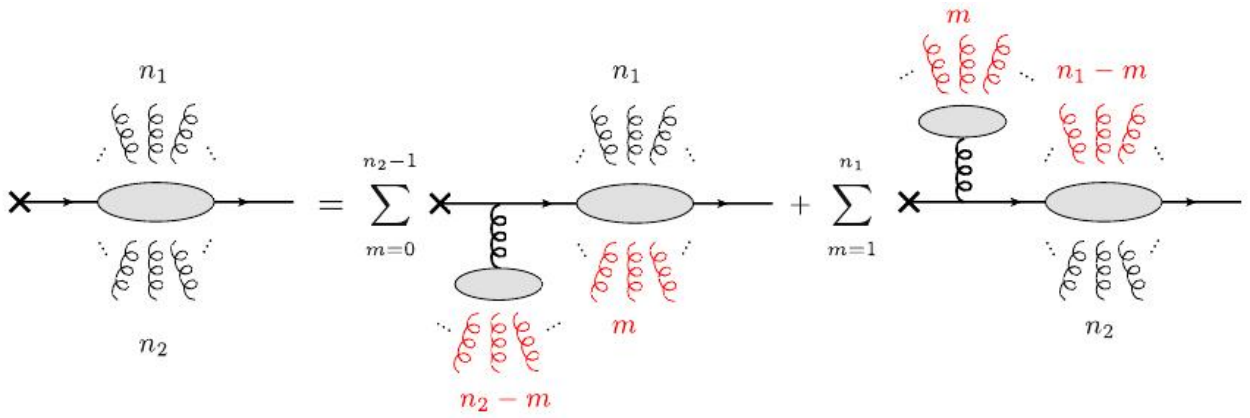
$$G^{\mu}(g_1,g_2,\ldots,g_n) = \sum_{m=1}^{n-1} V_3^{\mu\nu\rho}(-k_{1;n},k_{1;m},k_{m+1;n}) S^G_{\nu\nu'}(k_{1;m}) S^G_{\rho\rho'}(k_{m+1;n}) G^{\nu'}(g_1,g_2,\ldots,g_m) G^{\rho'}(g_{m+1},\ldots,g_n)$$
$$+ \sum_{m=1}^{n-1}\sum_{k=m+1}^{n-1} V_4^{\mu\nu\rho\sigma}(-k_{1;n},k_{1;m},k_{m+1;k},k_{k+1;n}) S^G_{\nu\nu'}(k_{1;m}) S^G_{\rho\rho'}(k_{m+1;k})$$
$$\times S^G_{\sigma\sigma'}(k_{k+1;n}) G^{\nu'}(g_1,\ldots,g_m) G^{\rho'}(g_{m+1},\ldots,g_k) G^{\sigma'}(g_{k+1},\ldots,g_n).$$

$G^{\mu}(g) = \epsilon^{\mu}$ is the initial condition for the gluon current . To obtain amplitude from the current, need to multiply with the gluon polarization vector provided that the external momentum is light-like.

$$m(g_{n+1},g_1,..g_n) = \epsilon^{\mu}(g_{n+1}) G^{\mu}(g_1,...g_n).$$

Calculation of color-ordered amplitudes

- Similar currents can be introduced for other amplitudes, for example qq + N gluons current satisfies the following relation



Note that for dealing with fermion amplitudes, color-ordering is not sufficient and one has to introduce the so-called left- and right- primitive ampltiudes

The Berends-Giele recursion relations for various currents provide a robust (masses are allowed, arbitrary space-time dimensionality is not a problem) way to calculate color-ordered amplitudes of arbitrary complexity.

Since modern incarnations of Fortran contain a notion of recursive functions, coding up recursion relations becomes very easy

BCFW relation for scattering amplitudes

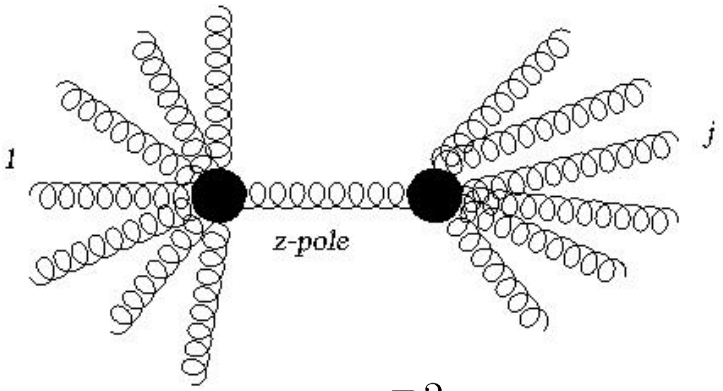
- As an example of how Berends-Giele recursion relations can be used, we will prove a particular relation for scattering amplitudes due to Britto, Cachazo, Feng and Witten .
- Consider a color-ordered N-gluon scattering amplitude $\mathcal{M}(g_1^-, g_2^{\lambda_2}, ...g_j^+, ..g_N^{\lambda_N})$

$$p_1 = \mu\,(1,0,0,1)\,, \quad p_j = \mu\,(1,0,0,-1)\,. \qquad \epsilon_1^- = \epsilon_j^+ = q/\sqrt{2} \qquad q = (0,1,i,0)$$

$$p_1 \to p_1(z) = p_1 + z\mu q, \quad p_j \to p_j(z) = p_j - z\mu q.$$

$$p_1(z)\cdot\epsilon_1^- = 0, \;\; p_j(z)\cdot\epsilon_j^+ = 0.$$

$$\int\limits_{|z|=\infty} \frac{\mathrm{d}z}{z}\mathcal{M}(z) = 0, \;\; \text{if} \;\; \lim_{z\to\infty}\mathcal{M}(z) \to 0 \qquad \Longrightarrow \qquad \mathcal{M}(0) = -\sum_{z=z_j\neq 0} \mathrm{Res}\frac{\mathcal{M}(z)}{z}.$$



$$z_j = -\frac{P^2_{\pi_1^\alpha}}{2P_{\pi_\alpha^1}\cdot q}$$

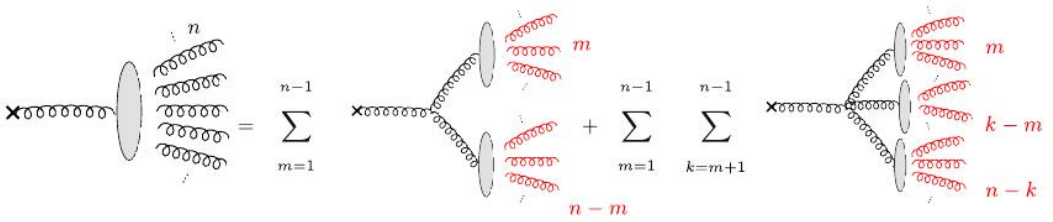
$$\mathcal{M}(0) = -i\sum_{\alpha=1}^{\alpha_{\max}}\sum_{\lambda_\pm}\frac{\mathcal{M}(\pi_1^\alpha,\lambda)\mathcal{M}(\pi_j^\alpha,-\lambda)}{P^2_{\pi_1^\alpha}}$$

The N-point amplitude is expressed through on-shell amplitudes of lower multiplicities, under the assumption that the amplitude vanishes at large z

BCFW relation for scattering amplitudes

- To prove that the amplitude vanishes at large values of z, we use Berends-Giele relations

$$\mathcal{M}(z) = \epsilon_1^\mu \mathcal{A}_1(z), \quad p_1(z)^\mu A_\mu(z) = 0, \quad \epsilon_\mu^1 \mathcal{A}^\mu(z) = -\frac{p_1^\mu \mathcal{A}_\mu(z)}{\sqrt{2}\mu z}.$$



$$H_\mu \sim (p_H + p_S)^{-2} \left(2(p_H \cdot S)H_\mu + (p_S - p_H)_\mu (H \cdot S) - 2(p_S \cdot H)S_\mu + V_{4,\mu}(S, H, S)\right)$$

$$p_H = -z\mu q + \mathcal{O}(1), \quad p_S \sim \mathcal{O}(1) \qquad \text{Assume that} \quad H \sim \mathcal{O}(1).$$

$$H_\mu \sim z^{-1} \left(2z(q \cdot S)H_\mu - z q_\mu (H \cdot S) - 2(p_s \cdot H)S_\mu\right).$$

$$H_\mu \sim (2q \cdot S)H_\mu - q_\mu (H \cdot S) + \mathcal{O}(z^{-1})$$

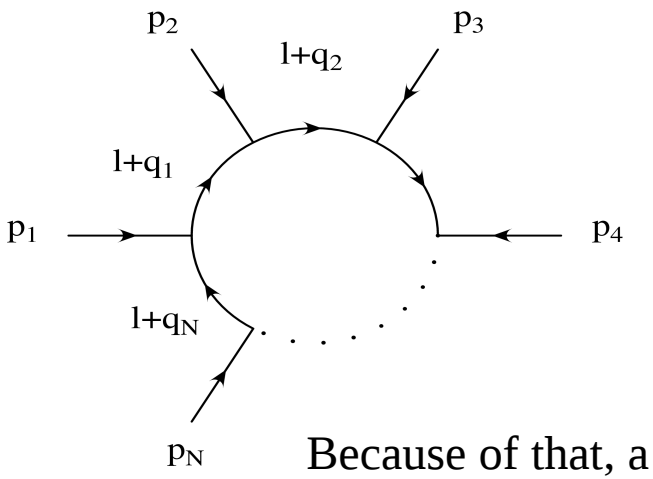
The initial condition $\quad H_\mu \sim e_\mu^j \sim q^\mu \qquad H_\mu \sim J q_\mu + \mathcal{O}(z^{-1})$

$$\mathcal{A}^\mu(z) \sim (p_H + p_S)^2 H^\mu \sim z J q^\mu + \mathcal{O}(1).$$

$$\mathcal{M}(z) = \epsilon_1^\mu \mathcal{A}^\mu(z) = -\frac{p_1^\mu \mathcal{A}^\mu(z)}{\sqrt{2}\mu z} = -\frac{J p_1 \cdot q}{\sqrt{2}\mu} + \mathcal{O}(z^{-1}) = \mathcal{O}(z^{-1}).$$

Colorless loop integrals

- The importance of color-ordering is that scattering amplitudes can be represented in a unique way. To see this note that because in a color-ordered amplitude external particles are physically ordered, there exists a well-defined ``parent diagram'' – a diagram with the largest number of propagators that depend on the loop momentum. All other diagrams can be obtained from the parent one by pinching and pulling (these operations do not change the ordering of external particles).



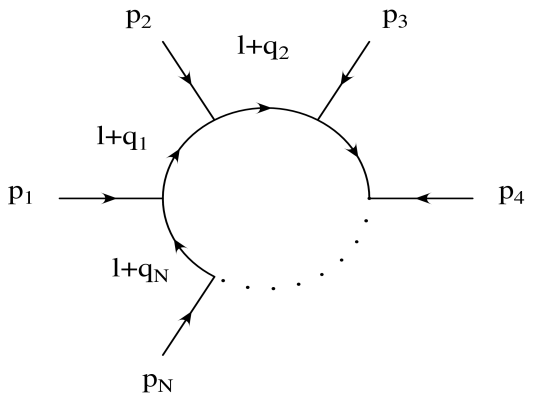
Because of that, a one-loop color-ordered amplitude has a well-defined integrand that we can write as

$$\mathcal{A}_N = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Num}(l,p,\epsilon)}{d_0 d_1 ... d_{N-1}}$$

At this point, we do know what the propagators are and we do not quite know what Num is. It turns out that we do not need to specify this explicitly since, for a one-loop computation, we need to know Num for very particular value of the loop momentum.

Colorless loop integrals, amplitudes and the OPP

- The key idea is to combine the existence of color-ordered representation for the amplitude and the OPP technology. These two things allow us to write



$$\mathcal{A}_N = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Num}(l,p,\epsilon)}{d_0 d_1 ... d_{N-1}} =$$

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \left(\sum_{i_1|i_5} \frac{\tilde{e}_{i_1 ... i_5}(l)}{d_{i_1} .. d_{i_5}} + \sum_{i_1 .. i_4} \frac{\tilde{d}_{i_1 .. i_4}(l)}{d_{i_1} .. d_{i_4}} + \sum_{i_1 .. i_3} \frac{\tilde{c}_{i_1 ... i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + ... \right)$$

Following our discussion of the OPP procedure, each of the reduction coefficients is computed from the loop momenta that forces relevant subset of propagators to vanish.

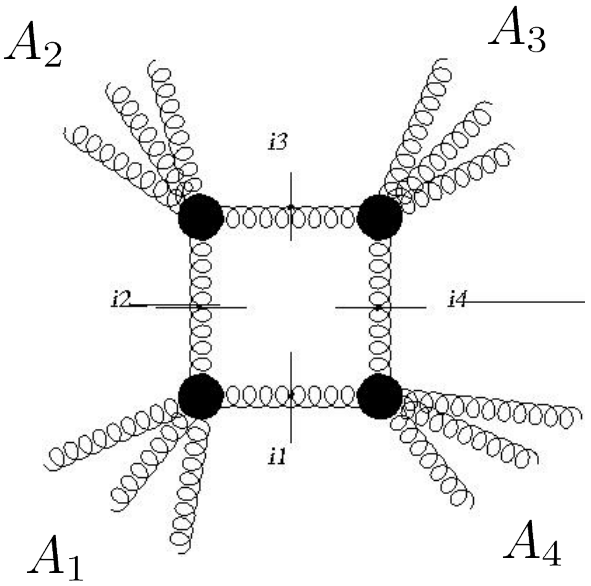
If this happens for the loop momentum l_c , the integrand factorizes into products of tree on-shell amplitudes, as the consequence of unitarity. These amplitudes are almost conventional except that they are needed for complex on-shell momenta and in D-dimensional space-time

We conclude that the OPP reduction coefficients for full color-ordered amplitudes can be obtained directly from tree-level amplitudes – no Feynman diagrams are required

$$\frac{\mathrm{Num}(k,p)}{\prod D_i} \to \prod_i A_i^{\mathrm{tree}}(\{p\},\{k_i\})$$

Four-point cut-constructible coefficient: BCF relation

- One of the beautiful consequences of this analysis is a very simple formula for the reduction coefficients of any four-point function that contributes to N-point scattering amplitude



$$d_{i_1}(l_c) = 0,\ d_{i_2}(l_c) = 0,\ d_{i_3}(l_c) = 0,\ d_{i_4}(l_c) = 0.$$

$$l_c = l_\pm = V^\mu \pm \sqrt{V^2 - m_{i_1}^2}\, n_4^\mu.$$

$$\tilde{d}_{i_1 i_2 i_3 i_4}(l) = \tilde{d}^{(0)}_{i_1 i_2 i_3 i_4} + \tilde{d}^{(1)}_{i_1 i_2 i_3 i_4}(l \cdot n_4) + \ldots$$

$$\tilde{d}_{i_1..i_4}(l_c) = \tilde{d}^0_{i_1 i_2 i_3 i_4} + \tilde{d}^1_{i_1 i_2 i_3 i_4}(n_4 \cdot l_c) = \prod_i^4 A_i(l_c)$$

$$n_4 \cdot l_\pm = \pm\sqrt{m_{i_1}^2 - V^2}$$

$$\tilde{d}^0_{i_1 i_2 i_3 i_4} = \frac{1}{2}\sum_{c=\pm} A_1(l_c) A_2(l_c) A_3(l_c) A_4(l_c)$$

We have derived a very general result for the box reduction coefficient(s) first pointed out by Britto, Cachazo and Feng

Getting the amplitude in D-dimensions

- To determine all relevant coefficients that are produced by a four-cut, we should allow the loop momentum to be D- or more precisely 5-dimensional. If we want to apply the same algorithm as before, we have to understand two things:
  - the D-dependence of the integrand
  - how calculation of tree-level scattering amplitudes in D-dim. space can be approached

D-dependence of the integrand (gluon scattering amplitudes) is linear. For one-loop computations, it is sufficient to know the function $N_1(l)$ (four-dimensional helicity scheme).

$$A_{1\text{loop}} = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{N_1(l) + (D-4)N_2(l)}{d_0 d_1 ... d_N}$$

The reduction procedure outlined above works for any integer D. To obtain N_1 we need to compute the integrand for two different values of D and take the difference . For example, D=5 and D=6 can be used.

From the described procedure, it follows that we need to be able to compute tree scattering amplitudes for higher-dimensional space-time to implement this construction

Getting the amplitude in D-dimensions

- We have the machinery to obtain amplitude in D-dimensions since Berends-Giele recursion relations are D-independent.
- The only thing we should carefully consider are the polarization degrees of freedom.
- Momenta are at most five-dimensional (external – four-dimensional, loop-momentum – five-dimensional).
- Gluon polarization states for 5-d momentum; three different polarization states

$$p = (E, p_4 \sin\theta \cos\phi, p_4 \sin\theta \sin\phi, p_4 \cos\theta, p_5), \qquad E^2 - p_4^2 = p_5^2, \qquad E > 0$$

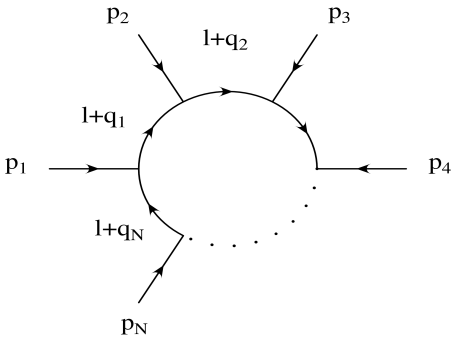
$$\epsilon^{\pm} = \frac{1}{\sqrt{2}} (0, \cos\theta \cos\phi \mp i \sin\phi, \cos\theta \sin\phi \pm i \cos\phi, -\sin\theta, 0)$$

$$\epsilon^0 = p_5^{-1} (p_4, E \sin\theta \cos\phi, E \sin\theta \sin\phi, E \cos\theta, 0) .$$

For fermions, very similar procedure can be employed ; it requires extension of the Dirac algebra and construction of spinors that are solutions of the Dirac equation in higher-dimensional space-times

The algorithm: numerical implementation

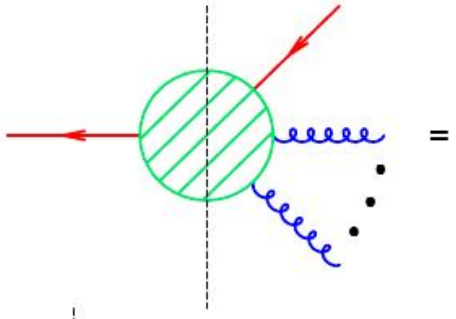
- For numerical implementation
  - fix the color-ordered amplitude; draw a parent diagram
  - specify all possible cuts that lead to non-vanishing contributions in dimensional regularization, starting with the quadruple cut
  - loop momentum on the cut assumes complex values
  - each cut produces a sum of products of certain number of tree amplitudes
  - tree-amplitudes for complex on-shell momenta are computed using Berends-Giele recursion relations
  - products of tree amplitudes provide reduction coefficients for master integrals
  - for proper treatment of ultraviolet structure of the theory, one needs to perform this procedure in higher-dimensional (integer) space-time. For pure Yang-Mills, for example, D=5 and D=6 is sufficient to reconstruct full one-loop scattering amplitude from on-shell unitarity cuts.



The procedure allows us to obtain an answer for a one-loop scattering amplitude without having to deal with Feynman diagrams AND off-shell degrees of freedom including ghosts !

Complications: charged massive particles

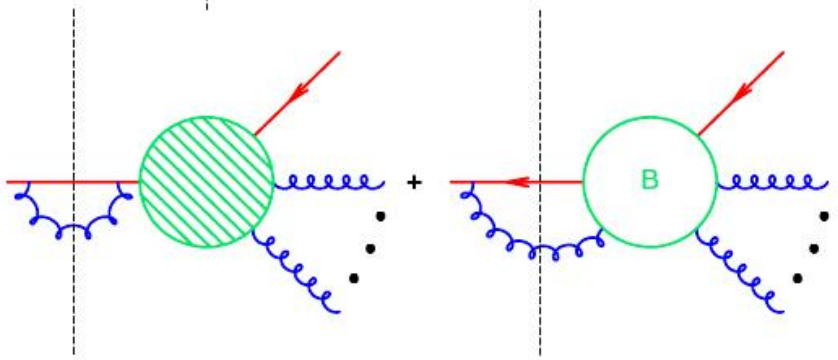
- The OPP reduction procedure is independent of whether external or internal particles have masses. However, when we put the OPP and the unitarity ideas together, a peculiarity appear.
- The reason is that one-particle reducible corrections to the external legs – which we so easily disregard when we do diagram-based computations – are part of many contributions that are required to get gauge-invariant amplitudes that contribute to a given cut.



$$\mathrm{Res}\left[\mathcal{M}(q,\{q,g\})\right] = \sum_{\text{states}} \mathcal{M}_0(q,q^*,g^*) \times \mathcal{M}(q^*,g^*,\{q,g\})$$

$$\mathcal{M}(q^*,g^*,\{q,g\}) = \frac{\mathcal{R}(p_{q^*},p_{g^*},\{q,g\})}{(p_{q^*}+p_{g^*})^2-m^2} + \mathcal{B}(p_{q^*},p_{g^*},\{q,g\}).$$

$$(p_{q^*}+p_{g^*})^2 = m^2 \ !$$

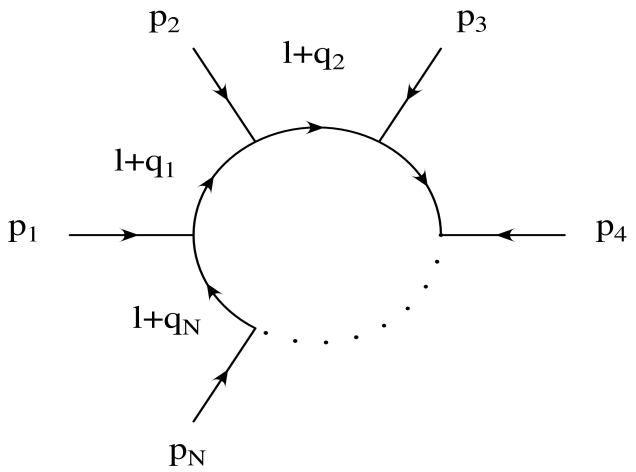


$$Z_2 = 1 - C_F g_s^2 c_\Gamma \left(\frac{\mu^2}{m^2}\right)^{\epsilon} \left(\frac{3}{\epsilon} + 5 - \eta\right)$$

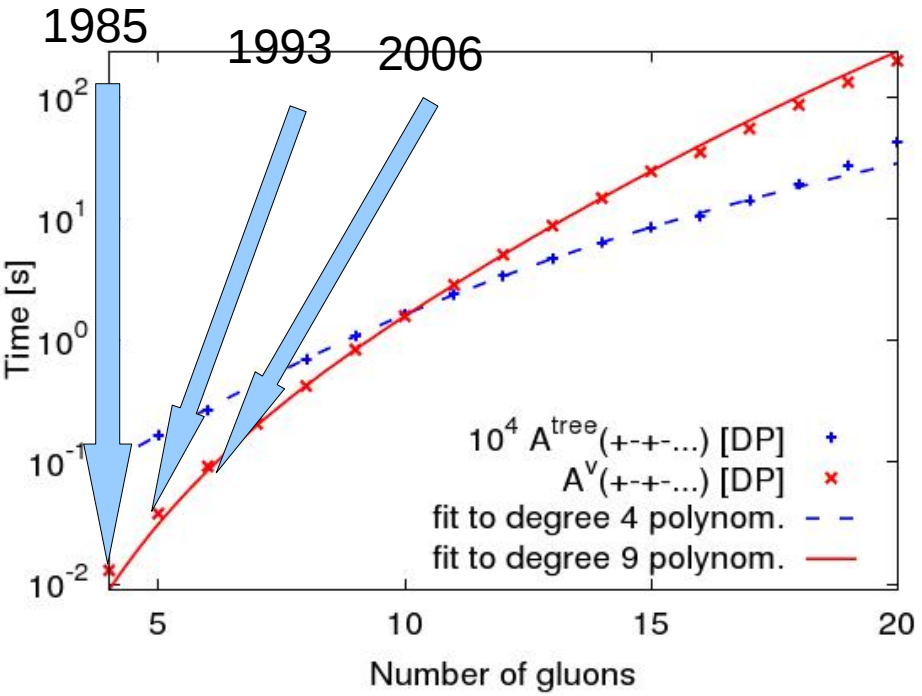
Disregarding the singular part is not a good option since the remainder is not gauge-invariant. One of the most appealing features of our construction gets, effectively, violated.

This problem is related to wave-function renormalization for massive external particles being not gauge-invariant (although it is gauge-parameter independent for covariant gauges in dim. reg.

The power of unitarity: gluon amplitudes



$$20! \approx 2.4 \times 10^{18}$$ diagrams



Giele, Zanderighi

N-gluon amplitudes can be calculated for arbitrary N. Explicit numerical results available for N through 20. Factorial growth in the number of Feynman diagrams makes this computation impossible with traditional methods.

Analytic methods

- Our discussion suggests that any scattering amplitude can be expressed as a linear combination of boxes, triangles etc. Not counting the rational part, we need to know one coefficient per master integral to derive the final result. Yet, we compute much more. Can we do better?

$$I_N = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Num}(l)}{\prod_i d_i(l)} = \sum_{i_1,i_2,i_3,i_4,i_5} \tilde{e}^{(0)}_{i_1,i_2,i_3,i_4,i_5} I_{i_1 i_2 i_3 i_4 i_5}$$
$$+ \sum_{i_1,i_2,i_3,i_4} \tilde{d}^{(0)}_{i_1,i_2,i_3,i_4} I_{i_1 i_2 i_3 i_4} + \sum_{i_1,i_2,i_3} \tilde{c}^{(0)}_{i_1,i_2,i_3} I_{i_1 i_2 i_3} + \sum_{i_1,i_2} \tilde{b}^{(0)}_{i_1,i_2} I_{i_1 i_2} + \sum_{i_1} \tilde{a}^{(0)}_{i_1} I_{i_1} + \mathcal{R}.$$

The answer to this question is provided by computational techniques developed by Forde, Mastrolia and Badger. These techniques, in princple, are more suitable for analytic, rather than numerical, construction that was the main focus of our discussion so far.

The very first example of such approaches is the BCF relation for the box reduction coefficient that we derived earlier. We would like to see if a similar formula exists for three- and two-point functions

$$\tilde{d}^0_{i_1 i_2 i_3 i_4} = \frac{1}{2} \sum_{c=\pm} A_1(l_c) A_2(l_c) A_3(l_c) A_4(l_c)$$

The three-point cut-constructible coefficient

- We consider a three-point function that corresponds to cut propagators d_0, d_1, d_2

Recall that on a three-point cut, the loop momentum is fixed up to a one-parameter ambiguity

$$l^\mu = V^\mu + (\cos\phi n_3^\mu + \sin\phi n_4^\mu) \qquad e^{i\phi} = t_+ \qquad l_+^\mu = V^\mu + l_\perp \left(t_+ n_-^\mu + t_+^{-1} n_+^\mu\right).$$

$$n_\mp = \frac{n_3 \mp i n_4}{2}, \quad n_-^2 = n_+^2 = 0, \quad 2n_- \cdot n_+ = 1$$

$$A_1(t_+)A_2(t_+)A_3(t_+) = \tilde{c}_{012i}(l_+) + \sum_{i=3}^{N} \frac{\tilde{d}^{(0)}_{012i} + \tilde{d}^{(1)}_{012i}\,(l_+ \cdot \tilde{n}_i)}{d_i(l_+)}$$

$$d_i(l_+) = (q_i + l_+)^2 = \Delta_i + 2l_\perp (q_i \cdot n_-) t_+ + 2l_\perp\,(q_i \cdot n_+)\, t_+^{-1}.$$

$$\Delta_i = q_i^2 + 2q_i \cdot V \qquad l_+ \cdot \tilde{n}_i = l_\perp \left(\tilde{n}_i \cdot n_- t_+ + \tilde{n}_i \cdot n_+ t_+^{-1}\right), \ \text{ since } \ V \cdot \tilde{n}_i = 0.$$

$$\frac{\tilde{d}^{(0)}_{012i} + \tilde{d}^{(1)}_{012i}\,(l_+ \cdot \tilde{n}_i)}{d_i(l_+)} = \tilde{d}^{(i)}_{0123} \frac{\tilde{n}_i \cdot n_-}{2q_i \cdot n_-} + \frac{r_{i,1}}{t_+ - t_i^{(1)}} + \frac{r_{i,2}}{t_+ - t_i^{(2)}}.$$

The three-point cut-constructible coefficient

- Therefore, we can write the triple-cut residue as $\tilde{c}_{012}(l_+) = \tilde{c}_{012}^{(0)} + \sum_{k=-3,k\neq 0}^{3} c_{012}^{(k)} t_+^k .$

$$A_1(t_+)A_2(t_+)A_3(t_+) = \sum_{i=3}^{N}\left(\sum_{j=1}^{2}\frac{r_{1,j}}{t_+ - t_i^{(j)}} + \tilde{d}_{012i}^{(1)}\frac{\tilde{n}_i\cdot n_-}{2q_i\cdot n_-}\right) + \tilde{c}_{012}^{(0)} + \sum_{k=-3,k\neq 0}^{k=3} c_{012}^{(k)} t_+^k$$

We see, therefore, that the triple cut is a rational function of the variable t_+. The Laurant expansion of this function at infinity can be used to obtain the required reduction coefficient of the relevant three-point function

$$\mathcal{L}_{t+,0}\left[A_1(t_+)A_2(t_+)A_3(t_+)\right] = \tilde{c}_{012}^{(0)} + \sum_{i=3}^{N}\tilde{d}_{012i}^{(1)}\frac{\tilde{n}_i\cdot n_-}{2q_i\cdot n_-}$$

$$\mathcal{L}_{t-,0}\left[A_1(t_-)A_2(t_-)A_3(t_-)\right] = \tilde{c}_{012}^{(0)} + \sum_{i=3}^{N}\tilde{d}_{012i}^{(1)}\frac{\tilde{n}_i\cdot n_+}{2q_i\cdot n_+}$$

$$\frac{\tilde{n}_i\cdot n_-}{q_i\cdot n_-} + \frac{\tilde{n}_i\cdot n_+}{q_i\cdot n_+} = \frac{\tilde{n}_i^{\mu}\omega_{\mu\nu}(q_1,q_2)q_i^{\nu}}{2(q_i\cdot n_-)(q_i\cdot n_+)} = \frac{\tilde{n}_i\cdot q_i}{2(q_i\cdot n-)(q_i\cdot n_+)} = 0$$

$$\tilde{c}_{012}^{(0)} = \frac{1}{2}\sum_{i=\pm}\mathcal{L}_{t_i,0}\left[A_1(t_i)A_2(t_i)A_3(t_i)\right].$$

As the result, we are able to find a compact result for the triplet cut of the reduction coefficient

The two-point cut-constructible coefficient

- We can use a similar framework to find the cut-constructible coefficient of the two-point function; the resulting formula is, however, more complicated.

$$b_{01}^{(0)} = \left[\mathcal{L}_{z,0}\left[\mathcal{L}_{y,\geq 0}\,[A_1A_2]^{y^m\to f_m}\right]\right] - \frac{1}{2}\sum_{i,\alpha=\pm}\left[\mathcal{L}_{z,\geq 0}[A_1A_2A_3]^{(i)}(z,y_\alpha^{(i)})\right]^{z^n\to Z(n)}$$

To understand this formula, we need to know the momentum parametrization for the double cut that is employed in writing it down. To write it down, we start from the canonical OPP formula

$$l^\mu = -\frac{1}{2}q_1^\mu + l_\perp\left(n_2^\mu\cos\theta + n_3^\mu\sin\theta\cos\phi + n_4^\mu\sin\theta\sin\phi\right)$$

$$\cos\theta = 1-2y, \quad z = t\sin\theta, \;\Rightarrow l^\mu = -\frac{q_1^\mu}{2} + l_\perp\left(n_2^\mu\,(1-2y) + zn_-^\mu + \frac{4y(1-y)}{z}n_+^\mu\right).$$

The formula for the double-cut coefficient implies that integration over y and z is performed in a particular way, by substituting powers of y and z by some constant terms. To understand why this is done, recall that dependence of the double-cut function on $\cos\theta$ is polynomial.

$$\tilde{b}_{01}(\theta) = \tilde{b}_{01}^{(0)} + \tilde{b}_{01}^{(1)}\cos\theta + \tilde{b}_{01}^{(2)}(3\cos^2\theta - 1). \qquad \text{Substitution} \qquad y^m \to \frac{1}{m+1}$$

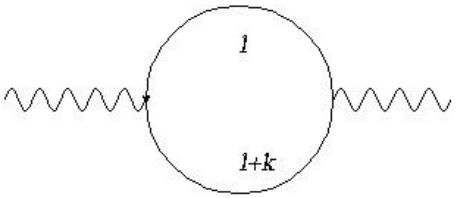
allows us to project $\tilde{b}_{01}(\theta)$ on to $\tilde{b}_{01}^{(0)}$. The z-substitutions are similar but more difficult to explain

Examples

- We will consider some examples that will illustrate concepts that we discussed in these lectures.
- We will talk about two examples that will illustrate what is needed for constructing numerical unitarity framework.
- We will be particularly interested in the desplaying the rational part which reminds very much what happens with various QFT anomalies in perturbation theory
- Time permitting, we will also talk about spinor-helicity methods and show an example of how these methods can be efficiently used in one-loop computations.

The photon mass in the Schwinger model

- We will compute the mass of the photon in two-dimensional QED (the Schwinger model)



$$\Pi_{12} = -e^2 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Tr}\left[\hat{\epsilon}_1 \hat{l} \hat{\epsilon}_2 (\hat{l}+\hat{k}))\right]}{l^2 (l+k)^2}$$

Since we deal here with massless theory, single cuts vanish identically; the non-vanishing result should come from the double-cut

$$l^2 = 0, \;\; (l+k)^2 = 0 \;\Rightarrow\;\; l^\mu = -\frac{1}{2}k^\mu + l_\perp^\mu + (l\cdot n_\epsilon) n_\epsilon^\mu, \;\;\; l_\perp^2 + (l\cdot n_\epsilon)^2 = -\frac{k^2}{4}.$$

$$\Pi_{12} = -2e^2 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{l^2 (l+k)^2} \left[2(l_\perp \cdot \epsilon)_1 (l_\perp \cdot \epsilon_2) + \frac{k^2}{2} \left(\epsilon_1 \cdot \epsilon_2 - \frac{\epsilon_1 \cdot k \epsilon_2 \cdot k}{k^2} \right) \right].$$

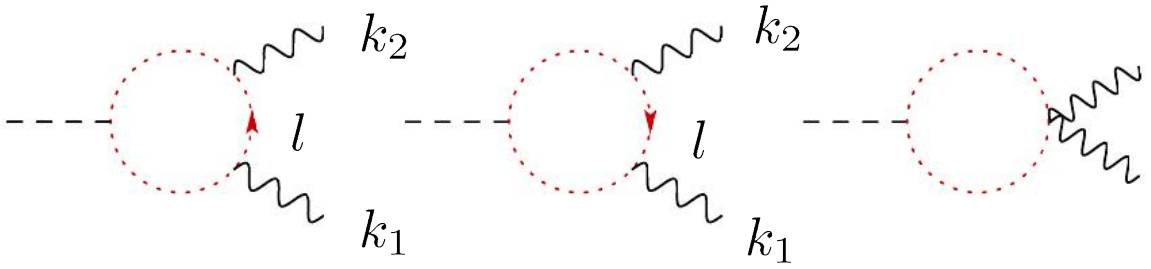
We can average over directions of the transverse momentum components

$$(l_\perp \cdot \epsilon_1)(l_\perp \cdot \epsilon_2) \to l_\perp^2 \left(\epsilon_1 \cdot \epsilon_2 - \frac{(\epsilon_1 \cdot k)(\epsilon \cdot k)}{k^2} \right) = \left(-\frac{k^2}{4} - (l\cdot n_\epsilon)^2 \right) \left(\epsilon_1 \cdot \epsilon_2 - \frac{(\epsilon_1 \cdot k)(\epsilon \cdot k)}{k^2} \right)$$

$$\Pi_{12} = -2e^2 \left(\epsilon_1 \cdot \epsilon_2 - \frac{\epsilon_1 \cdot k \epsilon_2 \cdot k}{k^2} \right) \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(l\cdot n_\epsilon)^2}{l^2 (l+k)^2} = \frac{ie^2}{\pi} \left(\epsilon_1 \epsilon_2 - \frac{\epsilon_1 \cdot k \epsilon_2 \cdot k}{k^2} \right)$$

The Higgs boson decay to two photons

- The Higgs boon (scalar particle) decay to two photons through a loop of massless scalars (the equivalence theorem regime). Note: no color ordering, so need to care about momentum assignments



$$k_{1,2} = (m_H/2, 0, 0, \pm m_H/2)\,, \quad e_{1,2} = \frac{1}{\sqrt{2}}\,(0, 1, \pm i, 0)\,.$$

The triple-cut condition for one of the diagrams: $l^2 = 0, \quad (l + k_1)^2 = 0, \quad (l - k_2)^2 = 0.$

$$l \cdot k_1 = 0, \;\; l \cdot k_2 = 0 \;\Rightarrow l^\mu = l_\perp^\mu + (l \cdot n_\epsilon) n_\epsilon^\mu, \;\; l_\perp \cdot k_{1,2} = 0.$$

$$l_\perp^2 + (l \cdot n_\epsilon)^2 = 0$$

The numerator on the triple cut is proportional to the product of $H\phi^2$ and $\phi^2\gamma$ vertices. Because photon polarization vectors are transverse, we find that the numerator is proportional to

$$4(l_\perp \cdot e_1)(l_\perp \cdot e_2) = 2 l_{\perp,\mu} l_{\perp,nu}\,(e_1^\mu e_2^\nu + e_1^\nu e_2^\mu) = -2 l_\perp^\mu l_\perp^\nu \omega_{\mu\nu}(k_1, k_2) = 2 l_\perp^2\,(e_1 \cdot e_2) = -2(l \cdot n_\epsilon)^2 (e_1 \cdot e_2).$$

$$\mathcal{A}_t = -2 g_H^2 e^2 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \left(\frac{(l \cdot n_\epsilon)^2 (e_1 \cdot e_2)}{l^2 (l + k_1)^2 (l - k_2)^2} + \frac{(l \cdot n_\epsilon)^2 (e_1 \cdot e_2)}{l^2 (l + k_2)^2 (l - k_1)^2} \right)$$

The Higgs boson decay to two photons

- Next, we need to compute the double-cut, with the Higgs boson to the left of the cut and the photons to the right

The double-cut conditions $l^2 = 0, \quad (l-K)^2 = 0, \quad K = k_1 + k_2.$

$$l^\mu = \frac{1}{2}K^\mu + l_\perp^\mu + (l\cdot n_\epsilon)n_\epsilon^\mu, \quad l_\perp \cdot K = l_\perp \cdot n_\epsilon = 0. \qquad l_\perp^2 + (l\cdot n_\epsilon)^2 = -\frac{K^2}{4} = -\frac{m_H^2}{4}.$$

$$\mathcal{I}^{(2)} = -g_H^2 e^2 \left[4(l\cdot e_1)(l\cdot e_2)\left(\frac{1}{2l\cdot k_1} + \frac{1}{2l\cdot k_2}\right) + 2e_1\cdot e_2 \right].$$

We can simplify this by using the following identities $2l\cdot k_{1,2} = \frac{m_H^2}{2} \pm m_H (l\cdot n_3).$

$$2(l\cdot e_1)(l\cdot e_2) = -e_1\cdot e_2\left[(l\cdot n_1)^2 + (l\cdot n_2)^2\right] = e_1\cdot e_2\left(l_\perp^2 + (l\cdot n_3)^2\right).$$

$$\mathcal{I}^{(2)} = 8 g_H^2 e^2 \frac{(l\cdot n_\epsilon)^2 e_1\cdot e_2}{m_H^2 - 4(n_3\cdot l)^2}$$

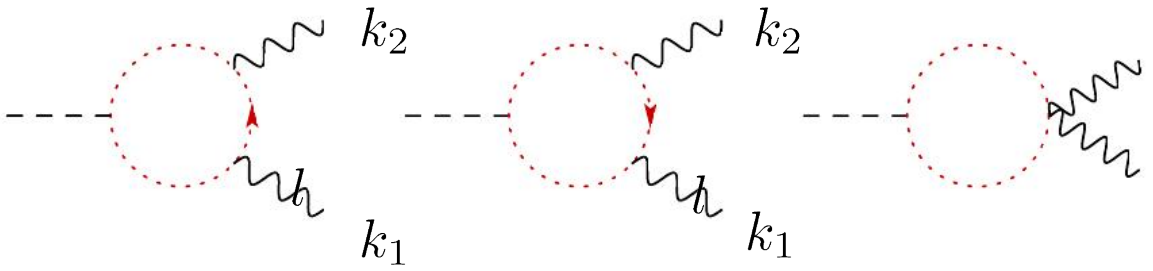
This is not yet a coefficient of the relevant two-point function because we need to subtract from it the contribution of the triple-cut computed previously

$$\mathcal{I}_{\rm subtr} = -2 g_H^2 e^2 e_1\cdot e_2 \left(\frac{(l\cdot n_\epsilon)^2}{l^2}\Big|_{l\to l-k_1} + \frac{(l\cdot n_\epsilon)^2}{l^2}\Big|_{l\to l-k_2} \right)$$

A simple computation yields $\mathcal{I}^{(2)} = \mathcal{I}_{\rm subtr}$ which implies that the reduction coefficient of the two-point function vanishes

The Higgs boson decay to two photons

- Therefore, we find that only triple cut contributes to the final result.



$$\mathcal{A}_t = -4g_H^2 e^2 (e_1 \cdot e_2) \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(l \cdot n_\epsilon)^2}{d_0 d_1 d_2} = i g_H \frac{\alpha}{2\pi} e_1 \cdot e_2$$

The amplitude for the Higgs boson to decay to two photons through the loop of massless particles is pure rational and can not be computed with four-dimensional set up . From this perspective it looks very similar to the one-loop axial anomaly (although it is not protected from higher-order corrections).

Spinor-helicity methods

- An approach that is ultimately analytics requires spinor-helicity methods. First, a reminder: in a massless theory, spinors with positive and negative helicity assume a very simple form in the Weyl representation

$$u_+(p) = \begin{bmatrix} \sqrt{p^+} \\ \sqrt{p^-}e^{i\phi_p} \\ 0 \\ 0 \end{bmatrix} \qquad u_-(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^-}e^{-i\phi} \\ -\sqrt{p^+} \end{bmatrix}$$

$$e^{\pm i\phi} = \frac{p^x \pm ip^y}{\sqrt{p_x^2 + p_y^2}} = \frac{p^x \pm ip^y}{\sqrt{p^+p^-}}, \quad p^\pm = E \pm p^z.$$

$$\bar{u_+}(p) = [0, 0, \sqrt{p^+}, \sqrt{p^-}e^{i\phi}], \quad \bar{u_-}(p) = [\sqrt{p^-}e^{i\phi}, -\sqrt{p^+}, 0, 0]$$

$$v_\pm(p) = u_\mp(p)$$

$$|i\rangle = u_+(p_i), \quad |i] = u_-(p_i) \qquad\qquad [i| = \bar{u_+}(p_i), \quad \langle i| = \bar{u_-}(p_i).$$

$$[ij\rangle = \langle ij] = 0 \qquad \langle ij\rangle = -\langle ji\rangle = \sqrt{|s_{ij}|}e^{i\phi_{ij}} \quad [ij] = -[ji] = -\sqrt{|s_{ij}|}e^{-i\phi_{ij}}$$

Useful identities

- The spinor products satisfy some useful identities

$$\langle ab\rangle\langle cd\rangle = \langle ac\rangle\langle bd\rangle + \langle ad\rangle\langle cb\rangle \qquad [ab][cd] = [ac][bd] + [ad][cb]$$

$$p_i = |i]|i\rangle + |i\rangle[i|$$

$$\langle pq\rangle = \langle p-|q+\rangle, \qquad [pq] = \langle p+|q-\rangle$$
$$\langle p\pm|\gamma_\mu|p\pm\rangle = 2p_\mu$$
$$\langle p+|q+\rangle = \langle p-|q-\rangle = \langle pp\rangle = [pp] = 0$$
$$\langle pq\rangle = -\langle qp\rangle, \qquad [pq] = -[qp]$$
$$2|p\pm\rangle\langle q\pm| = \frac{1}{2}(1\pm\gamma_5)\gamma^\mu\langle q\pm|\gamma_\mu|p\pm\rangle$$
$$\langle pq\rangle^* = -\mathrm{sign}(p\cdot q)[pq] = \mathrm{sign}(p\cdot q)[qp]$$
$$|\langle pq\rangle|^2 = \langle pq\rangle\langle pq\rangle^* = 2|p\cdot q| \equiv |s_{pq}|$$
$$\langle pq\rangle[qp] = 2p\cdot q \equiv s_{pq}$$
$$\langle p\pm|\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}}|q\pm\rangle = \langle q\mp|\gamma_{\mu_{2n+1}}\cdots\gamma_{\mu_1}|p\mp\rangle$$
$$\langle p\pm|\gamma_{\mu_1}\cdots\gamma_{\mu_{2n}}|q\mp\rangle = -\langle q\pm|\gamma_{\mu_{2n}}\cdots\gamma_{\mu_1}|p\mp\rangle$$
$$\langle a+|\gamma_\mu|b+\rangle\langle c-|\gamma^\mu|d-\rangle = 2[ad]\langle cb\rangle, \text{ (Fierz)}$$

$$\epsilon_\mu^\pm(k) = \pm\frac{\langle k\pm|\gamma_\mu|b\pm\rangle}{\sqrt{2}\langle b\mp|k\pm\rangle}$$

$$\gamma^\mu\epsilon_\mu^+(k,b) = \sqrt{2}\frac{(|k]\langle b| + |b\rangle[k|)}{\langle bk\rangle}$$

$$\gamma^\mu\epsilon_\mu^-(k,b) = \sqrt{2}\frac{(|k\rangle[b| + |b]\langle k|)}{[kb]}$$

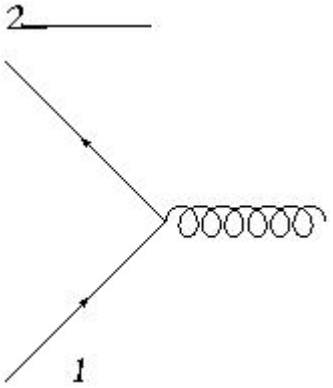
$$\langle a\pm|\gamma^\mu|b\pm\rangle\gamma_\mu = 2\Big[|a\mp\rangle\langle b\mp| + |b\pm\rangle\langle a\pm|\Big], \text{ (Fierz + Charge conjugation)}$$

$$\langle ab\rangle\langle cd\rangle = \langle ad\rangle\langle cb\rangle + \langle ac\rangle\langle bd\rangle. \text{ (Schouten)}$$

These and other identities can be used to simplify results of the calculations, in certain cases in a quite dramatic way

Three particle amplitudes

- The simplest thing we can do with these spinors is to compute the three-particle scattering amplitudes; they are important building blocks for unitarity-based computations.



$$m(\bar{q}_1^-, q_2^+, g_3^+) = \frac{ig_s}{\sqrt{2}}\bar{u}_+(p_2)\hat{\epsilon}_3^+ u_+(p_1) = ig_s\frac{[23]\langle b1\rangle}{\langle b3\rangle}.$$

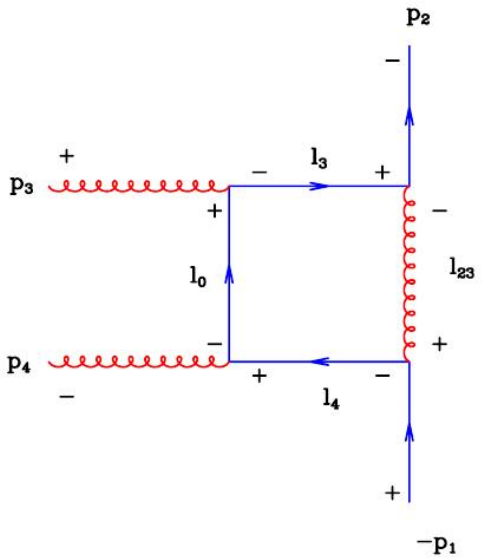
$$\hat{1} + \hat{2} + \hat{3} = 0 \Rightarrow \langle b1\rangle[12] + \langle b3\rangle[32] = 0 \Rightarrow \frac{\langle b1\rangle}{\langle b3\rangle} = -\frac{[32]}{[12]}$$

$$m(\bar{q}_1^-, q_2^+, g_3^+) = ig_s\frac{[23]^2}{[12]}, \quad m(\bar{q}_1^-, q_2^+, g_3^-) = ig_s\frac{\langle 31\rangle^2}{\langle 12\rangle}$$

$$m(\bar{q}_1^+, q_2^-, g_3^+) = -ig_2\frac{\langle 23\rangle^2}{\langle 12\rangle}, \quad m(\bar{q}_1^+, q_2^-, g_3^+) = -ig_s\frac{[31]^2}{[12]}$$

The primitive amplitude for qqgg scattering

- I would like to describe computation of the maximally abelian amplitude foe qqgg scattering using unitarity and the spinor helicity methods. The corresponding parent diagram for this primitive amplitude is shown below.
- We can deduce the master integrals that are needed to compute this amplitude: there is one box, two triangles, two bubbbles and the rational part.



$$
\begin{aligned}
I_4(0,0,0,0;s_{12},s_{23};0,0,0,0) &\sim D_0(p_1,p_2,p_3,0,0,0,0),\\
I_3(0,0,s_{12};0,0,0) &\sim C_0(p_1,p_2,0,0,0) = C_0(p_{12},p_3,0,0,0),\\
I_3(0,0,s_{23};0,0,0) &\sim C_0(p_2,p_3,0,0,0) = C_0(p_1,p_{23},0,0,0),\\
I_2(s_{12},0,0) &\sim B_0(p_{12},0,0),\\
I_2(s_{23},0,0) &\sim B_0(p_{23},0,0).
\end{aligned}
$$

$$
\begin{aligned}
-i\tilde{m}_4^{(1)}(\bar{q}_1,g_4,g_3,q_2) = {}& \tilde{d}_0\, I_4(0,0,0,0;s_{12},s_{23};0,0,0,0)\\
&+ \tilde{c}_0^{(12)}\, I_3(0,0,s_{12};0,0,0) + \tilde{c}_0^{(23)}\, I_3(0,0,s_{23};0,0,0)\\
&+ \tilde{b}_0^{(12)}\, I_2(s_{12};0,0) + \tilde{b}_0^{(23)}\, I_2(s_{23};0,0) + \mathcal{R},
\end{aligned}
$$

The reduction coefficients can be further constrained using known divergences of the amplitude and divergences of integrals

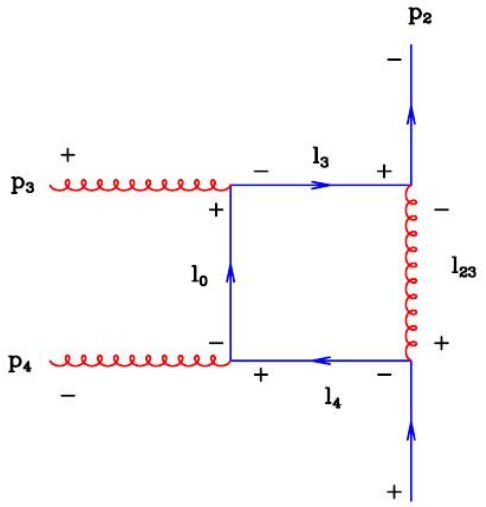
$$
\tilde{m}_4^{(1)}(\bar{q}_1,g_4,g_3,q_2) = -m_4(\bar{q}_1,q_2,g_3,g_4)\left[\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{3}{2} + L_{12}\right)\right)\right] + \mathcal{O}(\epsilon^0),
$$

where $L_{12} = \ln\left(\mu^2/(-s_{12}-i0)\right), s_{12} = 2p_1\cdot p_2.$

Divergences fix coefficients of all integrals except the box and one two-point function. These are the two things that we need to compute

The box reduction coefficient

- We will discuss how to compute the box reduction coefficient. The general principle we know – the result should be given by the product of four tree amplitudes computed at the four-cut



$$l_0 = l, \quad l_3 = l_0 - p_3, \quad l_{23} = l_0 - p_2 - p_3 \quad l_4 = l_0 + p_4.$$

$$l_0^\mu = \alpha p_3^\mu + \beta p_4^\mu + \gamma \epsilon_{34}^\mu + \delta \epsilon_{43}^\mu \qquad \epsilon_{ij} = \frac{1}{2}\langle i|\gamma^\mu|j]$$

$$e_{34} \cdot e_{34} = 0, \quad e_{34} \cdot e_{43} = -s_{ij}/2.$$

$$l_3^2 = 0 \Rightarrow \beta = 0 \qquad l_4^2 = 0 \Rightarrow \alpha = 0$$

$$l_0^2 = 0 \Rightarrow \gamma\delta = 0$$

$$\Rightarrow \delta = 0 \Rightarrow \qquad l_{23}^2 = 0 \Rightarrow \gamma = \frac{[23]}{[24]} \qquad l_0 = \gamma \epsilon_{34}$$

$$\Rightarrow \gamma = 0 \Rightarrow l_{23}^2 = 0 \Rightarrow \delta = \frac{\langle 23\rangle}{\langle 24\rangle} \qquad l_0 = \delta \epsilon_{43}$$

Consider first the case of equal gluon helicities – the box reduction coefficient vanishes

$$m\left(\bar{q}, q^-, \epsilon_3^+\right) \sim [4l_4]^2, \quad m\left(\bar{q}, q^-, \epsilon^4+\right) \sim [3l_0]^2.$$

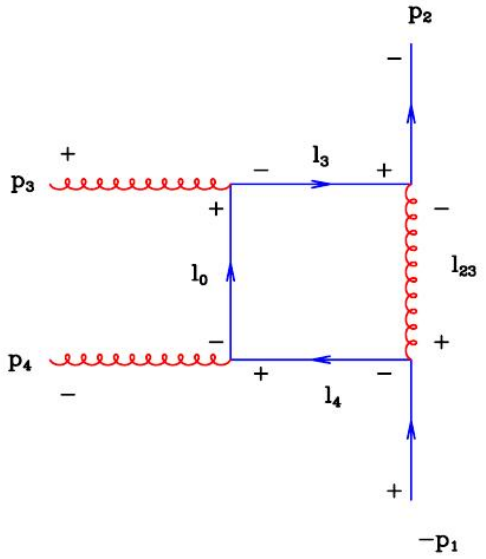
The box reduction coefficient is the product of these amplitudes. We will now show that this product must vanish

The box reduction coefficient

- Vanishing of the box coefficient for equal heliciities:

$$m\left(\bar{q},q^{-},\epsilon_{3}^{+}\right)\sim[4l_4]^2,\quad m\left(\bar{q},q^{-},\epsilon^{4}+\right)\sim[3l_0]^2.\quad \tilde{d}_0\sim[4l_4]^2[3l_0]^2=[4l_0]^2[3l_0]^2$$

$$l_0=\frac{\gamma}{2}\langle 3|\gamma^{\mu}|4]\Rightarrow p_{3,4}\cdot l_0=0\Rightarrow[4l_0]\langle l_0 4\rangle=0 \text{ and } \Rightarrow[3l_0]\langle l_0 3\rangle=0.$$



Hence, to have d_0 none zero, $\langle l_0 4\rangle$ and $\langle l_0 3\rangle$ have to vanish.
But this is not possible for generic p_3 and p_4

$$\hat{l}_0=|l_0]\langle l_0|+|l_0\rangle[l_0|=\frac{\gamma}{2}\langle 3|\gamma^{\mu}|4]\gamma_{\mu}=\gamma\left[|3\rangle[4|+|4]\langle 3|\right).$$

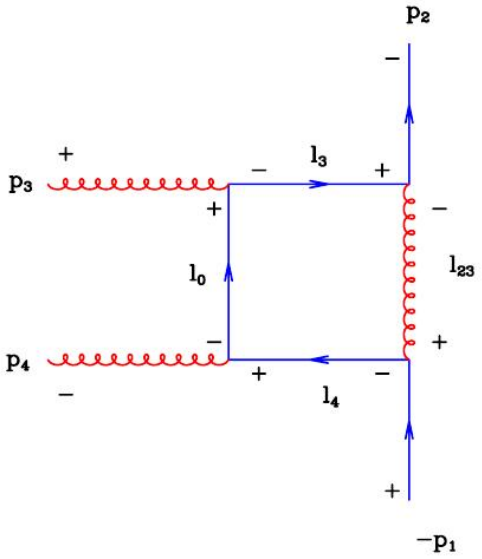
$$\hat{l}_0|3\rangle=|l_0]\langle l_0 3\rangle=0,\quad \hat{l}_0|4\rangle=|l_0]\langle l_0 4\rangle=\gamma|4]\langle 34\rangle$$

$$\tilde{d}_0\sim[4l_4]^2[3l_0]^2=[4l_0]^2[3l_0]^2=0$$

We conclude that the box reduction coefficient of the (- + + -) ampltiude vanishes

The box reduction coefficient

- We will now compute the reduction coefficient for (+ - + -) amplitude



$$\tilde{d}_0 = \frac{1}{2}\frac{\langle l_2 32\rangle^2}{\langle 2 l_3\rangle} \times \frac{[3 l_0]^2}{[l_3 l_0]} \times \frac{\langle l_0 4\rangle^2}{\langle l_0 l_4\rangle} \times \frac{[1 l_{23}]^2}{[1 l_4]}$$

This expression can be simplified using the completeness relation, e.g.

$$[3 l_0]\langle l_0|4\rangle = [3|\hat{l}_0 4\rangle = \gamma[3|\left(|3\rangle[4| + |4]\langle 3|\right)|4\rangle = \gamma[34]\langle 34\rangle.$$

$$\begin{aligned}\tilde{d}_0(\bar{q}_1^+, g_4^-, g_3^+, q_2^-) &= \frac{\langle 2|\not{l}_{23}|1]^2\langle 4|\not{l}_0|3]^2}{\langle 2|\not{l}_3\not{l}_0\not{l}_4|1]} = \frac{\langle 2|\not{l}_0-\not{\beta}|1]^2\langle 4|\not{l}_0|3]^2}{\langle 2\,3\rangle[3|\not{l}_0|4\rangle[41]} \\ &= \frac{(\gamma\langle 2\,3\rangle[4\,1] - \langle 2\,3\rangle[3\,1])^2\gamma\langle 4\,3\rangle[4\,3]}{\langle 2\,3\rangle[4\,1]},\end{aligned}$$

$$\begin{aligned}\tilde{d}_0(\bar{q}_1^+, g_4^-, g_3^+, q_2^-) &= \frac{1}{2}\frac{\langle 2\,3\rangle^2([2\,3][4\,1] - [2\,4][3\,1])^2[2\,3]\langle 4\,3\rangle[4\,3]}{\langle 2\,3\rangle[2\,4]^3[4\,1]} \\ &= \frac{1}{2}\frac{\langle 2\,3\rangle[2\,1]^2[2\,3]\langle 4\,3\rangle[4\,3]^3}{[2\,4]^3[4\,1]} \\ &= \frac{1}{2}\frac{\langle 2\,3\rangle\langle 2\,1\rangle[2\,1]^2[2\,3]\langle 4\,3\rangle[4\,3]^3}{[2\,4]^3[4\,1]\langle 2\,1\rangle} \\ &= \frac{1}{2}\frac{\langle 2\,1\rangle[2\,1]^2[2\,3]\langle 4\,3\rangle[4\,3]^2}{[2\,4]^3} \\ &= \frac{1}{2}\frac{s_{12}^2 s_{23}[2\,1][4\,3]}{\langle 2\,3\rangle[2\,4]^3} = \frac{1}{2}\frac{s_{12}s_{23}[2\,1]^2[3\,4]^2}{[2\,4]^3[4\,1]}.\end{aligned}$$