Part 2:
Cosmological perturbations and CMB

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Outline of lecture 1 of part 2

- Preliminaries
  - Conformal times of various epochs
  - Perturbations: helicity (Lifshitz) decomposition
- Warm up: tensor modes
  - Superhorizon and subhorizon regimes
  - Superhorizon regime: constant and decaying modes
  - Solution inside the horizon
- Scalar perturbations: equations.
- First glimpse: perturbations in dominant component
  - Radiation at radiation domination
  - Dark matter at matter domination
- Summary

Preliminaries

- Conformal time η:

$$dt = a(\eta) d\eta$$

Metric in conformal coordinates:

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2 = a^2(\eta)[d\eta^2 - d\mathbf{x}^2]$$

Convenient: light travels along light cone $ds = 0 \Longrightarrow dx = d\eta$, exactly like in Minkowski space-time.

  - η = coordinate size of horizon at time η. Physical size at that time $= a(\eta) \cdot \eta$. Comoving size (seen today) $= a_0 \cdot \eta$.
  - Hubble parameter

$$H = \frac{da/dt}{a} = \frac{a'}{a^2}$$

- Friedmann equation in conformal time (prime $= \partial/\partial\eta$)

$$\frac{a'^2}{a^4} = \frac{8\pi}{3} G\rho$$

- Solutions:
  - Radiation domination, RD: $\rho \propto a^{-4} \Longrightarrow a(\eta) = \text{const} \cdot \eta$, $a(t) \propto t^{1/2}$ NB: $p_{rad} = \rho_{rad}/3$
  - Matter domination, MD: $\rho \propto a^{-3} \Longrightarrow a(\eta) = \text{const} \cdot \eta^2$, $a(t) \propto t^{2/3}$

In either case,

$$\frac{a'}{a} \sim \frac{1}{\eta}, \qquad H \sim \frac{1}{a\eta}$$

Conformal times of various epochs

$$\eta = \int_0^a \frac{da}{a^2} \frac{1}{H(a)} = \int_\infty^z \frac{dz}{a_0 H(z)}$$

where $1 + z = a_0/a$. Use

$$H = H_0 \sqrt{\Omega_\Lambda + \Omega_M \left(\frac{a}{a_0}\right)^3 + \Omega_{rad} \left(\frac{a}{a_0}\right)^4}$$

and find

$$\eta = \frac{1}{a_0 H_0} \int_\infty^z \frac{1}{\sqrt{\Omega_\Lambda + \Omega_M (1+z)^3 + \Omega_{rad} (1+z)^4}}$$

Recall

$$\Omega_\Lambda = 0.72\,, \qquad \Omega_M = 0.28\,, \qquad \Omega_{rad} = 8.4 \cdot 10^{-5}$$

(neutrinos are massless for our purposes).

- Equality: transition from radiation domination to matter domination, $1 + z_{eq} = \Omega_M/\Omega_{rad} = 3200$,

$$\eta_{eq} a_0 = 120 \text{ Mpc}$$

- Photon last scattering $\approx$ recombination: $z = 1100$,

$$\eta_r a_0 = 280 \text{ Mpc}$$

- Today

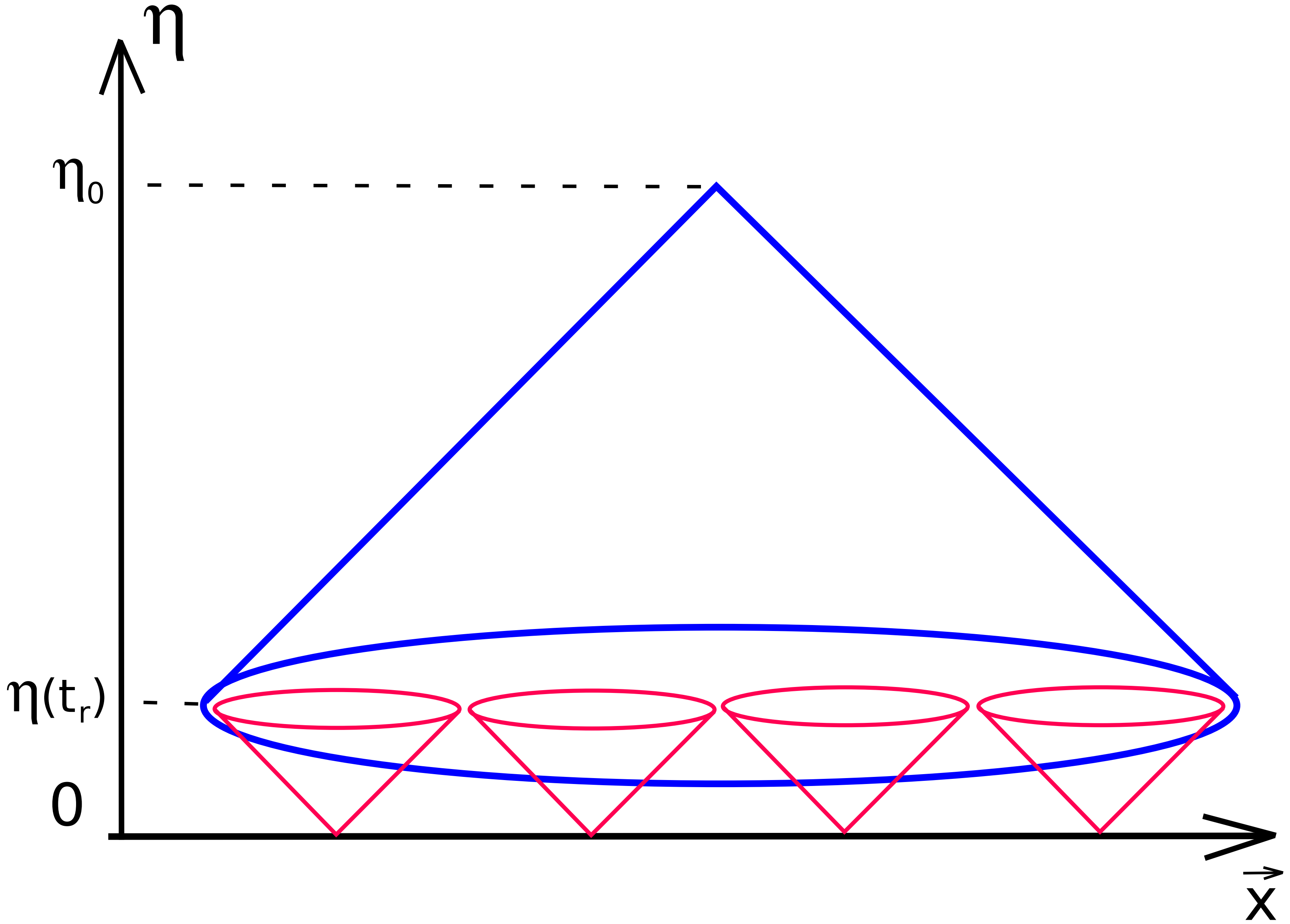
$$\eta_0 a_0 = 14\,000 \text{ Mpc}$$

NB: 1 Mpc = 3 M light yrs.

Important numbers:

$$\frac{\eta_0}{\eta_r} = 50\,, \qquad \frac{\eta_0}{\eta_{eq}} = 120$$

NB: We see 50^3 regions that had horizon size at recombination.



η
η₀
η(t_r)
0
x⃗

Perturbations: helicity (Lifshitz) decomposition

Perturbations are small in amplitude until structure starts forming. Definitely small at recombination, $\delta\rho/\rho \sim \delta T/T \sim 10^{-4} - 10^{-5} \Longrightarrow$ Linearized theory appropriate

Linearized Einstein equations

$$\delta R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\, \delta R = 8\pi G\, \delta T^{\mu}_{\nu}$$

Plus linearized equations of covariant conservation of energy-mometum

$$\delta(\nabla_{\sigma} T^{\sigma}_{\mu}) = 0$$

NB: Several components interacting gravitationally only: right hand side of Einstein eqs. involves sum of all components; covariant conservation holds for each component separately.

- Perturbations in energy-momentum tensor of matter, in ideal fluid approximation (otherwise spatial components T_{ij} contain anisotropic stress Π_{ij}, with $\mathrm{Tr}\Pi \equiv \Pi_{ii} = 0$)

$$T_{\mu\nu} = (\rho + p)u^\mu u^\nu - p g_{\mu\nu}$$

  Perturbations of energy density $\delta\rho$, pressure δp and physical velocity $v^i = a(\eta)u^i = a(\eta)dx^i/ds$, $i = 1,2,3$ (since $g_{\mu\nu}u^\mu u^\nu = 1$, component u^0 is not independent).

NB: Effects beyond ideal fluid approximation important, especially for short wavelengths and for neutrinos. Some will be pointed out later on.

- Perturbations of metric

$$ds^2 = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$$

- Background is invariant under spatial translations $\Longrightarrow$ go to 3d Fourier space,

$$h_{\mu\nu}(\eta,\mathbf{x}) = \int d^3k \, \mathrm{e}^{i\mathbf{kx}} h_{\mu\nu}(\eta,\mathbf{k}) \; , \qquad \text{same for} \;\; \delta\rho \; , \;\; \delta p \; , \;\; \mathbf{v}$$

  NB: $\mathbf{k}$ is conformal (coordinate) momentum, constant in time. Physical momentum $\mathbf{p} = \mathbf{k}/a(\eta)$ gets redshifted.

- For given $\mathbf{k}$, there remains unbroken $SO(2)$ of rotations around $\mathbf{k}$ $\Longrightarrow$ decompose into its representations $\Longrightarrow$ helicity decomposition.

- Helicity ± 2: tensor modes, transverse traceless 3d tensors. Only h_{ij} (property of ideal fluid approximation),

$$k_i h_{ij}^{TT} = 0 \qquad h_{ii}^{TT} = 0$$

  Two polarizations, $h_{ij} = e_{ij}^{(\times)} h^{(\times)} + e_{ij}^{(+)} h^{(+)}$.

- Helicity ± 1: vector modes, transverse 3d vectors. v_i^T, h_{0i}^T, $h_{ij} = k_i W_j^T + k_j W_i^T$, with $k_i v_i^T = 0$, etc., two polarizations.

  Vector modes = rotational motion of cosmic medium. Parametrized by vorticity v_i^T. Its amplitude (if present initially) decays as $v_i^T \propto a^{-1}(\eta)$ (angular momentum conservation in expanding Universe) $\Longrightarrow$ Vector modes most probably irrelevant. We are not going to consider vector modes.

- Helicity 0: scalar modes, 3d scalars. $\delta\rho$, δp, $v_i = ik_i v$, $h_{00} = 2\Phi$, $h_{0i} = k_i Z$, $h_{ij} = -2\Psi \cdot \delta_{ij} + k_i k_j E$.

  NB: v = velocity potential, $v_i(\mathbf{x}) = \partial_i v(\mathbf{x})$

Warm up: tensor modes

Tensor modes: $\delta\rho = \delta p = 0,\ v_i = 0,\ h_{0i} = h_{00} = 0,\ \Longrightarrow\ \delta T^{\mu}_{\nu} = 0$ (in ideal fluid approximation only).

$$h_{ij} = h_{ij}^{TT} = \sum_{A=\times,+} h^{(A)} e_{ij}^{(A)}$$

Each polarization has the same action as massless scalar field in expanding Universe (modulo prefactor):

$$S = \frac{1}{64\pi G} \int d^4x \sqrt{-\bar{g}}\, \bar{g}^{\mu\nu} \partial_\mu h^{(A)} \partial_\nu h^{(A)}$$

where $\bar{g}_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$ = unperturbed metric.

Explicitly

$$S = \frac{1}{64\pi G}\int d^4x\, a^2(\eta)\left[\left(\partial_\eta h^{(A)}\right)^2 - \left(\partial_i h^{(A)}\right)^2\right]$$

Field equation

$$\partial_\eta^2 h^{(A)} + 2\frac{a'}{a}\partial_\eta h^{(A)} - \partial_i\partial_i h^{(A)} = 0$$

or in 3d momentum representation

$$\partial_\eta^2 h^{(A)} + 2\frac{a'}{a}\partial_\eta h^{(A)} + k^2 h^{(A)} = 0$$

Different behaviour for $k \ll a'/a$ and $k \gg a'/a$.

Recall physical momentum $p = k/a$ and $H = a'/a^2 \Longrightarrow$

These are regimes $p \ll H$ and $p \gg H$, or $\lambda \gg H^{-1}$ and $\lambda \ll H^{-1}$, subhorizon and superhorizon, respectively.

At RD, MD epochs

$$a(\eta) \propto \eta, \eta^2 \quad \Longrightarrow \quad a'/a \propto \eta^{-1} \; ,$$

large at early times $\Longrightarrow$ mode of given conformal momentum $\mathbf{k}$ is first superhorizon and later superhorizon.

In other words, $H(t) \propto t^{-1}$ decreases faster than $p(t) = k/a(t) \propto t^{-1/2}, \;\; t^{-2/3} \Longrightarrow p \ll H$ at early times

Fig.

NB: Cosmologically interesting scales entered horizon quite late: at horizon crossing time $\eta_\times$

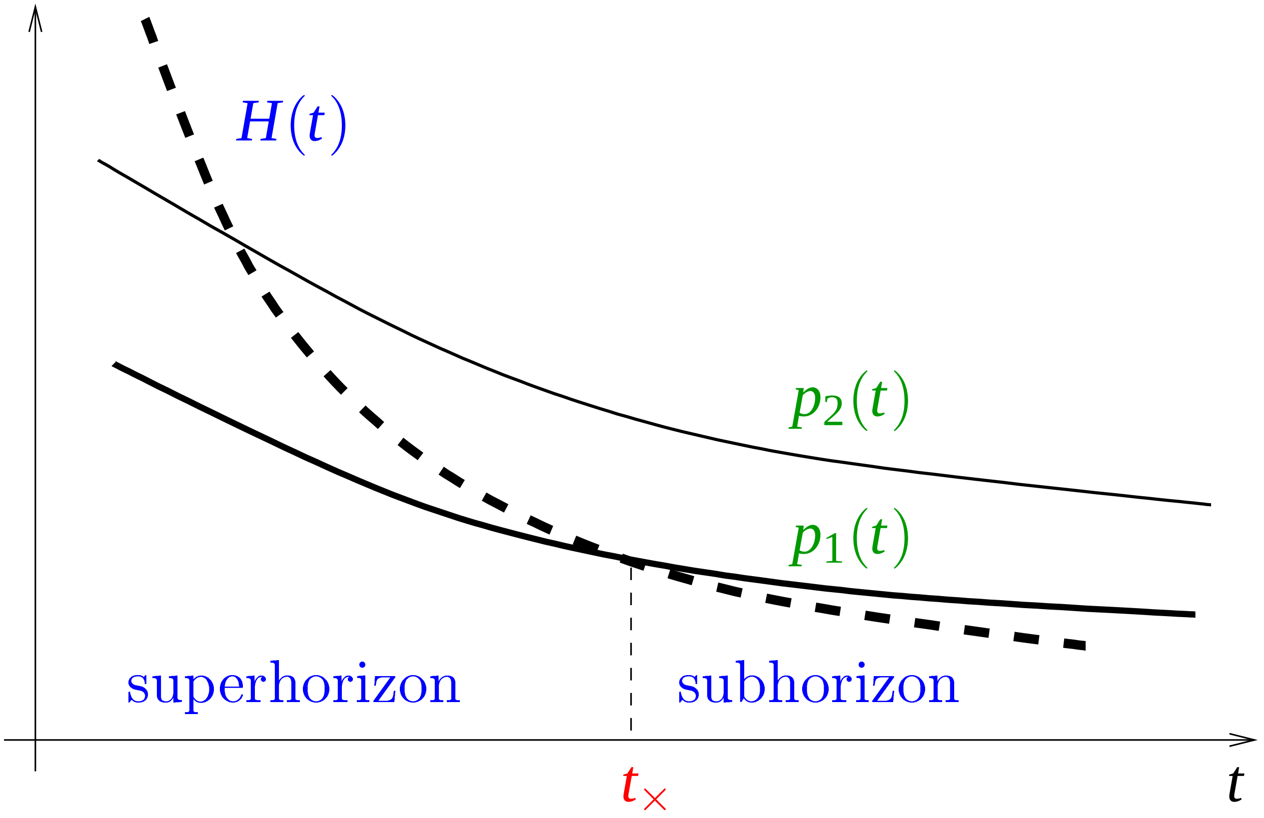
$$\frac{k}{a(\eta_\times)} \sim H(\eta_\times) \Longrightarrow \frac{k}{a_0} \frac{a_0}{a(\eta_\times)} \sim H(\eta_\times) \Longrightarrow p_0 \frac{T_\times}{T_0} \sim \frac{T_\times^2}{M_{Pl}^*} \Longrightarrow T_\times \sim p_0 \frac{M_{Pl}}{T_0}$$

For $p_0 \sim (10 \text{ kpc})^{-1}$ (halos of first stars) get

$$T_\times \sim 30 \text{ keV}$$

much later than Big Bang Nucleosynthesis.

Regimes at radiation and matter domination



$p_2 > p_1$

- Early times: superhorizon regime, $k \to 0$ (e.g., at RD, $a \propto \eta$)

$$\partial_\eta^2 h^{(A)} + \frac{2}{\eta}\partial_\eta h^{(A)} = 0$$

Two solutions: constant mode $h^{(A)}(\eta) = \text{const}$

decaying mode $h^{(A)}(\eta) \propto \eta^{-1}$.
sometimes called growing mode

Decaying mode: strongly inhomogeneous and anisotropic Universe at early times.

Must not be present !!!

In absence of decaying mode, solution is unique, up to overall amplitude.

- Late times: subhorizon regime, WKB. General solution

$$h^{(A)}(\eta) = \frac{c}{a(\eta)} \sin(k\eta + \varphi)$$

Matching to constant mode (by solving the exact equation) $\Longrightarrow \varphi = 0$,

$$h^{(A)}(\eta) = \frac{c}{a(\eta)} \sin k\eta$$

Oscillations (gravity waves) with well defined phase

Story repeats for scalar perturbations: acoustic waves with well defined phase determined by absence of decaying mode

NB: Gravity wave amplitude decreases as $a^{-1}(\eta)$ after horizon entry. This is not true for acoustic waves.

Scalar perturbations

More complicated story.

- Gauge fixing

  Gauge invariance of General Relativity

$$g^{\mu\nu} \to \tilde{g}^{\mu\nu} = g^{\mu\nu} + \nabla^{\mu}\xi^{\nu} + \nabla^{\nu}\xi^{\mu}$$

  ξ^{μ} = gauge functions (small).

  Can be used to eliminate h_{0i} and longitudinal part of $h_{ij} \propto \partial_i \partial_j E$
  $\Longrightarrow$ Conformal Newtonian gauge

$$ds^2 = a^2(\eta)\left[(1+2\Phi)\,d\eta^2 - (1+2\Psi)\,d\mathbf{x}^2\right]$$

  Longitudinal part of perturbed $\{ij\}$ Einstein equation:
  $\Psi = -\Phi$, for ideal fluid only.

  The only gravitational potential Φ.

Complicated composition

Independent components

λ = photons, baryons, dark matter, neutrinos

Coupled via common gravitational potential $\boldsymbol{\Phi}$

To simplify:

- disregard neutrinos (sometimes possible to include them into radiation component)
- Treat baryons and photons as single fluid before recombination (tight coupling approximation).

NB: Life becomes tuff beyond these approximations!

Write and solve Boltzmann equations for particle distribution functions.

Complete set of equations in the ideal fluid approximation, conformal Newtonian gauge

- Equations for background
  - Einstein equations:

$$\frac{a'^2}{a^4} = \frac{8\pi}{3} G \sum_\lambda \rho_\lambda$$

$$2\frac{a''}{a^3} - \frac{a'^2}{a^4} = -8\pi G \sum_\lambda p_\lambda$$

  - Covariant energy conservation for each component, $\nabla_\sigma T^{\sigma\mu} = 0$, $\mu = 0$

$$\rho'_\lambda = -3\frac{a'}{a}(\rho_\lambda + p_\lambda)$$

NB: Dependence only on $\eta \Longrightarrow$ ordinary diff. eqs.

- Perturbed Einstein equations

$$k^2\Phi + 3\frac{a'}{a}\Phi' + 3\frac{a'^2}{a^2}\Phi = -4\pi G a^2 \cdot \sum_\lambda \delta\rho_\lambda \ ,$$

$$\Phi' + \frac{a'}{a}\Phi = -4\pi G a^2 \cdot \sum_\lambda [(\rho + p)v]_\lambda \ ,$$

$$\Phi'' + 3\frac{a'}{a}\Phi' + \left(2\frac{a''}{a} - \frac{a'^2}{a^2}\right)\Phi = 4\pi G a^2 \cdot \sum_\lambda \delta p_\lambda$$

- Covariant energy-momentum conservation for perturbations in each component (continuity equation and Euler equation in expansing Universe; recall $v_i = \partial_i v$)

$$\delta\rho'_\lambda + 3\frac{a'}{a}(\delta\rho_\lambda + \delta p_\lambda) - (\rho_\lambda + p_\lambda)(k^2 v_\lambda + 3\Phi') = 0 \ ,$$

$$[(\rho_\lambda + p_\lambda)v_\lambda]' + 4\frac{a'}{a}(\rho_\lambda + p_\lambda)v_\lambda + \delta p_\lambda + (\rho_\lambda + p_\lambda)\Phi = 0$$

NB: — system of linear ordinary diff. eqs. for given k;

— more equations than unknowns, not all equations independent (because of gauge invariance of original system)

— Additionally, need equation of state for each component,

$$p_\lambda = p_\lambda(\rho_\lambda)$$

In particular

$$\frac{\delta p_\lambda}{\delta \rho_\lambda} = u_s^2$$

$u_s =$ sound velocity in component λ.

First glimpse:
perturbations in dominant component

Radiation at radiation domination; dark matter at matter domination (in approximation $\rho_{DM} \gg \rho_B$).

- Forget about other components $\Longrightarrow$ single component Universe with ρ, p; $\delta\rho$, δp, v, Φ
- Set $p = u_s^2 \rho$ (not always possible), $\delta p = u_s^2 \delta\rho$
- Combine perturbed Einstein eqs.

$$k^2\Phi + 3\frac{a'}{a}\Phi' + 3\frac{a'^2}{a^2}\Phi = -4\pi G a^2 \cdot \delta\rho \ ,$$

$$\Phi'' + 3\frac{a'}{a}\Phi' + \left(2\frac{a''}{a} - \frac{a'^2}{a^2}\right)\Phi = 4\pi G a^2 \cdot \delta p = 4\pi G a^2 u_s^2 \delta\rho$$

with eqs. for background

Result

$$\Phi'' + 3\frac{a'}{a}(1+u_s^2)\Phi' + u_s^2 k^2 \Phi = 0$$

- Perturbations in radiation at RD stage: $u_s = 1/\sqrt{3}$, $a = \text{const} \cdot \eta$
  - Superhorizon regime (early times): again constant and decaying modes,

    $$\Phi = \Phi_i = \text{const} \quad \text{and} \quad \Phi \propto \eta^{-3} \propto a^{-3}$$

    Forbid decaying mode. Then from Einstein eqn.

    $$\delta_{rad} \equiv \frac{\delta\rho_{rad}}{\rho_{rad}} = -2\Phi_i$$

    also constant in superhorizon regime.
  - Without decaying mode initially, solution unique, expressed through $J_{3/2}(ku_s\eta)$.

After horizon entry (assuming this happens at RD stage)

$$\Phi = -3\Phi_i \frac{\cos(ku_s\eta)}{(ku_s\eta)^2}$$

Phase is uniquely determined by initial absence of decaying mode.

These are acoustic oscillations. Einstein eq. in subhorizon regime

$$\frac{k^2}{a^2}\Phi = -4\pi G \cdot \delta\rho \quad \Longleftrightarrow \quad \text{“}\Delta\text{”}\ \Phi = 4\pi G\, \delta\rho\ , \quad \text{Poisson eq.}$$

$\rho \propto H^2 = (a\eta)^{-2} \Longrightarrow$

$$\delta_{rad} \equiv \frac{\delta\rho_{rad}}{\rho_{rad}} = 6\Phi_i \cos(ku_s\eta)$$

Acoustic oscillations with time-independent amplitude and well defined phase.

NB: Oscillations in subhorizon regime can be obtained also in standard way, from energy-momentum conservation eqs. with $\Phi \to 0$.

- Perturbations in dark matter in matter dominated Universe (neglecting baryons); $u_s = 0$, $a = \mathrm{const} \cdot \eta^2$:

$$\Phi'' + 3\frac{a'}{a}\Phi' = \Phi'' + \frac{6}{\eta}\Phi' = 0$$

Solutions $\Phi = \mathrm{const}$ and $\Phi \propto 1/\eta^5$. Constant solution relevant at late times.

Again use Poisson eqn.,

$$\frac{k^2}{a^2}\Phi = -4\pi G \cdot \delta\rho = -4\pi G\rho\frac{\delta\rho}{\rho}$$

but now with $\rho \propto a^{-3}$. Find at matter domination

$$\delta_{DM} \equiv \frac{\delta\rho_{DM}}{\rho_{DM}} \propto a(\eta)$$

Gravitational instability in matter dominated Universe.

To summarize:

- At early times at the hot stage, perturbations are in superhorizon regime, $p \ll H$. Assuming that the Universe was not strongly inhomogeneous in the beginning of hot Big Bang epoch, there is constant mode only in this regime.

  NB: Long modes were still in superhorizon regime at recombination/last scattering epoch. They determine low l region of CMB angular spectrum.

- Assuming that perturbations were there before they entered horizon, density perturbations of shorter wavelengths in baryon-photon component experience acoustic oscillations after horizon entry with well defined phase

$$\delta_{rad} \equiv \frac{\delta\rho_{rad}}{\rho_{rad}} = 6\Phi_i \cos(k u_s \eta)$$

  These oscillations continue to recombination epoch, and in the end give rise to oscillations in CMB angular spectrum.

- These assumptions would not be valid if density perturbations were generated at hot stage by some causal mechanism (e.g., tological defects). That mechanism could only work inside the horizon, i.e., no perturbations would exist before horizon entry. Phases of acoustic oscillations would be random in that case, this would yield non-oscillatory CMB angular spectrum. Such a scenario is ruled out, since there are oscillations in CMB angular spectrum.

- Perturbations in dark matter (and in baryons after recombination) grow as

$$\delta_{DM} \equiv \frac{\delta\rho_{DM}}{\rho_{DM}} \propto a(\eta)$$

They eventually become large, $\delta_{DM} \sim 1$ (and $\delta_B = \delta_{DM}$ soon after recombination), and form structure.

In linear regime, their gravitational potential is time-independent,

$$\Phi = \text{const} .$$

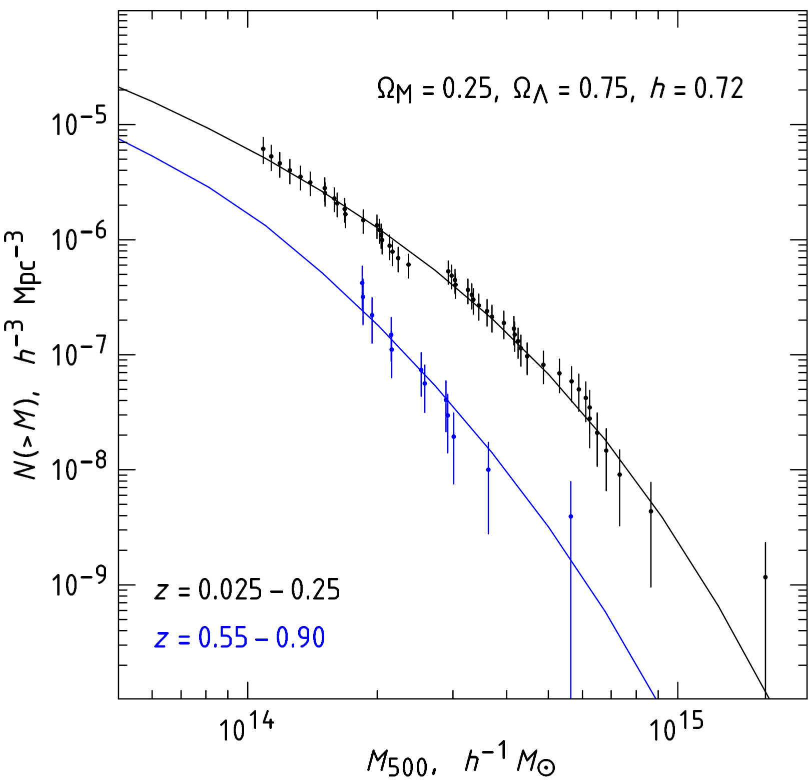
NB: Due to effect of dark energy, growth of δ_M has slowed down recently, and potential Φ started to decrease. This applies to large wavelengths, which are in linear regime (or have become non-linear only recently).

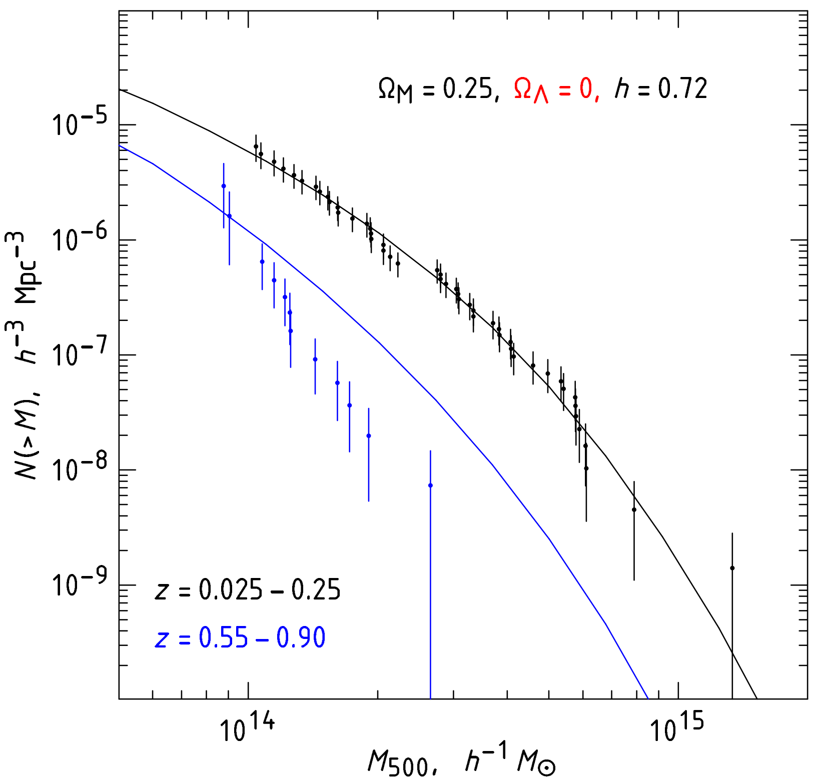
Way to measure Ω_Λ

Fig.

- Tensor perturbations, if any, decay as $a^{-1}(\eta)$ after horizon entry. They are most important for CMB at fairly low l.

Cluster counting





Vikhlinin et.al. '2008

Outline of lecture 2, part 2

- Initial conditions
  - Adiabatic mode in superhorizon regime
  - Entropy (isocurvature) modes
- Dark matter at radiation domination
- Baryons and photons at matter domination before recombination
- Summary of adiabatic perturbations
- Silk damping
- Baryon acoustic oscillations
- Effect of perturbations on CMB: general formulae

Initial conditions

From now on: assume that perturbations were superhorizon and that there was no decaying mode.

Off hand: various kinds of initial conditions for multi-component cosmic medium, set up deep in superhorizon regime

- Adiabatic perturbations = perturbations in energy density with constant in space composition

$$\frac{n_{DM}}{s} = \frac{n_B}{s} = \text{const in space}$$

(similarly for neutrinos).

In this case $n_{DM} = \text{const} \cdot T^3$, and $\rho_{DM} = m_{DM} n_{DM}$, hence

$$\frac{\delta \rho_{DM}}{\rho_{DM}} \equiv \delta_{DM} = 3 \frac{\delta T}{T}$$

while $\rho_{rad} \propto T^4$ and therefore $\frac{\delta \rho_{rad}}{\rho_{rad}} \equiv \delta_{rad} = 4 \frac{\delta T}{T}$

Integral of motion in superhorizon regime, $k\eta \to 0$:

Continuity equation with $k = 0$:

$$\delta\rho_\lambda' + 3\frac{a'}{a}(\delta\rho_\lambda + \delta p_\lambda) - 3(\rho_\lambda + p_\lambda)\Phi' = 0$$

recall $\rho_\lambda' = -3\frac{a'}{a}(\rho_\lambda + p_\lambda)$ and use $p_\lambda'/\rho_\lambda' = \delta p_\lambda/\delta\rho_\lambda = u_s^2$ to obtain

$$\zeta_\lambda = -\Phi + \frac{\delta\rho_\lambda}{3(\rho_\lambda + p_\lambda)} = \text{const in time}$$

NB: this has been generalized beyond ideal fluid

S.Weinberg' 2003

- Adiabatic perturbation:

$$\zeta_{DM} = \zeta_B = \zeta_\gamma = \zeta_\nu \equiv \zeta$$

The only initial condition for given $\mathbf{k}$.
Another notation: $\mathscr{R} = \zeta + O(k\eta)$ in superhorizon regime.

All other quantities in superhorizon regime are expressed through ζ. Expressions slightly different for RD and MD epochs.

At radiation domination $\delta_{rad} \equiv \delta\rho_{rad}/\rho_{rad} = -2\Phi_i$ and $p_{rad} = \rho_{rad}/3$. Use

$$\zeta = \zeta_{rad} = -\Phi + \frac{1}{4}\delta_{rad}$$

to get

$$\Phi_i = -\frac{2}{3}\zeta \quad \Longrightarrow \quad \delta_{rad} = \frac{4}{3}\zeta, \quad \delta_{DM} = \delta_B = \zeta$$

At matter domination we have instead $\delta_M = -2\Phi$ (again from Einstein eq. in superhorizon regime), and

$$\zeta = \zeta_{DM} = -\Phi + \frac{1}{3}\delta_{DM}$$

hence

$$\Phi = -\frac{3}{5}\zeta \quad \Longrightarrow \quad \delta_{rad} = \frac{8}{5}\zeta, \quad \delta_{DM} = \delta_B = \frac{6}{5}\zeta$$

- Isocurvature (entropy) modes. E.g. dark matter entropy mode: No perturbation in energy density, only in composition, n_{DM}/s varies in space

  Deep at radiation domination this means that $\delta\rho_{rad} = 0$, $\delta\rho_{DM} \neq 0$, or

$$\mathscr{S}_{DM} \equiv \zeta_{DM} = \frac{\delta(n_{DM}/s)}{n_{DM}/s} \neq 0 \, , \qquad \zeta_B = \zeta_\gamma = \zeta_\nu = 0$$

  Also, $\Phi_i = 0$ deep at RD.

  Similarly for baryon entropy mode.

Generally speaking, initial condition is a linear combination of adiabatic and entropy modes (plus neutrino isocurvature modes of two types, but unlikely on physical grounds).

If dark matter and baryon asymmetry were generated at hot stage, adiabatic mode only. But it is up to experiment to decide.

Existing data: consistent with adiabatic mode only.

DM isocurvature (entropy) mode constrained

$$\frac{\mathscr{S}_{DM}^2}{\zeta^2} < 0.07$$

(this is to be understood as ratio of power spectra, see below for defintion of power spectrum).
Constraint on baryon entropy mode worse by a factor $(\Omega_{DM}/\Omega_B)^2 \sim 20$.

What does dark matter do at radiation domination?

Use conservation equations for dark matter, with gravitational potential generated by radiation. These can be written as

next slide

$$\delta'_{DM} - k^2 v_{DM} = 3\Phi'$$

$$v'_{DM} + \frac{1}{\eta} v_{DM} = -\Phi$$

Solution to homogeneous equation ($\Phi = 0$):

$$v_{DM} = \frac{c_1}{\eta}\,, \qquad \delta_{DM} = c_1 k^2 \log\eta + c_2$$

- CDM isocurvature mode: $\Phi = 0$ at radiation domination $\Longrightarrow$ $c_1 = 0$ (no mode growing towards $\eta \to 0$!) $\Longrightarrow$ $\delta_{DM} =$ const in time
- Adiabatic mode: $\Phi \neq 0$, produced by δ_{rad}, but Φ decays as η^{-2} after horizon entry $\Longrightarrow$ gives kick to dark matter; $\delta_{DM} \propto \log\eta$ after horizon entry.

Another form of conservation equations

$$\delta'_\lambda + 3\frac{a'}{a}(u^2_{s,\lambda} - w_\lambda)\delta_\lambda - (1+w_\lambda)k^2 v_\lambda = 3(1+w_\lambda)\Phi'$$

$$[(1+w_\lambda)v_\lambda]' + \frac{a'}{a}(1-3w_\lambda)(1+w_\lambda)v_\lambda + u^2_{s,\lambda}\delta_\lambda = -(1+w_\lambda)\Phi$$

where

$$w_\lambda = \frac{p_\lambda}{\rho_\lambda} \quad = \quad \text{barotropic index}$$

$$u^2_{s,\lambda} = \frac{\delta p_\lambda}{\delta \rho_\lambda} \quad = \quad \text{sound velocity squared}$$

Adiabatic mode: initial condition for dark matter perturbations right after equality epoch (short wavelengths, enter horizon at RD, $k\eta_{eq} \gg 1$)

$$\delta_{DM} = 9\zeta \log(0.15 k\eta_{eq})$$

NB: enhanced both logarithmically and numerically compared to initial $\delta_{DM} = \zeta$. Just right for structure formation.

After equality epoch, δ_{DM} grow as $a(\eta)$, starting from this value. $\mathbf{\Phi}$ stays constant in time. Use Poisson equation and Friedmann eqs. to get at equality and later (using $H_{eq}^2 = \frac{1}{2}\frac{8\pi}{3}G\rho_{DM}$)

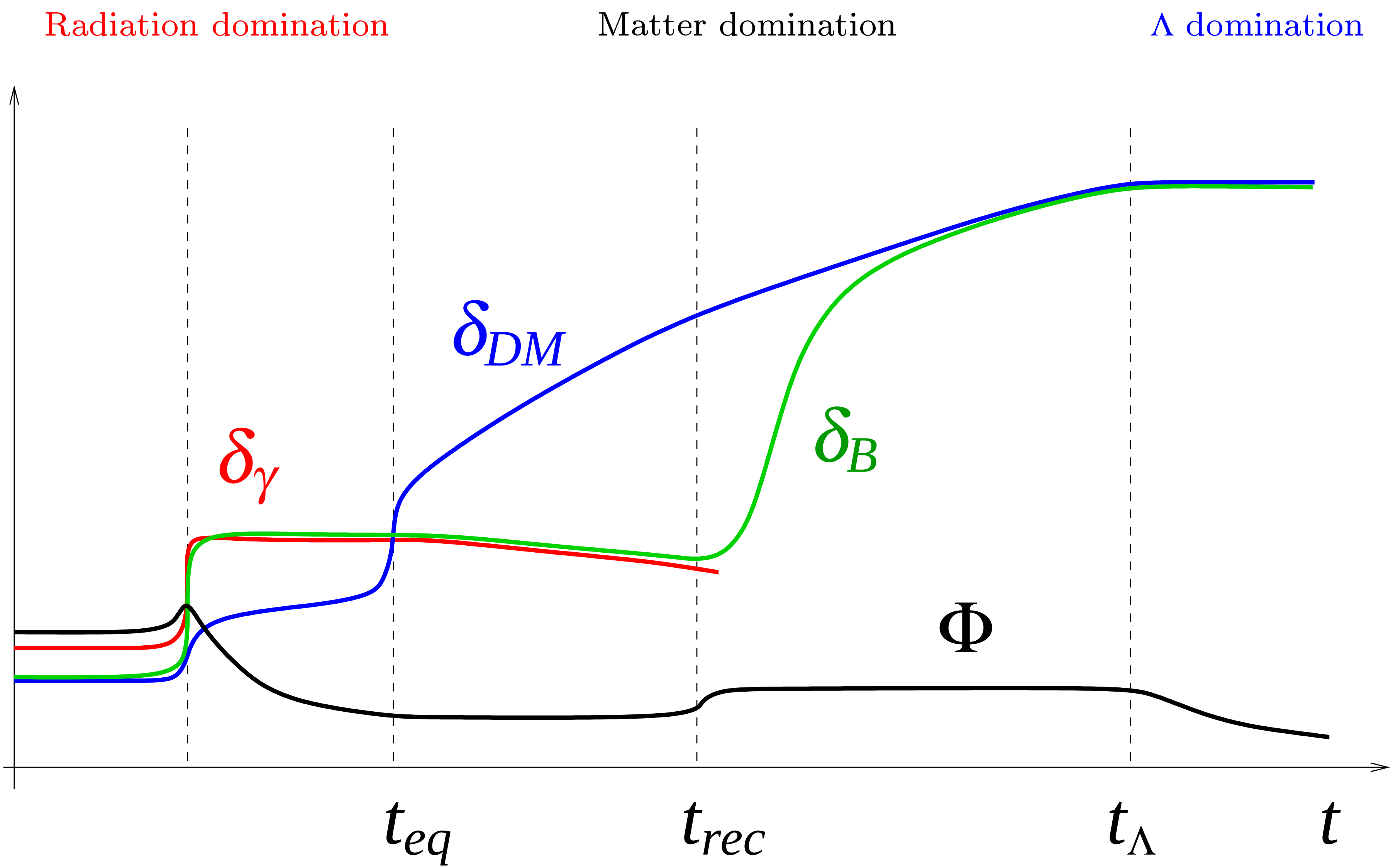
$$\Phi_{DM} = -\frac{a_{eq}^2}{k^2} \cdot 4\pi G\rho_{DM}\delta_{DM} = -\frac{27}{4}\zeta\frac{H_{eq}^2 a_{eq}^2}{k^2}\log(0.15k\eta_{eq})$$

NB: Sign important for CMB.

NB: $\mathbf{\Phi}$ decays as function of k.

Smaller spatial scales enter horizon earlier $\Longrightarrow$ have more time for log growth $\Longrightarrow$ smaller structures get formed earlier.

Growth of perturbations (linear regime)



What does baryon-photon component do at matter domination (but before recombination)?

Acoustic oscillations continue,

$$\delta_\gamma = 6\Phi_i \cos\left(\int_0^\eta ku_s \, d\eta\right) = -4\zeta \cos\left(\int_0^\eta ku_s \, d\eta\right)$$

NB: $u_s \neq 1/\sqrt{3}$ because of baryons. Prefactor actually also gets small correction.

Density contrast in baryons

$$\delta_B = \frac{3}{4}\delta_\gamma$$

since

$$\frac{\delta\rho_B}{\rho_B} = 3\frac{\delta T}{T}\,, \qquad \frac{\delta\rho_\gamma}{\rho_\gamma} = 4\frac{\delta T}{T}$$

Need also velocity. Take continuity equation and apply to photon (or baryon) component $\Longrightarrow$ in absense of gravitational potential (has decayed away)

$$kv_{\gamma B} = \frac{3}{4k}\delta_\gamma' = 3u_s\zeta \sin\left(\int_0^\eta ku_s\, d\eta\right)$$

New effect:

Just before recombination: matter domination. There is gravitational potential Φ_{DM} due to dark matter. Photons and baryons feel it.

Euler equation for baryon-photon component:

$$\left[(\rho_{\gamma B} + p_\gamma)v_{\gamma B}\right]' + 4\frac{a'}{a}(\rho_{\gamma B} + p_\gamma)v_{\gamma B} + \delta p_\gamma + (\rho_{\gamma B} + p_\gamma)\Phi_{DM} = 0$$

Particular solution for time-indpendent Φ_{DM}: $v_{\gamma B} = 0$ and

$$\delta p_\gamma = -(\rho_\gamma + \rho_B + p_\gamma)\Phi_{DM}$$

Recall $p_\gamma = \rho_\gamma/3$, $\delta p_\gamma = \delta\rho_\gamma/3$ and get

$$\delta_\gamma \equiv \frac{\delta\rho_\gamma}{\rho_\gamma} = -4(1+R_B)\Phi_{DM}$$

where

$$R_B = \frac{3\rho_B}{4\rho_\gamma} \quad = 0.48 \quad \text{at} \;\; \text{recombination}$$

NB: protons only, $\rho_B = 0.75\, n_{Btot}\, m_p$: helium is neutral at recombination of hydrogen

R_B is the parameter directly measured by CMB observations $\Longrightarrow$ determination of $\Omega_B h^2$.

Adiabatic perturbations at recombination: summary

- Long modes, still superhorizon at recombination:

$$\Phi = -\frac{3}{5}\zeta \qquad \delta_\gamma = \frac{8}{5}\zeta$$

- Short modes, enter sound horizon at radiation domination:

  - Perturbation in photon energy density/local temperature

$$\delta_\gamma \equiv \frac{\delta\rho_\gamma}{\rho_\gamma} = 4\left(\frac{\delta T}{T}\right)_{loc} \simeq -4\zeta\cos\left(\int_0^{\eta_r} k u_s \, d\eta\right) - 4(1+R_B)\Phi_{DM}$$

  - Velocity

$$k v_{\gamma B} \simeq 3 u_s \zeta \sin\left(\int_0^{\eta} k u_s \, d\eta\right)$$

  - Gravitational potential (produced by dark matter)

$$\Phi_{DM} \simeq -\frac{27}{4}\zeta \frac{H_{eq}^2 a_{eq}^2}{k^2} \log(0.15 k \eta_{eq})$$

- Intermediate modes, enter horizon between equality and recombination: qualitatively similar behavior to short modes

Properties of dark matter isocurvature perturbations entirely different (baryon and DM isocurvature perturbations are in fact very similar)

- No initial perturbations in baryon-photon component.

  Acoustic oscillations triggered by gravitational potential of dark matter. Initial condition $\delta_{\gamma B} = 0 \Longrightarrow$ oscillatory part

  $$\delta_\gamma = \mathscr{S}_{DM} \cdot A(k) \sin\left(\int_0^\eta k u_s \, d\eta \right)$$

  Phase differs by $\pi/2$.

  Short wavelengths enter horizon when ρ_{DM} small compared to radiation $\Longrightarrow$ oscillations suppressed at short scales,

  $$A(k) \propto k^{-1}$$

- No log enhancement of δ_{DM} and Φ_{DM} (not very important).

Shorter wavelengths: Silk damping

Beyond ideal fluid/tight coupling approximation

Photon mean free path λ_γ is finite $\Longrightarrow$ photons diffuse away $\Longrightarrow$ acoustic oscillations get smeared out.

Diffusion length in Hubble time

$$l_S \sim \lambda_\gamma \sqrt{N_{coll}}$$

$N_{coll} \sim H^{-1}/\tau_\gamma = H^{-1}/\lambda_\gamma$ — number of photon collisions with electrons in Hubble time H^{-1} $\Longrightarrow$

$$l_S \sim \sqrt{\lambda_\gamma H^{-1}} \sim \sqrt{(\sigma_T n_e)^{-1} H^{-1}}$$

$\sigma_T = 0.67 \cdot 10^{-24}\ \mathrm{cm}^2$, Thomson cross section;

$$n_e = 0.75 \frac{\rho_B}{m_p} = 8 \cdot 10^{-6} \Omega_B h^2 (1+z)^3\ \mathrm{cm}^{-3}$$

$$\simeq 230\ \mathrm{cm}^{-3} \quad \text{just before recombination}$$

This gives for comoving scale

$$(1+z_r)l_S \sim 20 \text{ Mpc}$$

More accurate analysis (beyond ideal fluid) gives

$$\frac{k_S}{a_0} \simeq 0.1 \text{ Mpc}^{-1} \, ,$$

and oscillatory part of δ_γ

$$\delta_{\gamma,osc} \simeq -4\zeta \mathrm{e}^{-\frac{k^2}{k_S^2}} \cos\left(\int_0^{\eta_r} k u_s \, d\eta \right)$$

same effect for velocity.

Baryon acoustic oscillations

Before recombination, baryons oscillate in time together with photons.

Immediately after recombination, oscillations in time freeze out (no pressure $\Longrightarrow$ no oscillations = acoustic waves) at

$$\delta_B = \frac{3}{4}\delta_\gamma \simeq -3\zeta \mathrm{e}^{-\frac{k^2}{k_S^2}} \cos\left(\int_0^{\eta_r} k u_s \, d\eta\right)$$

These are oscillations in momentum k.

Furthermore, just before recombination baryon-photon component has non-zero velocity

$$k v_{\gamma B} = 3 u_s \zeta \mathrm{e}^{-\frac{k^2}{k_S^2}} \sin\left(\int_0^{\eta_r} k u_s \, d\eta\right)$$

These are initial conditions for evolution after recombination.

Soon after recombination baryons and dark matter equalize, $\delta_B = \delta_{DM}$ (baryons fall into potential wells produced by dark matter, and vice versa)

Hence, the total matter density some time after recombination is a linear combination of smooth and oscillating functions of momentum (solve conservation eqs. for baryons and dark matter; grav. potential Φ is produced by both and obeys Poisson eqn.)

$$\delta_{CDM} = \delta_B = \frac{a(\eta)}{a(\eta_r)} \left[\frac{\Omega_{CDM}}{\Omega_M} \delta_{CDM}(\eta_r) + \frac{\Omega_B}{\Omega_M} \left(\frac{3}{5} \delta_B(\eta_r) + \frac{k\eta_r}{5} k v_B(\eta_r) \right) \right]$$

where $\Omega_M = \Omega_{CDM} + \Omega_B$.

Oscillating part: small, since Ω_B is small, while δ_{CDM} is enhanced.

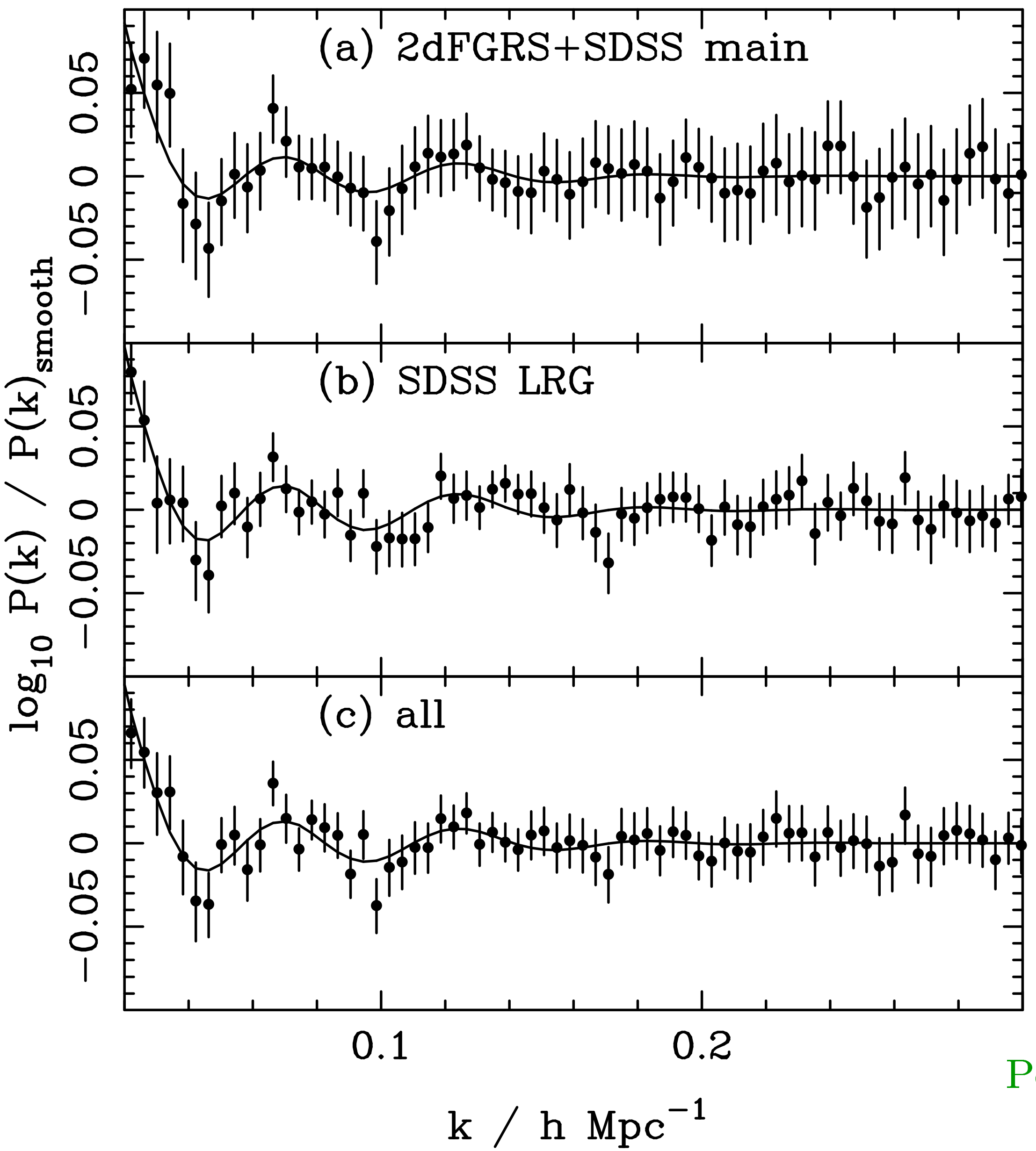
Yet observed in power spectrum of galaxy distribution.

Fig.

NB: δ_{DM}, δ_B and v_B are all proportional to one and the same $\zeta \Longrightarrow$ interference bewtween smooth and oscillating parts

NB: Silk damping at $k > 0.1\ \mathrm{Mpc}^{-1}$

BAO in power spectrum



Percival et. al. '2007

Interpretation: If a region is overdense, dark matter stays there, baryons and photons move away as sound wave $\Longrightarrow$ correlation of mass densities at coordinate distance

$$r_s = \int_0^{\eta_r} u_s \, d\eta$$

sound horizon at recombination.

Fig.

Comoving size

$$a_0 r_s \simeq 155 \text{ Mpc}$$

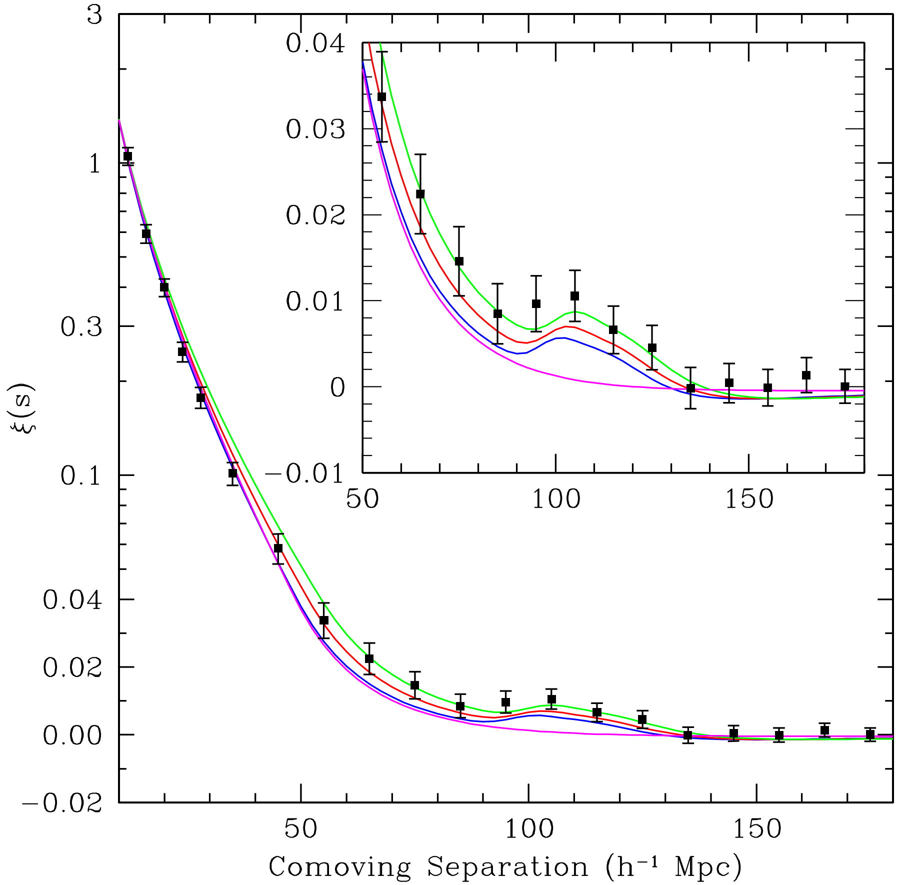
Well defined absolute length scale, standard ruler.
In principle, can measure angle at which this scale is seen at different z (angular diameter distance, $\Delta\theta = a_0 r_s / D_a(z) = r_s/(\eta_0 - \eta(z))$), and Hubble parameter at diffetent z $\Longrightarrow$ expansion history and geometry of the Universe. In practice, a combination of $D_a(z)$ and $H(z)$.

r_s slightly depends on Ω_M and Ω_B, since $a_0 d\eta = H^{-1} dz$ depends on Ω_M, and

$$u_s^2 = \frac{\delta p_\gamma}{\delta\rho_\gamma + \delta\rho_B} = \frac{(1/3)\rho_\gamma\delta_\gamma}{\rho_\gamma\delta_\gamma + \rho_B \frac{3}{4}\delta_\gamma} = \frac{1}{3(1 + R_B(\eta))}$$

But R_B is well measured, and dependence on Ω_M is weak.

BAO in correlation function



NB: $h^{-1} = (0.7)^{-1} = 1.43$

Eisenstein et.al., SDSS '2005

Effect of perturbations on CMB: general formula

Propagation of a photon in metric (linear order in perturbations)

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$$

(scale factor $a(\eta)$ $\Longleftrightarrow$ overall redshift $\Longleftrightarrow$ conformal invariance of Maxwell's action $\Longrightarrow$ forget).

Geodesic equation

$$\frac{dP^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma} P^\nu P^\sigma = 0$$

$P^\mu = dx^\mu / d\lambda$, λ = affine parameter. Use η as a parameter instead (time along world line); $d\eta/d\lambda = P^0$ $\Longrightarrow$

$$\frac{dP^\mu}{d\eta} + \Gamma^\mu_{\nu\sigma} \frac{P^\nu}{P^0} \frac{P^\sigma}{P^0} = 0$$

Take $\mu = 0$ $\Longrightarrow$ evolution of photon energy.

Scalar perturbations, conformal Newtonian gauge

$$ds^2 = a^2(\eta)\left[(1+2\Phi)\,d\eta^2 - (1+2\Psi)\,d\mathbf{x}^2\right]$$

hence

$$\frac{dP^0}{d\eta} = \left(\Phi' - \Psi'\right)P^0 - 2\left(\Phi' + \mathbf{n}\nabla\Phi\right)P^0$$

where $\mathbf{n} = \mathbf{P}/P^0$ is unit vector along photon trajectory.

Last term = total derivative along trajectory $\Longrightarrow$

$$\frac{P^0(\eta_a) - P^0(\eta_e)}{P^0} = \int_{\eta_e}^{\eta_a}\left(\Phi' - \Psi'\right)d\eta + 2\Phi(\eta_e)$$

η_e, η_a: times of emission and absorption; ignore $\Phi(\eta_a)$, as it gives overall red/blueshift, independent on photon arrival direction.

Now, let $\boldsymbol{\Omega}$ be photon energy in locally Lorentz rest frame of cosmic plasma at photon emission. Then

$$\boldsymbol{\Omega} = u_\mu P^\mu$$

u^μ = 4-velocity of plasma. In locally Lorentz rest frame $u^\mu = (1,0,0,0)$, while in cosmic frame

$$u^\mu = (1 - \Phi, v^i)$$

(from $g_{\mu\nu} u^\mu u^\nu = (\eta_{\mu\nu} + h_{\mu\nu}) u^\mu u^\nu = 1$) $\Longrightarrow$

$$u_\mu = (1 + \Phi, -\mathbf{v})$$

and

$$\boldsymbol{\Omega} = [1 + \Phi(\eta_e) - \mathbf{n}\mathbf{v}(\eta_e)] P^0(\eta_e)$$

Finally, $\boldsymbol{\Omega} \propto \bar{T} + (\delta T)_{loc} = \bar{T}(1 + \delta_\gamma/4)$.

Collect all terms, set $\eta_e = \eta_r$ and get

$$\frac{\delta T}{T}(\mathbf{n_{obs}}, \eta_0) = \frac{1}{4}\delta_\gamma + \Phi \qquad \text{Sachs–Wolfe}$$

$$- \mathbf{n_{obs}} \mathbf{v} \qquad \text{Doppler}$$

$$+ \int_{\eta_r}^{\eta_0} (\Phi' - \Psi')\, d\eta \qquad \text{Integrated SW}$$

$\mathbf{n_{obs}} = -\mathbf{n}$: direction in the sky;

All quantities in the right hand side taken at photon emission position $\mathbf{x} = \mathbf{n_{obs}}(\eta_0 - \eta_e)$, integral runs along photon world line.

Key formula for CMB temperature anisotropy.

Likewise, effect of tensor perturbations (ISW only)

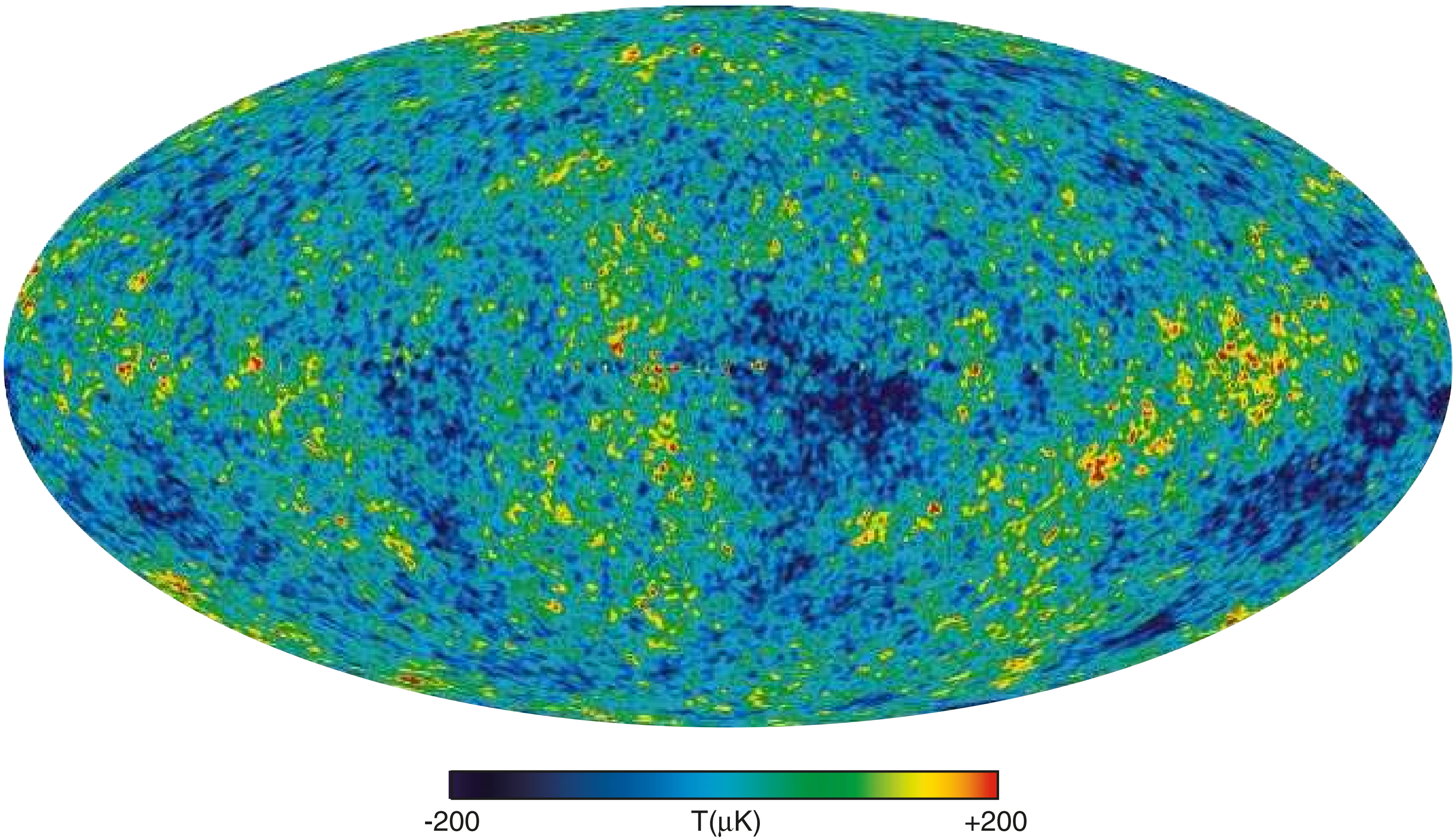
$$\frac{\delta T}{T}(\mathbf{n_{obs}}, \eta_0) = \frac{1}{2}\int_{\eta_r}^{\eta_0} d\eta \;\; n_i \, (h_{ij}^{TT})' \, n_j$$

Outline of lecture 3, part 2

- CMB temperature anisotropy: preliminaries
- What do we want to know — to zeroth order?
- Understanding CMB temperature spectrum
  - Small l, long waves.
  - Acoustic peaks
  - How tensor modes and entropy modes would show up
  - Examples of sensitivity to cosmological parameters
- CMB polarization
- Conclusion

CMB temperature anisotropy

$$T = 2.726^\circ K, \quad \frac{\delta T}{T} \sim 10^{-4} - 10^{-5}$$



WMAP

- Decompose temperature fluctuation in spherical harmonics (starting from $l = 2$; dipole $\Longleftrightarrow$ Earth's motion)

$$\delta T(\theta,\varphi) = \sum_{l,m} a_{lm} Y_{lm}(\theta,\varphi)$$

Large l $\Longleftrightarrow$ small angular scales

Working hypothesis: temperature fluctuations = isotropic Gaussian random field $\Longrightarrow$ a_{lm}: Gaussian random variables,

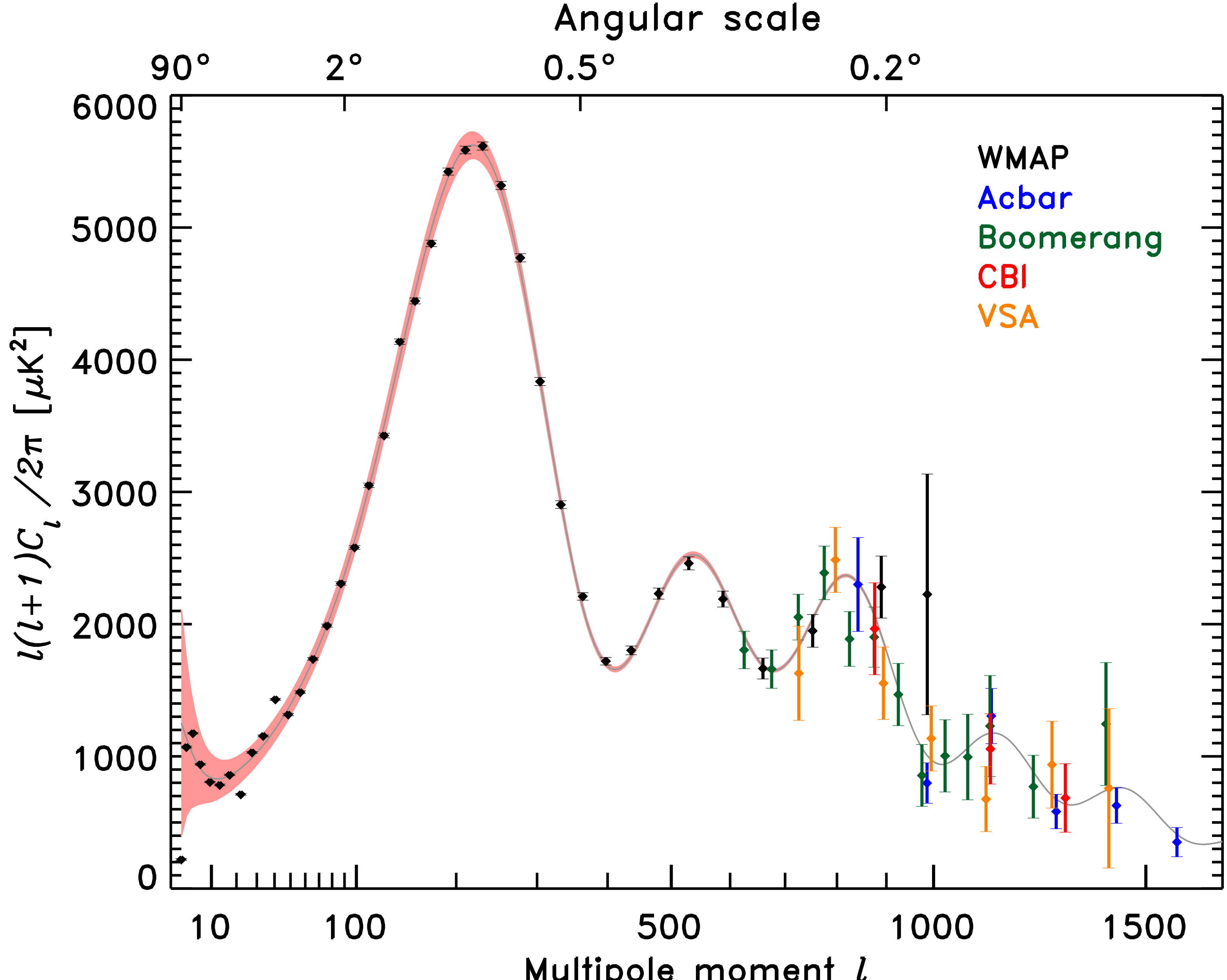
$$\langle a_{lm} a^*_{l'm'} \rangle = C_{lm} \delta_{ll'} \delta_{mm'}$$

Average over ensemble of Universes like ours.
Isotropy: $C_{lm} = C_l$ independent of m

- Temperature fluctuation

$$\langle \delta T^2(\mathbf{n}) \rangle = \sum_l \frac{2l+1}{4\pi} C_l \approx \int \frac{dl}{l} \frac{l(l+1)}{2\pi} C_l$$

CMB anisotropy spectrum



NB: Note funny scale on horizontal axis

NB: $\delta T \propto$ primordial scalar perturbations (and tensor, if any) $\Longleftrightarrow$ hypothesize that ζ is isotropic Gaussian random field (h_{ij}^{TT} also, if any).

In general, δT inherit correlation properties of ζ $\Longleftrightarrow$ search for non-Gaussianities, statistical anisotropy, etc.

NB: Cosmic variance: we observe only one Universe.

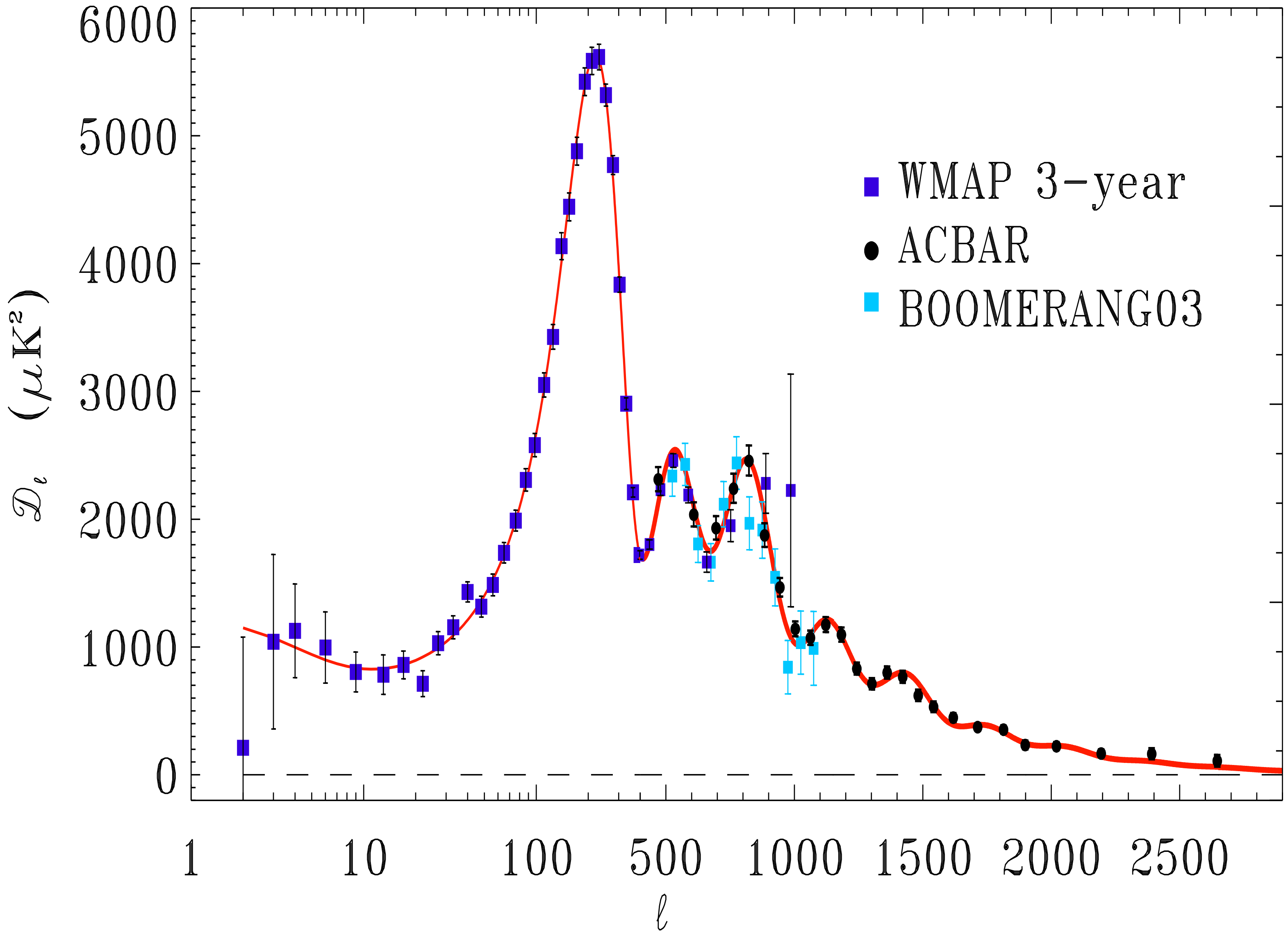
$2l+1$ measurements of a_{lm} for given l $\Longrightarrow$ small l $\Leftrightarrow$ large intrinsic uncertainly,

$$\frac{\Delta C_l}{C_l} = \frac{1}{\sqrt{l+1/2}}$$

No cure.

Fig.

$$\mathscr{D}_l = \frac{l(l+1)}{2\pi} C_l$$



What do we want to know — to zeroth order?

I. Properties of primordial perturbations.

- Adiabatic scalar perturbations
  Assuming isotropy and Gaussianity (Wick theorem for correlation functions)

$$\langle \zeta(\mathbf{x})\zeta(\mathbf{x}')\rangle = \int \frac{d^3k}{4\pi k^3} \mathrm{e}^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \mathscr{P}_\zeta(k)$$

$\mathscr{P}_\zeta(k) =$ power spectrum. Parametrization

$$\mathscr{P}_\zeta(k) \equiv \Delta^2_\zeta(k) \equiv \Delta^2_{\mathscr{R}}(k) = A_s \left(\frac{k}{k_0}\right)^{n_s-1}$$

$A_s = \Delta^2_{\mathscr{R}}(k_0) =$ scalar amplitude;
$n_s =$ scalar spectral index (for historical reason); $n_s - 1 =$ scalar tilt;
$k_0 =$ fiducial momentum (WMAP choice: $k_0/a_0 = 0.002$ Mpc^{-1} (NB: usual choice $a_0 = 1$).

NB: n_s close to 1. WMAP: $n_s = 0.963 \pm 0.012$ @ 68% C.L.

Also: running spectral index $n_s(k) = n_s(k_0) + \frac{dn_s}{d\log k} \cdot \log\frac{k}{k_0}$.

NB: fluctuation

$$\langle \zeta^2(\mathbf{x}) \rangle = \int_0^\infty \frac{dk}{k} \mathscr{P}_\zeta(k)$$

$n_s = 1 \iff$ Flat (Harrison–Zeldovich) spectrum.

- Similarly for tensor modes:

$$\langle h^{(A)}(\mathbf{x}) h^{(B)}(\mathbf{x}') \rangle = \frac{1}{2}\delta^{AB} \int \frac{d^3k}{4\pi k^3} \mathrm{e}^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \mathscr{P}_T(k)$$

$$\mathscr{P}_T(k) = A_T \left(\frac{k}{k_0}\right)^{n_T}$$

  Tensor-to-scalar ratio $r = A_T / A_s$.

- Admixture of isocurvature (entropy) modes.

II. Properties of the late Universe:

H_0, Ω's, dark matter equation of state, spatial curvature
$\Omega_k = 1/(a_0 H_0)^2$

Also: optical depth due to re-ionization, i.e., z_{rei} ; neutrino mass.

Understanding CMB temperature angular spectrum

$$\frac{\delta T}{T}(\mathbf{n},\eta_0) = \left(\frac{1}{4}\delta_\gamma + \Phi\right) - \mathbf{n}\mathbf{v} + \int_{\eta_r}^{\eta_0} (\Phi' - \Psi')\, d\eta$$

Sachs-Wolfe Doppler Integrated SW

$\mathbf{n}$ = direction in the sky, all quantities in the right hand side taken at photon emission position $\mathbf{x} = \mathbf{n}(\eta_0 - \eta_r)$, integral runs along photon world line.

Begin with Sachs–Wolfe effect (set $\eta_0 - \eta_r = \eta_0$)

$$\frac{\delta T}{T}(\mathbf{n},\eta_0) = \int d^3k \; e^{i\mathbf{k}\mathbf{n}\eta_0} \varphi_{SW}(\mathbf{k}) \,, \qquad \varphi_{SW}(\mathbf{k}) \equiv \frac{1}{4}\delta_\gamma(\mathbf{k}) + \Phi(\mathbf{k})$$

Expand in spherical harmonics in $\mathbf{n}$. Make use of the fact that $\varphi_{SW}(\mathbf{k})$ is random field with

$$\langle \varphi_{SW}(\mathbf{k}) \varphi^*_{SW}(\mathbf{k}')\rangle = \delta(\mathbf{k} - \mathbf{k}') \frac{1}{4\pi k^3} \mathscr{P}_{SW}(k)$$

Calculate $C_l = \frac{1}{2l+1} \sum_m \langle a_{lm} a^*_{lm} \rangle$.

Outcome

$$C_l/T_0^2 = 4\pi \int_0^\infty \frac{dk}{k} \mathscr{P}_{SW}(k) j_l^2(k\eta_0)$$

where j_l is spherical Bessel function.

Next slide for simple calculation

Properties:

- $j_l(k\eta_0)$ almost vanishes at $k\eta_0 < l$ (for $l \gtrsim 5$).

  Interpretation: expansion in Y_{lm} on a sphere of radius $\eta_0 \Longleftrightarrow$ Fourier expansion in plane, normal to line of sight, with 2d momentum $q \simeq l/\eta_0$ (cf. Laplacians q^2 and $l(l+1)/\eta_0^2$). Perturbation contributes, if its momentum is $\mathbf{k} = (\mathbf{q}, k_T)$. Hence, $k^2 > q^2 \simeq l^2/\eta_0^2$.

- $j_l(k\eta_0)$ decays as $(k\eta_0)^{-1}$ at $k\eta_0 \gg l$.

Corollary: Most relevant for multipole l are perturbations of momenta $k \sim l/\eta_0$.

Trick

$$\langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \rangle = \int \frac{d^3k}{4\pi k^3} \mathscr{P}_{SW}(k) \ \mathrm{e}^{i\mathbf{kn}\eta_0} \ \mathrm{e}^{-i\mathbf{kn}'\eta_0}$$

Perform calculation for given $\mathbf{k}$. Its contribution to C_l is independent of the choice of coordinate frame on the sphere $\Longrightarrow$ choose frame with $\mathbf{k}$ along 3d axis. Then

$$\mathrm{e}^{i\mathbf{kn}\eta_0} = \mathrm{e}^{ik\eta_0 \cos\theta} = \sum_l i^l (2l+1) P_l(\cos\theta) j_l(k\eta_0) = \sum_l i^l \sqrt{4\pi(2l+1)} Y_{l0} j_l(k\eta_0)$$

and similarly for $\mathrm{e}^{-i\mathbf{kn}'\eta_0}$. Thus, the only non-vanishing contribution to C_l comes from $m = 0$ in this frame, and

$$C_l/T_0^2 = \frac{1}{2l+1} \sum_m \langle a_{lm} a_{lm}^* \rangle = \frac{1}{2l+1} \int \frac{d^3k}{4\pi k^3} \mathscr{P}_{SW}(k) \cdot 4\pi(2l+1)$$

This trick — calculation of C_l in different frames for different $\mathbf{k}$ — is particularly convenient for calculating effect of tensor modes.

Small l $\Longleftrightarrow$ long waves

Still superhorizon at recombination:

$$\Phi = -\frac{3}{5}\zeta \qquad \delta_\gamma = \frac{8}{5}\zeta \qquad v = 0$$

Sachs–Wolfe only (ISW small, see below).

$$\varphi_{SW} = \frac{1}{4}\delta_\gamma + \Phi = -\frac{1}{5}\zeta$$

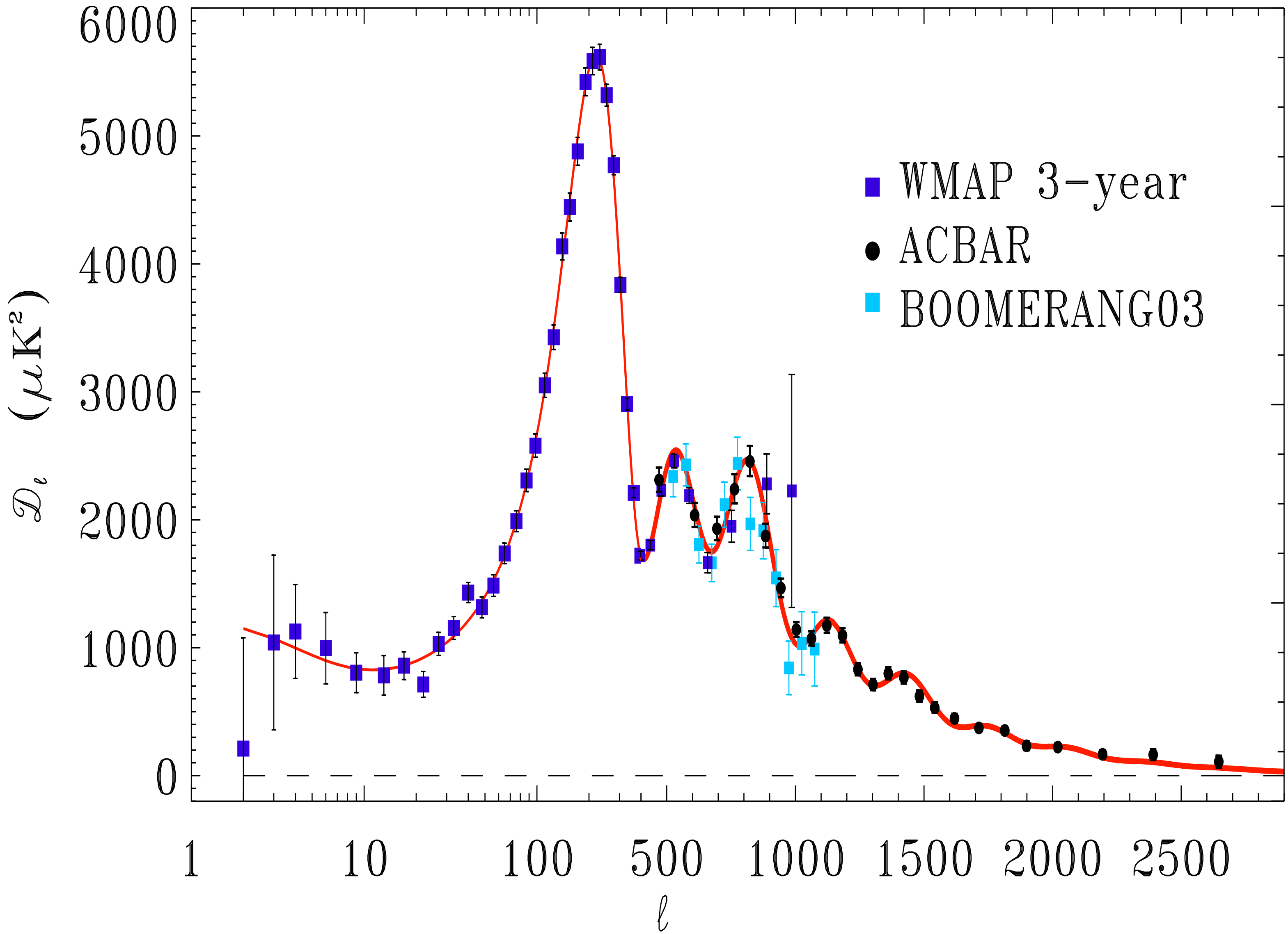
Thus,

$$C_l/T_0^2 = 4\pi \int_0^\infty \frac{dk}{k}\frac{1}{25}\mathscr{P}_\zeta(k)\, j_l^2(k\eta_0) = \frac{2\pi}{25}\frac{1}{l(l+1)}A_s\left(\frac{l}{l_0}\right)^{n_s-1}$$

$\mathscr{D}_l = \frac{l(l+1)}{2\pi}C_l$ is independent of l for $n_s = 1$. Almost no dependence on cosmological parameters.

Validity: $k \sim l/\eta_0 \ll \eta_r^{-1} \Longrightarrow l \ll \eta_0/\eta_r = 50$

$$\mathscr{D}_l = \frac{l(l+1)}{2\pi} C_l$$



Excersise:

Take COBE "quadrupole" (in fact, inferred by COBE from several low multipoles), defined as

$$Q^2 = \frac{5}{4\pi} C_2$$

and according to COBE

$$Q = 18\ \mu\mathrm{K}$$

Calculate scalar amplitude A_s for $n_s = 1$. Compare with WMAP result

$$A_s \equiv \Delta^2_{\mathcal{R}} = (2.44 \pm 0.09) \cdot 10^{-9}$$

[was Nobel Committee right about COBE?]

Integrated Sachs–Wolfe effect

$$\frac{\delta T}{T}(\mathbf{n})_{ISW} = \int_{\eta_r}^{\eta_0} \left(\Phi' - \Psi'\right) d\eta$$

$\Phi = -\Psi$ time-independent at matter domination. $\Longrightarrow$ ISW relevant right after recombination (matter domination not exact, early ISW) and recently (effect of dark energy, late ISW).

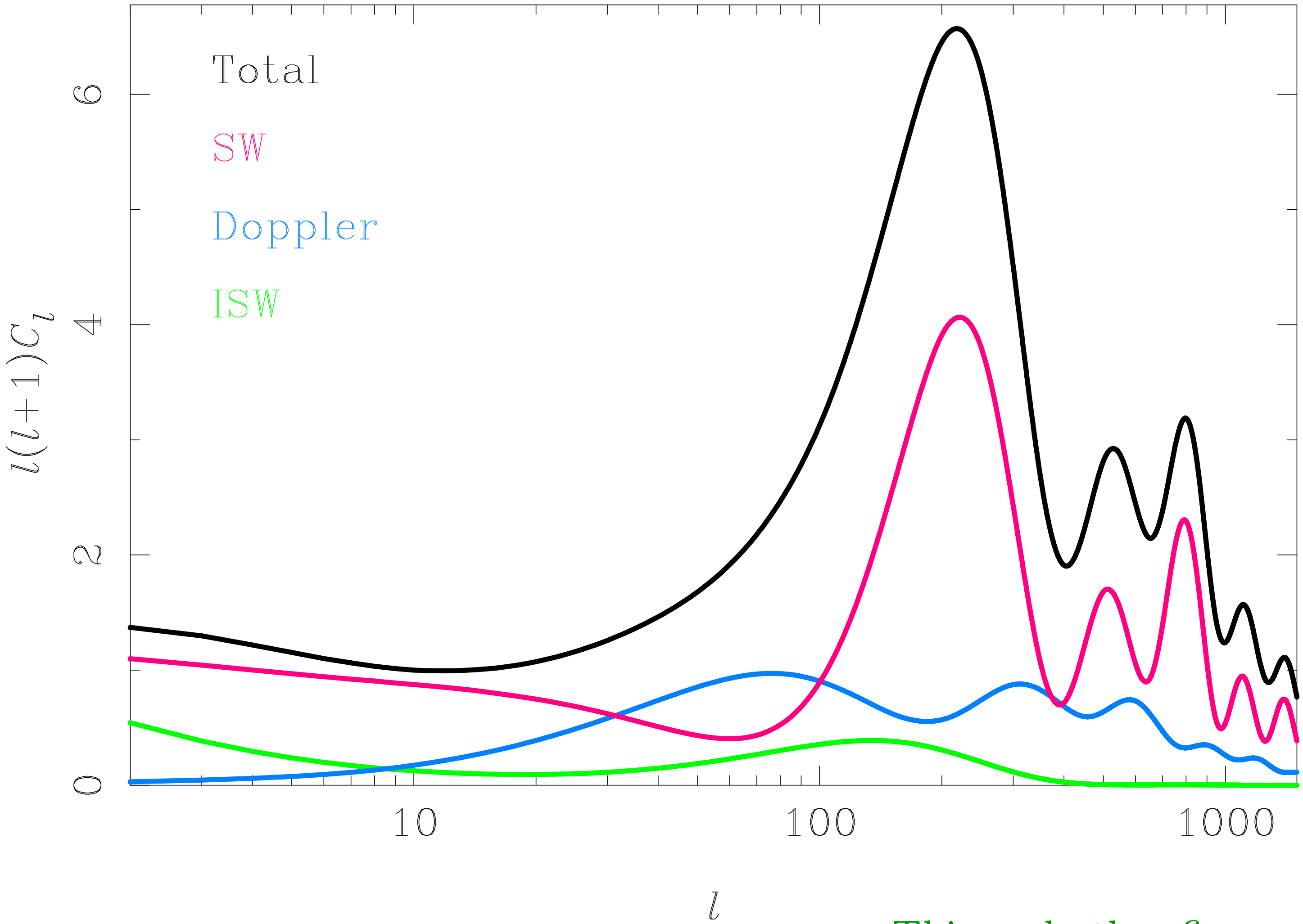
- Early ISW suppressed by
  ρ_{rad}/ρ_M $(\eta > \eta_r) < (1+z_r)/(1+z_{eq}) \sim 0.3$ in amplitude.
  Relatively large for $l \sim (2-4)\eta_0/\eta_r = 100-200$, where SW is quite large.

- Late ISW effect works for largest angular scales, but numerically small since dark energy has not yet diluted Φ substantially.

- There must be correlations of temperature with large structures, due to ISW.

  Detected. In principle, a tool for measuring expansion rate $\Longrightarrow$ properties of dark energy. Not at this stage yet.

Calculated angular spectrum. Adiabatic perturbations.



This and other figs.: see Challinor '2004

Acoustic peaks

Sometimes called Doppler peaks — wrong name.

Major player: Sachs–Wolfe effect. Doppler effect numerically smaller, since waves traveling normal to line of sight do not contribute.

- Short and intermediate scales, $l \gg 50$:

$$\rho_\gamma = -4A(k)\zeta \cos\left(\int_0^{\eta_r} k u_s \, d\eta\right) - 4(1+R_B)\Phi$$

$$\Phi = \Phi_{DM} = -\frac{B(k)}{k^2}\zeta$$

with $A(k) \simeq 1$, $B(k) \simeq \frac{27}{4} H_{eq}^2 a_{eq}^2 \log(0.15 k\eta_{eq})$ at large k

- Sachs–Wolfe term

$$\varphi_{SW} = \frac{1}{4}\delta_\gamma + \Phi = \text{osclliatory part} - R_B \, \Phi_{DM}$$

If not for $R_B \equiv 3\rho_B/(4\rho_\gamma)$, non-oscillating term with Φ_{DM} would cancel out.

Physics: before recombination, temperature is the same everywhere, even though local temperature is higher in potential well. If not for baryons, photons escaped from the well would have the same temperature as away from the well.

Mismatch: in thermal equilibrium are photons and baryons, but only photons move out of potential well.

Anyway,

$$\varphi_{SW} = \zeta \cdot \left(-A(k)\cos kr_s + \frac{B(k)}{k^2} R_B \right)$$

and

$$C_l/T_0^2 = 4\pi \int_0^\infty \frac{dk}{k} \mathscr{P}_\zeta(k) \left| -A(k)\cos kr_s + \frac{B(k)}{k^2} R_B \right|^2 j_l^2(k\eta_0)$$

Corrspondence* $k \leftrightarrow l/\eta_0 \Longleftrightarrow$ oscillations in k $\longleftrightarrow$ oscillations in l

Maxima at $kr_s \simeq \pi n \Longrightarrow l \simeq \pi n \eta_0 / r_s$.

Recall $a_0 r_s \simeq 155$ Mpc, $a_0 \eta_0 \simeq 14\,000$ Mpc $\Longrightarrow$ maxima at $l \simeq 290n$ (all sligtly shifted to the left).

Fig.

Interference between oscillating and non-oscillating terms: constructive for odd n; destructive for even n $\Longrightarrow$ odd peaks more pronounced.

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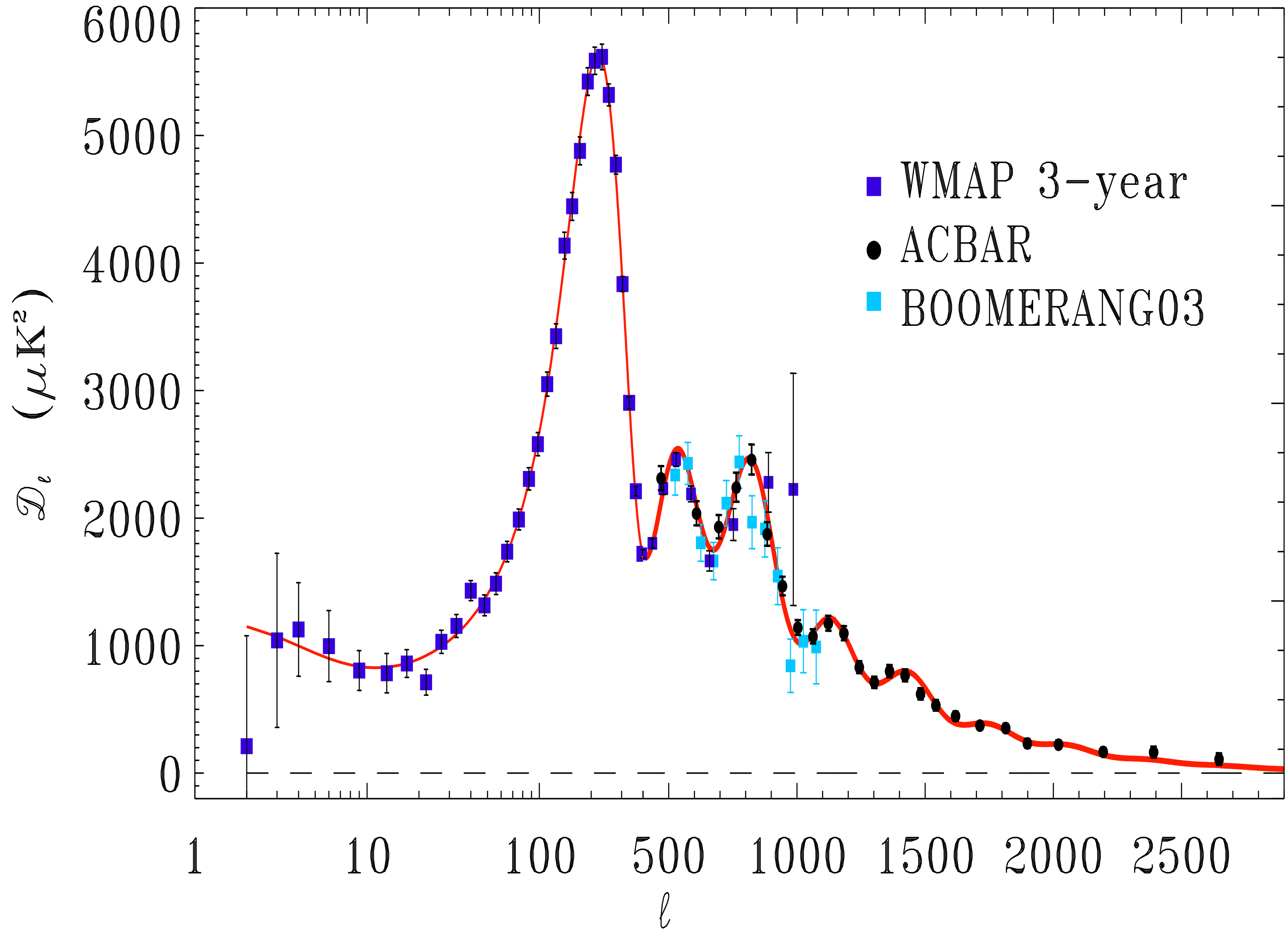
*Even more so for oscillating part:
$\int dk \cos(kr_s) j_l^2(k\eta_0)$ is saturated very near $k = l/\eta_0$; higher momenta get averaged out.



6000
5000
4000
3000
2000
1000
0
$\mathcal{D}_\ell$ (μK^2)
WMAP 3-year
ACBAR
BOOMERANG03
1
10
100
500
1000
1500
2000
2500
ℓ

NB: Oscillations get damped due to Silk effect, damping factor e^{2k^2/k_s^2} in amplitude squared $\Longrightarrow$ suppression for for $k \gtrsim k_s/\sqrt{2} \Longrightarrow$ $l \gtrsim l_s = k_s \eta_0/\sqrt{2}$. Recall $k_s/a_0 \simeq 0.1$ Mpc^{-1} $\Longrightarrow$ $l_s \simeq 1000$.

Overall decline due to B/k^2.

- What would tensor perturbations do?
  Recall that they decay as $a^{-1}(\eta)$ after horizon entry $\Longrightarrow$ maximum effect for $k \lesssim 1/\eta_r$, $l < \eta_0/\eta_r$

  Fig.

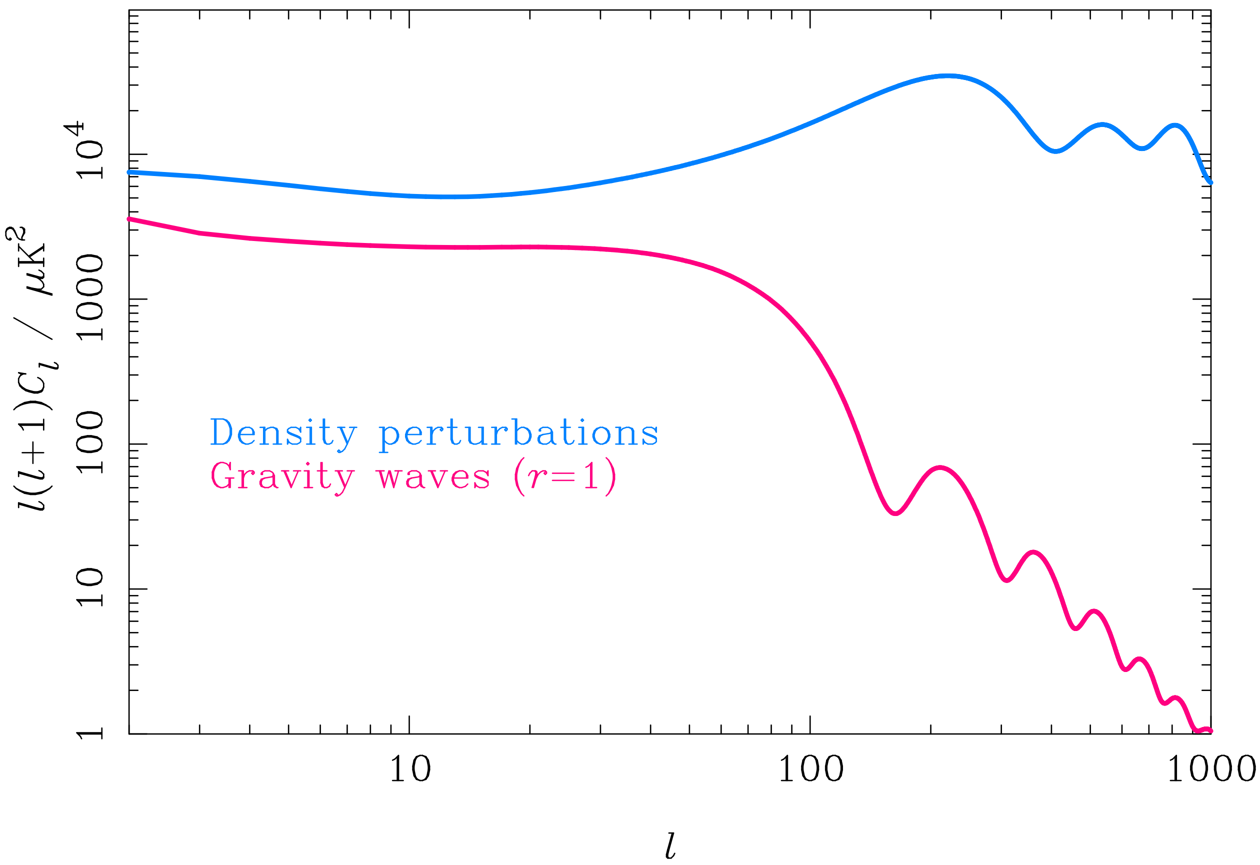
  Difficult to discriminate between tensor perturbations and red scalar tilt $n_s < 1$.

  Fig.

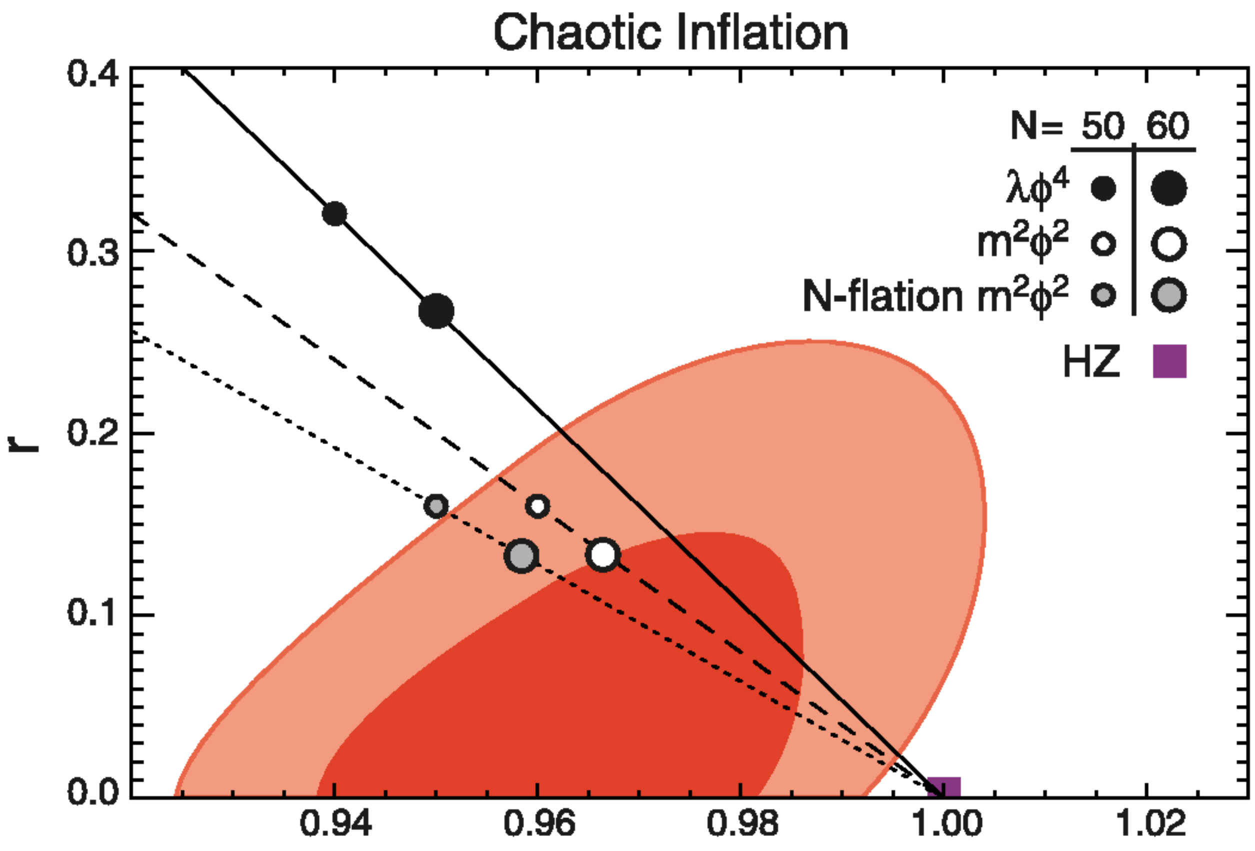
- What would CDM entropy perturbations do?
  Grossly different picture: $\sin(kr_s)$ instead of $\cos(kr_s)$, minima $\longleftrightarrow$ maxima.
  Rapid decrease of amplitude at large k.

  Fig.

Effect of tensor perturbations

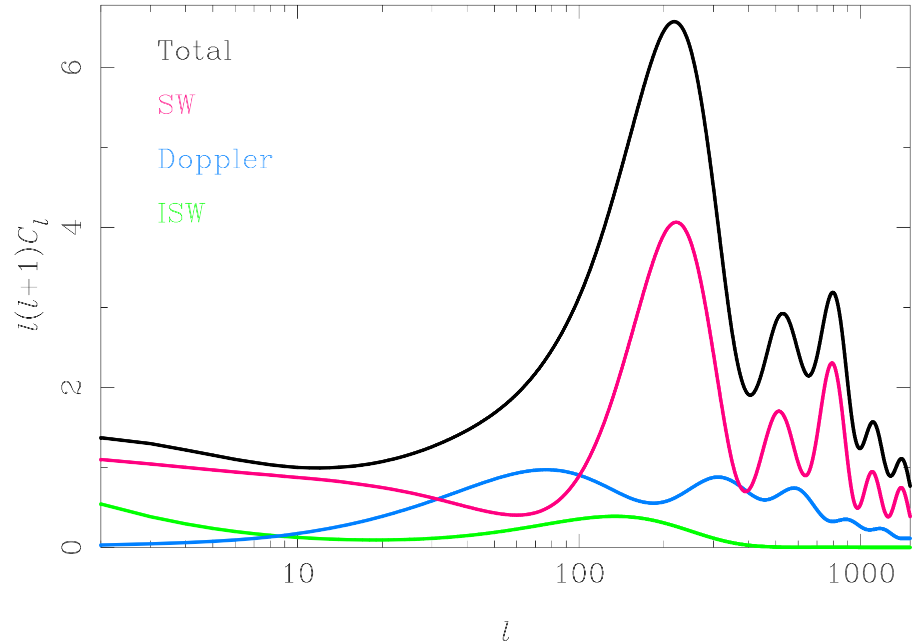


Scalar tilt vs tensor power

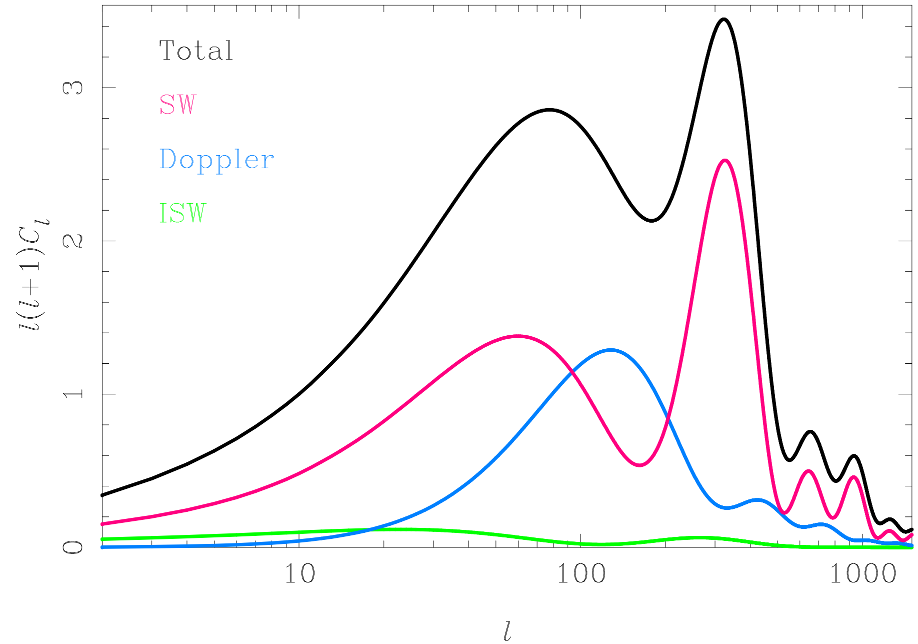


WMAP

Effects of adaiabatic and entropy perturbations



adiabatic perturbations



entropy perturbations

Examples of sensitivity to cosmological parameters

- Very sensitive to Ω_B through $R_B = 3\rho_B/(4\rho_\gamma)$. The larger Ω_B, the stronger interference effect, enhancement of odd peaks and suppression of even peaks.

  Fig.

- Peak positions very sensitive to spatial curvature: r_s is standard ruler at recombination, seen at different angles in open, flat and closed Universes.

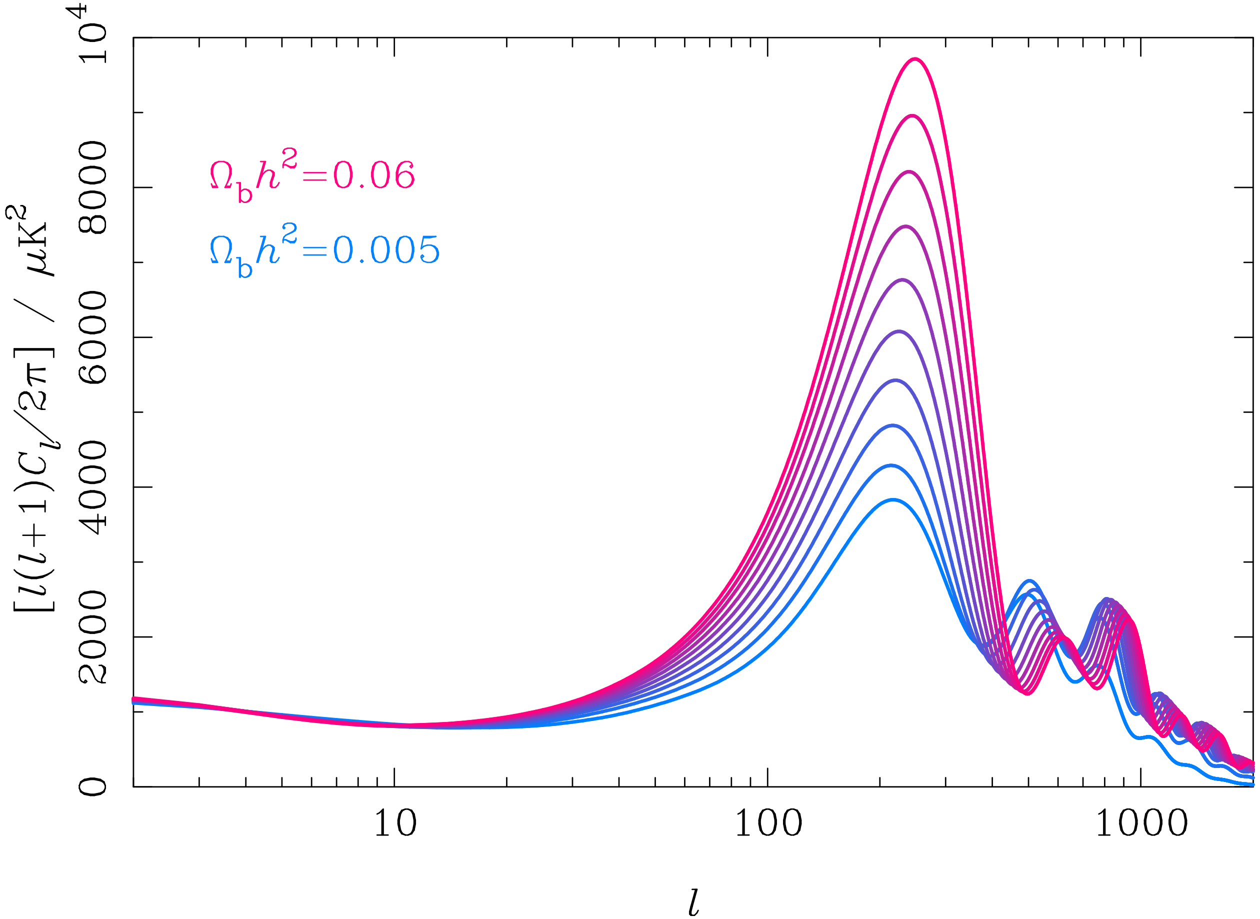
  Some degeneracy with Ω_Λ that determines conformal lifetime $\eta_0 \Longrightarrow$ distance to surface of last scattering.

  Fig.

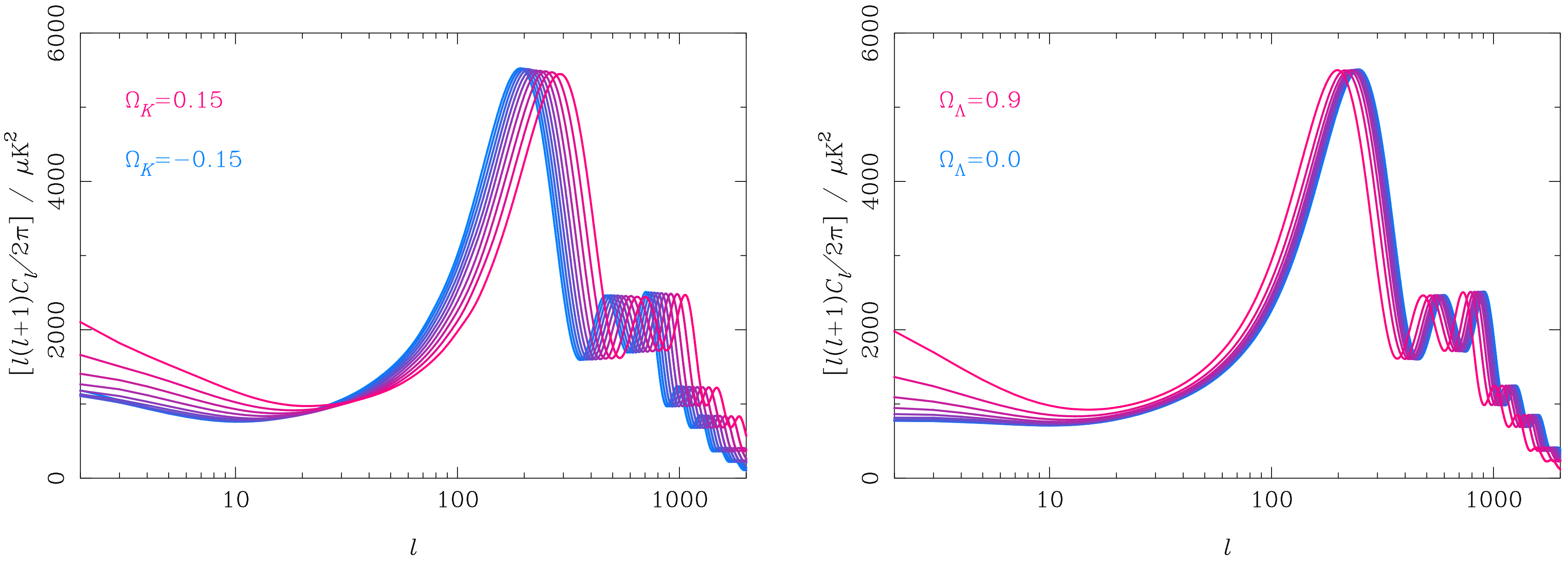
  Degeneracies lifted by other data.

- Updated fit of parameters: see Particle Data Group.

Effect of baryons



Effect of curvature (left) and $\mathbf{\Lambda}$



CMB polarization

Polarization in Thomson scattering:

$$\frac{d\sigma}{d\Omega} \propto \vec{\varepsilon}_i \cdot \vec{\varepsilon}_f$$

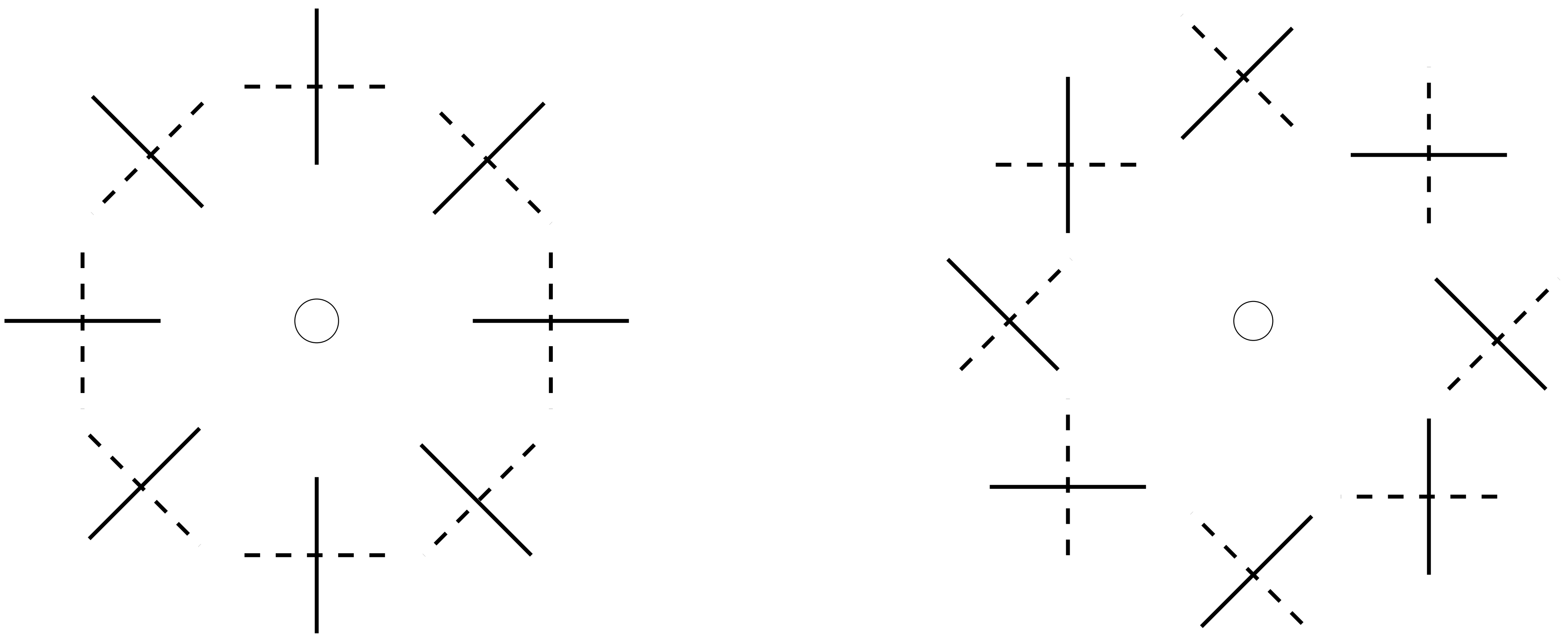
$\vec{\varepsilon}_{i,f}$ = polarization vectors of incoming and outgoing photons.

Photons with polarization normal to scattering plane scatter at larger cross section than in-plane polarized photons.

Unpolarized radiation coming to electron before the very last scattering from the right or left, is polarized in vertical direction after the very last scattering.

Temperature anisotropy of radiation incident on electron before the very last scattering results in linear polarization of radiation we see today.

This temperature anisotropy is generated similarly to δT we observe now, but locally, at time just preceding last scattering.



E-mode: hot (dashed) and cold (solid) spot. B-mode

E- and B-modes

Polarization tensor on celestial sphere

$$P_{ab} = \frac{\langle E_a E_b - \frac{1}{2}\delta_{ab}\vec{E}^2 \rangle}{\langle \vec{E}^2 \rangle}$$

$\vec{E}$ = electric field, normal to line of sight, $a, b = 1, 2$.

Can be written in terms of scalar P_E and pseudoscalar P_B:

$$P_{ab}(\mathbf{n}) = -\left(\nabla_a \nabla_b - \frac{1}{2}\delta_{ab}\right) P_E(\mathbf{n}) - \varepsilon^c{}_{(a}\nabla_{b)}\nabla_c \, P_B(\mathbf{n})$$

Repeat the story: decompose P_E and P_B in spherical harmonics, define a^E_{lm}, a^B_{lm} and correlation and cross-correlation spectra:

$$\langle a^E_{lm} a^{E\,*}_{l'm'} \rangle = \delta_{ll'}\delta_{mm'} C^{EE}_l \ ,$$

$$\langle a^T_{lm} a^{E\,*}_{l'm'} \rangle = \delta_{ll'}\delta_{mm'} C^{TE}_l \ , \quad \text{etc.}$$

On symmetry grounds: do not expect EB and TB cross-correlations.

EE and TE spectra measured (with rather large errors)

- Point: scalar perturbations produce only E-mode. Tensor perturbations produce both E- and B-mode
- Small effect: exists to the extent that photon experiences integrated Sachs–Wolfe effect when traveling between last-before-last scattering to last scattering events (true also for E-mode; re-ionization helps for very long waves).
  Suppression factor

$$\frac{\lambda_\gamma}{\lambda_{pert}} \sim k\Delta\eta \lesssim \frac{\Delta\eta}{\eta_r} \simeq 0.04 \quad \text{of} \quad \frac{\delta T}{T}$$

  $\lambda_\gamma =$ photon mean free path before the very last scattering (thickness of last scattering shell $\Delta\eta$),
  $\lambda_{pert} = 2\pi a/k =$ wavelength of perturbation at recombination.

- Yet the most promising way of detecting tensor perturbations

To conclude:

- CMB encodes a lot of information about late Universe and primordial perturbations
- Primordial perturbations is a window to pre-hot cosmological epoch.
- No doubt that this epoch existed: CMB properties can only be explained by assuming that perturbations were built in at the very beginning of the hot stage.
- Still we know only very basic facts about primordial perturbations.

- More to come
  - Precise determination of scalar tilt (Planck)
  - Primordial tensor perturbations (maybe Planck)
  - Non-Gaussianity (maybe already observed, watch out Planck)
  - Statistical anisotropy (maybe already observed, watch out Planck)
  - Isocurvature perturbations (will be great surprize)

Hopefully, not only limits....