Bloch–Nordsieck effective theory and HQET

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Velocity

$$\vec{v} = \frac{\partial \varepsilon(\vec{p})}{\partial \vec{p}} = \frac{\vec{p}}{M} \to 0$$

Lagrangian

$$L = h^{+} i\partial_0 h$$

equation of motion

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$$\varepsilon = -eA_0$$

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Not Lorentz-invariant

Lagrangian

+ Lagrangian of the photon field

$$\partial_\mu F^{\mu\nu} = j^\nu$$
$$j^0 = -eh^+h$$

The electron produces the Coulomb field

Spin symmetry

At the leading order in $1/M$, the electron spin does not interact with electromagnetic field

We can rotate it without affecting physics

In addition to the $U(1)$ symmetry $h \to e^{i\alpha} h$, also the $SU(2)$ spin symmetry

$$h \to U h$$

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The electron magnetic moment $\vec{\mu} = \mu\vec{\sigma}$
interacts with magnetic field: $-\vec{\mu}\cdot\vec{B}$
By dimensionality $\mu \sim e/M$
(Bohr magneton $e/(2M)$ up to radiative corrections)

$$L_m = -\frac{e}{2M} h^+ \vec{B}\cdot\vec{\sigma} h$$

Violates the $SU(2)$ spin symmetry at the $1/M$ level

Spin-flavour symmetry

n_f flavours of heavy fermions

$$L = \sum_{i=1}^{n_f} h_i^+ i D_0 h_i$$

$U(1) \times SU(2n_f)$ symmetry
Broken at $1/M_i$ by kinetic energy and magnetic interaction

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$U(1) \times SU(2n_f)$ symmetry
Broken at $1/M_i$ by kinetic energy and magnetic interaction
At the leading order in $1/M$, not only the spin direction but also its magnitude is irrelevant
We can, for example, switch the electron spin off:

$$L = \varphi^* i D_0 \varphi$$

Superflavour symmetry

The scalar and the spinor fields together

$$L = \varphi^* i D_0 \varphi + h^+ i D_0 h$$

$U(1) \times SU(3)$ symmetry

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The superflavour $SU(3)$ symmetry:

- $\varphi \to e^{2i\alpha}\varphi$, $h \to e^{-i\alpha} h$
- $SU(2)$ spin rorations
- $$\delta \begin{pmatrix} \varphi \\ h \end{pmatrix} = i \begin{pmatrix} 0 & \varepsilon^+ \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ h \end{pmatrix}$$

  ε — an infinitesimal spinor

Broken at $1/M$

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Broken at $1/M$
We can consider, e. g., spins $\frac{1}{2}$ and 1
$SU(5)$ superflavour symmetry

Feynman rules

Leading order in $1/M$

$$L = \varphi_0^* i D_0 \varphi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^\mu)^2$$

The usual photon propagator

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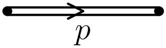
The momentum-space free electron propagator



$$= iS_0(p) \qquad S_0(p) = \frac{1}{p_0 + i0}$$

depends only on p_0, not on $\vec{p}$

(spin-$\frac{1}{2}$ field h_0 — the unit 2×2 spin matrix)

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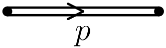

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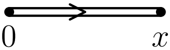
The coordinate-space propagator



$$= iS_0(x) \qquad S_0(x) = S_0(x_0)\delta(\vec{x}\,) \qquad S_0(t) = -i\theta(t)$$

Static electron does not move

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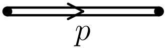

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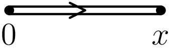

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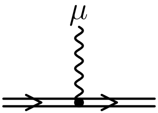
Solving the equation

$$i\partial_0 S_0(x) = \delta(x)$$

Feynman rules

Vertex



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$$v^\mu = (1, \vec{0}\,)$$

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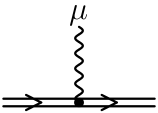

$= ie_0 v^\mu$

$v^\mu = (1, \vec{0}\,)$

The static field φ_0 (or h_0) describes only particles, there are no antiparticles.
No loops formed by static-electron propagators.
The electron propagates only forward in time;
the product of θ functions for a loop vanishes.
In the momentum space: all poles of the propagators are in the lower p_0 half-plane;
closing the integration contour upwards, we get 0.

Wilson line

In an external field $A^\mu(x)$

$$iD_0 S(x,x') = (i\partial_0 + e_0 A^0(x))S(x,x') = \delta(x-x')$$

Solution

$$S(x,x') = S(x_0,x'_0)\delta(\vec{x}-\vec{x}\,') \qquad S(x_0,x'_0) = S_0(x_0-x'_0)W(x_0,x'_0)$$

Wilson line from x' to x (along v)

$$W(x_0,x'_0) = \exp ie_0 \int_{x'_0}^{x_0} A^\mu(t,\vec{x}\,)v_\mu dt$$

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The HQET Lagrangian has been introduced as a device to investigate of Wilson lines

Gauge $A^0 = 0$

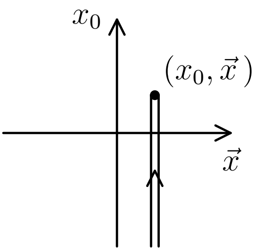
The field $\varphi_0(x)$ does not interact with the electromagnetic field (and thus becomes free).
However, this gauge is rather pathological.
The static electron creates the Coulomb electric field $\vec{E}$.
In the $A^0 = 0$ gauge, $\vec{A}$ has to depend on t linearly.

Gauge $A^0 = 0$

We can formally express the field $\varphi_0(x)$ in any gauge via a free field $\varphi^{(0)}(x)$:

$$\varphi_0(x) = W(x)\varphi^{(0)}(x)$$

$$W(x_0, \vec{x}\,) = P \exp i \int_{-\infty}^{x_0} A_0^\mu(t, \vec{x}\,) v_\mu dt$$



Then $W^{-1}(x) D_0 W(x) = \partial_0$, and

$$L = \varphi^{(0)+} i\partial_0 \varphi^{(0)}$$

Residual momentum

The full-theory energy M is the HEET zero level

$$E = M + \varepsilon$$

ε — the residual energy

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$$P^\mu = M v^\mu + p^\mu$$

- P^μ — 4-momentum of some state (containing a single electron) in the full theory
- p^μ — its momentum in HEET (the residual momentum)

v^μ — 4-velocity of a reference frame in which the electron always stays approximately at rest

Reparametrization invariance

HEET is applicable if there exists such v that

$$p^\mu \ll M \qquad p^\mu_{\gamma i} \ll M$$

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This condition does not fix v uniquely: $v \to v + \delta v$, $\delta v \sim p/M$.
Effective theories corresponding to different choices of v must produce identical physical predictions: reparametrization invariance.
Relations between quantities at different orders in $1/M$.

Relativistic notation

Lagrangian

$$L = \varphi_0^* i v \cdot D \varphi_0 + (\text{light fields})$$

Free propagator

$$S_0(p) = \frac{1}{p \cdot v + i0}$$

Mass shell

$$p \cdot v = 0$$

Spin $\frac{1}{2}$

4-component spinor field

$$\not{v} h_v = h_v$$

Lagrangian

$$L = \bar{h}_{v0} i v \cdot D h_{v0} + (\text{light fields})$$

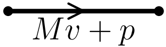
Propagator

$$S_0(p) = \frac{1 + \not{v}}{2} \frac{1}{p \cdot v + i0}$$

Vertex $i e_0 v^\mu$

Qedland

$$S_0(Mv+p) = \frac{M + M\not{v} + \not{p}}{(Mv+p)^2 - M^2 + i0} = \frac{1+\not{v}}{2}\,\frac{1}{p\cdot v + i0} + \mathcal{O}\left(\frac{p}{M}\right)$$



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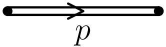

$+\,\mathcal{O}\left(\frac{p}{M}\right)$

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$$\frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2} = \frac{1+\not{v}}{2}v^\mu\frac{1+\not{v}}{2}$$

We may insert the projectors $(1+\not{v})/2$ before $u(P_i)$ and after $\bar{u}(P_i)$, too, because

$$\not{v}u(Mv+p) = u(Mv+p) + \mathcal{O}\left(\frac{p}{M}\right)$$

Qedland

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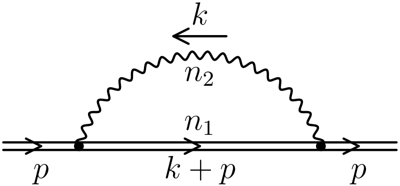
We have derived the HEET Feynman rules from the QED ones at $M \to \infty$. Therefore, we again arrive at the HEET Lagrangian which corresponds to these Feynman rules. We have thus proved that at the tree level any QED diagram is equal to the corresponding HEET diagram up to $\mathcal{O}(p/M)$ corrections. This is not true at loops, because loop momenta can be arbitrarily large. Renormalization properties of HEET (anomalous dimensions, etc.) differ from those in QED.

One-loop diagrams



$$\frac{1}{i\pi^{d/2}} \int \frac{d^d k}{\left[-2(k+p)_0 - i0\right]^{n_1} \left[-k^2 - i0\right]^{n_2}}$$
$$= I(n_1, n_2)(-2\omega)^{d-n_1-2n_2}$$

Depends only on $\omega = p_0$, not $\vec{p}$

$\omega > 0$ — real pair production, cut

- integer $n_1 \leq 0$ — massless vacuum diagram $I(n_1, n_2) = 0$
- integer $n_2 \leq 0$ — HEET loop $I(n_1, n_2) = 0$

Coordinate space

$$\int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{(-2\omega - i0)^n} \frac{d\omega}{2\pi} = \frac{i}{2\Gamma(n)} \left(\frac{it}{2}\right)^{n-1} e^{-0t}\theta(t)$$

$$\int_{0}^{\infty} e^{(i\omega - 0)t} \left(\frac{it}{2}\right)^{n-1} dt = -\frac{2i\Gamma(n)}{(-2\omega - i0)^n}$$

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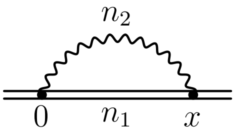
$$\int \frac{e^{-ip\cdot x}}{(-p^2 - i0)^n} \frac{d^d p}{(2\pi)^d} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(d/2 - n)}{\Gamma(n)} \left(\frac{4}{-x^2 + i0} \right)^{d/2-n}$$

$$\int \left(\frac{4}{-x^2 + i0} \right)^n e^{ip\cdot x} d^d x = -i(4\pi)^{d/2} \frac{\Gamma(d/2 - n)}{\Gamma(n)} \frac{1}{(-p^2 - i0)^{d/2-n}}$$

Coordinate space



Product of propagators ($x = vt$, $-x^2/4 = -t^2/4 = (it/2)^2$)

$$-\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)}\left(\frac{it}{2}\right)^{n_1+2n_2-d-1}\theta(t)$$

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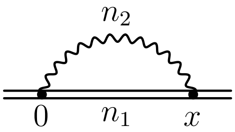

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Inverse Fourier transform

$$\frac{i}{(4\pi)^{d/2}} I(n_1,n_2)(-2\omega)^{d-n_1-2n_2}$$

$$I(n_1,n_2) = \frac{\Gamma(n_1+2n_2-d)\Gamma\left(\frac{d}{2}-n_2\right)}{\Gamma(n_1)\Gamma(n_2)}$$

α parametrization

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\alpha\, \alpha^{n-1}\, e^{-a\alpha}$$

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$$\frac{1}{\Gamma(n_1)\Gamma(n_2)} \int d\alpha\, \alpha^{n_2-1}\, d\beta\, \beta^{n_1-1}\, d^d k\, e^X$$

$$X = \alpha k^2 + 2\beta(k+p)\cdot v$$

α has dimensionality $1/m^2$, β — $1/m$

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$k = k' - \frac{\beta}{\alpha} v$

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Wick rotation $k_0 = i k_{E0}$
Euclidean momentum space $(k^2 = -k_E^2)$

$$\int d^d k\, e^{\alpha k^2} = i \int d^d k_E\, e^{-\alpha k_E^2} = i \left(\frac{\pi}{\alpha}\right)^{d/2}$$

α parametrization

$$(-2\omega)^{d-n_1-2n_2} I(n_1, n_2) = \frac{1}{\Gamma(n_1)\Gamma(n_2)}$$

$$\times \int d\alpha \, \alpha^{n_2-1} \, d\beta \, \beta^{n_1-1} \, \alpha^{-d/2} \exp\left(-\frac{\beta^2}{\alpha} + 2\beta\omega\right)$$

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$$\times \int d\alpha\, \alpha^{n_2-1}\, d\beta\, \beta^{n_1-1}\, \alpha^{-d/2} \exp\left(-\frac{\beta^2}{\alpha} + 2\beta\omega\right)$$

$\beta = \alpha y$, integrate in α

$$\frac{\Gamma\left(n_1+n_2-\frac{d}{2}\right)}{\Gamma(n_1)\Gamma(n_2)} \int_0^\infty dy\, y^{n_1-1} \left[y(y-2\omega)\right]^{d/2-n_1-n_2}$$

HQET Feynman parameter y has the dimensionality of energy and varies from 0 to ∞

HQET Feynman parametrization

$$\frac{1}{a^n b^m} = \frac{1}{\Gamma(n)\Gamma(m)} \int d\alpha\, \alpha^{n-1}\, d\beta\, \beta^{m-1}\, e^{-a\alpha - b\beta}$$

$\beta = \alpha y$, integrate in α

$$\frac{1}{a^n b^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^\infty \frac{y^{m-1} dy}{(a+by)^{n+m}}$$

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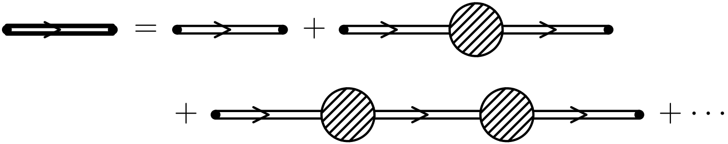
$$\frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int \frac{y^{n_1-1}\, dy\, d^d k}{(-k^2 - 2y(k+p)\cdot v)^{n_1+n_2}}$$

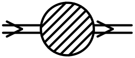
$k = k' - yv$

Static-electron propagator

The full propagator $S(p)$ depends only $\omega = p_0$, not $\vec{p}$



Static-electron self-energy



$= -i\Sigma(\omega)$

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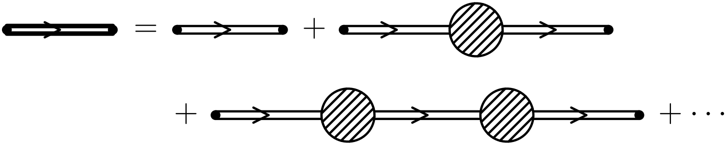

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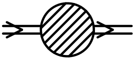


$$= -i\Sigma(\omega)$$

$$iS(\omega) = iS_0(\omega) + iS_0(\omega)(-i)\Sigma(\omega)iS_0(\omega) + iS_0(\omega)(-i)\Sigma(\omega)iS_0(\omega)(-i)\Sigma(\omega)iS_0(\omega) + \cdots$$

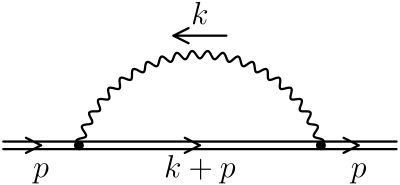
$S_0(\omega) = 1/\omega$ — free propagator

Static-electron propagator

$$S(\omega) = S_0(\omega) + S_0(\omega)\Sigma(\omega)S(\omega)$$
$$S^{-1}(\omega) = S_0^{-1}(\omega) - \Sigma(\omega)$$

$$S(\omega) = \frac{1}{\omega - \Sigma(\omega)}$$

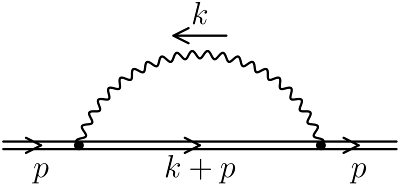
1 loop



$$\Sigma(\omega) = i \int \frac{d^d k}{(2\pi)^d} i e_0 v^\mu \frac{1}{k_0 + \omega} i e_0 v^\nu \frac{-i}{k^2} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

$\xi = 1 - a_0$

1 loop



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$\xi = 1 - a_0$
Numerator $(k \cdot v)^2 = (k_0 + \omega - \omega)^2 \to \omega^2$

$$\Sigma(\omega) = \frac{e_0^2 (-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}} \left[2 I(1,1) + \frac{\xi}{2} I(1,2) \right]$$

$$= \frac{e_0^2 (-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)}{d-4} \left(\xi + \frac{2}{d-3} \right)$$

Vanishes in the d-dimensional Yennie gauge

$$a_0 = \frac{2}{d-3} + 1$$

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x space

Free photon propagator

$$D^0_{\mu\nu}(x) = \frac{i\Gamma(d/2-1)}{8\pi^{d/2}} \frac{(1+a_0)x^2 g_{\mu\nu} + (d-2)(1-a_0)x_\mu x_\nu}{(-x^2+i0)^{d/2}}$$

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$$\begin{aligned}\Sigma(x) &= -e_0^2 D^0_{\mu\nu}(vt) v^\mu v^\nu \theta(t) \\ &= ie_0^2 \frac{\Gamma(d/2-1)}{8\pi^{d/2}} (d-3) \left(\xi + \frac{2}{d-3}\right) (it)^{2-d} \theta(t)\end{aligned}$$

Transform to p space

Propagator to 1 loop

$$S(\omega) = S_0(\omega)\left[1 - \frac{e_0^2(-2\omega)^{-2\varepsilon}}{(4\pi)^{d/2}} \frac{2\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)}{d-4} \left(\xi + \frac{2}{d-3}\right) + \mathcal{O}(e_0^4)\right]$$

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x space

$$S(t) = S_0(t)\left[1 - \frac{e_0^2}{(4\pi)^{d/2}}\left(\frac{it}{2}\right)^{2\varepsilon}\Gamma(-\varepsilon)\left(\xi + \frac{2}{d-3}\right) + \mathcal{O}(e_0^4)\right]$$

$(S_0(t) = -i\theta(t))$
Real in the Euclidean space $t = -i\tau$

Renormalization

$$S(\omega) = S_0(\omega)\left[1 + \frac{\alpha}{4\pi\varepsilon} e^{-2L\varepsilon}\left(3 - a + 4\varepsilon + \mathcal{O}(\varepsilon^2)\right) + \mathcal{O}(\alpha^2)\right]$$

$$L = \log\frac{-2\omega}{\mu}$$

Renormalization

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$$L = \log \frac{-2\omega}{\mu}$$

Should be $Z_h(\alpha(\mu), a(\mu)) S_r(\omega; \mu)$

$$Z_h(\alpha, a) = 1 - (a - 3)\frac{\alpha}{4\pi\varepsilon} + \mathcal{O}(\alpha^2)$$

Renormalization

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Should be $Z_h(\alpha(\mu), a(\mu))S_r(\omega; \mu)$

$$Z_h(\alpha, a) = 1 - (a-3)\frac{\alpha}{4\pi\varepsilon} + \mathcal{O}(\alpha^2)$$

Anomalous dimension of the static electron field

$$\gamma_h(\alpha, a) = 2(a-3)\frac{\alpha}{4\pi} + \mathcal{O}(\alpha^2)$$

Vanishes in the Yennie gauge $a = 3$

x space

$$S(t) = S_0(t)\left[1 + \frac{\alpha}{4\pi\varepsilon}e^{2L_t\varepsilon}\left(3 - a + 4\varepsilon + \mathcal{O}(\varepsilon^2)\right) + \mathcal{O}(\alpha^2)\right]$$

$$L_t = \log\frac{i\mu t}{2} + \gamma_E$$

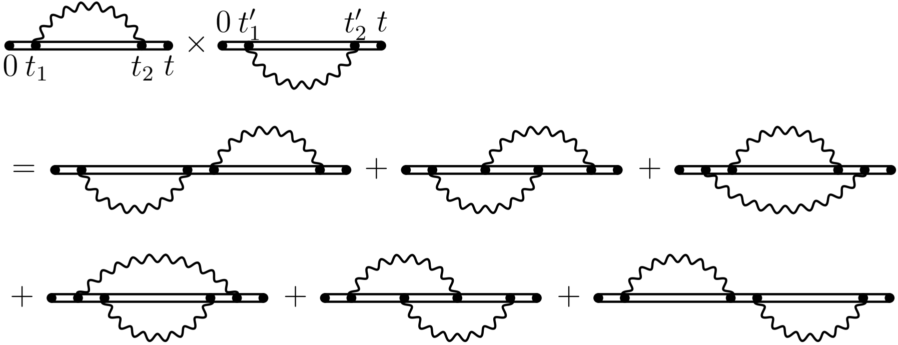
Exponentiation

1-loop correction to x-space propagator, multiply by itself

Integral in t_1, t_2, t_1', t_2' with $0 < t_1 < t_2 < t$, $0 < t_1' < t_2' < t$

Ordering of primed and non-primed t's can be arbitrary

6 regions corresponding to 6 diagrams



Exponentiation

This is 2× the 2-loop correction
1-loop correction cubed is 3!× the 3-loop correction, . . .

$$S(t) = S_0(t) \exp \left[-\frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{it}{2} \right)^{2\varepsilon} \Gamma(-\varepsilon) \left(\xi + \frac{2}{d-3} \right) \right]$$

In the d-dimensional Yennie gauge the exact propagator is free

Exponentiation

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1-loop correction cubed is $3!\times$ the 3-loop correction, ...

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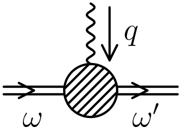
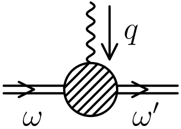
In the d-dimensional Yennie gauge the exact propagator is free
No corrections to the photon propagator: $Z_A = 1$,
Therefore, the photon field is not renormalized: $Z_A = 1$,
$a = a_0$

Vertex



$$= ie_0 v^\mu \Gamma(\omega, \omega') \qquad \Gamma(\omega, \omega') = 1 + \Lambda(\omega, \omega')$$

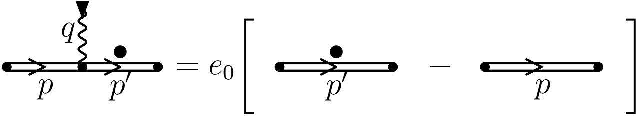
Vertex



$$= ie_0 v^\mu \Gamma(\omega, \omega') \qquad \Gamma(\omega, \omega') = 1 + \Lambda(\omega, \omega')$$

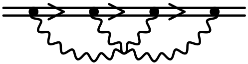
Identity

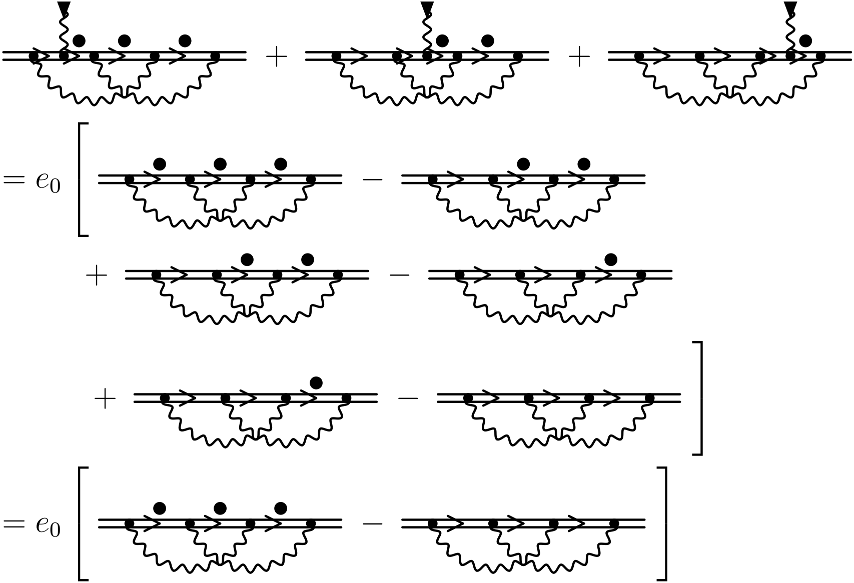
$$iS_0(p')\; ie_0 v \cdot q\; iS_0(p) = ie_0 \left[S_0(p') - S_0(p)\right]$$



Ward identity

Starting from each diagram for Σ, we can obtain a set of diagrams for Λ by inserting the external photon vertex into each electron propagator. After contracting with q_μ, each diagram in this set becomes a difference. For example,

 $\Rightarrow$



= e0
= e0

Ward identity

$$\Lambda(\omega, \omega') = -\frac{\Sigma(\omega') - \Sigma(\omega)}{\omega' - \omega}$$

$$\Gamma(\omega, \omega') = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega}$$

Ward identity

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$Z_\Gamma Z_h = 1$, $Z_\alpha = (Z_\Gamma Z_h)^{-2} Z_A^{-1} = Z_A^{-1} = 1$ — the electron charge is not renormalized in HEET

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$e_0 \to e$, $a_0 \to a$ in the bare propagator

$$Z_h = \exp\left[-(a-3)\frac{\alpha}{4\pi\varepsilon}\right]$$

$$\gamma_h = 2(a-3)\frac{\alpha}{4\pi}$$

Vanishes in the Yennie gauge

Operators

Full QED operators — series in $1/M$
via HEET operators

$$O(\mu) = C(\mu)\tilde{O}(\mu) + \frac{1}{2M}\sum_i B_i(\mu)\tilde{O}_i(\mu) + \cdots$$

Matching on-shell matrix elements

Electron field

$$\psi_0(x) = e^{-iMv\cdot x}\left[z_0^{1/2}h_{v0}(x) + \cdots\right]$$

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On-shell matrix elements

$$<0|\psi_0|e(p)> = \left(Z_\psi^{\mathrm{os}}\right)^{1/2} u(p)$$
$$<0|h_{v0}|e(p)> = \left(Z_h^{\mathrm{os}}\right)^{1/2} u_v(k)$$

Bare matching coefficient $(Z_h^{\mathrm{os}} = 1)$

$$z_0 = \frac{Z_\psi^{\mathrm{os}}(e_0^{(1)})}{Z_h^{\mathrm{os}}(e_0^{(0)})}$$

Electron field

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Bare matching coefficient ($Z_h^{\text{os}} = 1$)

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Renormalized matching coefficient

$$z(\mu) = \frac{Z_h(\alpha^{(0)}(\mu), a^{(0)}(\mu))}{Z_\psi(\alpha_s^{(1)}(\mu), a^{(1)}(\mu))} z_0$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

$$S(x) = S_L(x)$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta(k) k_\mu k_\nu$$

$$S(x) = S_L(x)\, e^{-ie_0^2(\tilde{\Delta}(x) - \tilde{\Delta}(0))}$$

$$\tilde{\Delta}(x) = \int \Delta(k) e^{-ikx} \frac{d^d k}{(2\pi)^d}$$

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$$\Delta(k) = \frac{a_0}{(k^2)^2} \qquad \tilde{\Delta}(0) = 0 \text{ in dim. reg.}$$

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Landau, Khalatnikov (1955)
Fradkin (1955)
Zumino (1960)
Fukuda, Kubo, Yokoyama (1980)
Bogoliubov, Shirkov (1980)

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$S(x) = S_0(x) e^{\sigma(x)}$$

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$$S(x) = S_0(x) e^{\sigma(x)}$$

$$\sigma(x) = \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^\varepsilon \Gamma(-\varepsilon)$$

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$$= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4} \right)^{\varepsilon} e^{\gamma_E \varepsilon} \Gamma(-\varepsilon)$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$
\begin{aligned}
S(x) &= S_0(x) e^{\sigma(x)} \\
\sigma(x) &= \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^{\varepsilon} \Gamma(-\varepsilon) \\
&= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4} \right)^{\varepsilon} e^{\gamma_E \varepsilon} \Gamma(-\varepsilon) \\
&= \log Z_\psi + \sigma_r
\end{aligned}
$$

Gauge dependence of Z_ψ, γ_ψ

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$$= \log Z_\psi + \sigma_r$$

$$\log Z_\psi(\alpha, a) = \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon}$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

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S(x) &= S_0(x) e^{\sigma(x)} \\
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&= \log Z_\psi + \sigma_r \\
\log Z_\psi(\alpha, a) &= \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon} \\
\gamma_\psi(\alpha, a) &= 2a \frac{\alpha}{4\pi} + \gamma_L(\alpha)
\end{aligned}$$

$d \log(a(\mu)\alpha(\mu)) / d \log \mu = -2\varepsilon$ exactly
$\gamma_L(\alpha)$ starts from α^2

Gauge dependence of Z_ψ, γ_ψ

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$$

$d \log(a(\mu)\alpha(\mu))/d \log \mu = -2\varepsilon$ exactly
$\gamma_L(\alpha)$ starts from α^2
4 loops: Chetyrkin, Rétey (2000)

Gauge independence of $z(\mu)$ in QED

- $z_0 = Z_\psi^{\text{os}}$ gauge invariant
- $\log Z_h = (3 - a^{(0)}) \dfrac{\alpha^{(0)}}{4\pi\varepsilon}$
  $\alpha^{(0)} = \alpha_{\text{os}} \approx 1/137$
- $\log Z_\psi = -a^{(1)}(\mu) \dfrac{\alpha^{(1)}(\mu)}{4\pi\varepsilon} + (\text{gauge invariant})$
- Decoupling $a^{(1)}\alpha^{(1)} = a^{(0)}\alpha^{(0)}$
  Gauge dependence cancels in $\log(\tilde{Z}_\psi / Z_\psi)$

Result

$$z(M) = 1 - \frac{\alpha}{\pi} + \left(\pi^2 \log 2 - \frac{3}{2}\zeta_3 - \frac{55}{48}\pi^2 + \frac{5957}{1152}\right)\left(\frac{\alpha}{\pi}\right)^2 + \cdots$$

Electron propagator near the mass shell

On-shell mass $M = M_0 + \delta M$, $\omega \ll M$

$$p = (M + \omega)v \qquad \Sigma(p) = \Sigma_0(\omega) + \Sigma_1(\omega)(\not{v} - 1)$$

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$$\begin{aligned} S(p) &= \frac{1}{\not{p} - M_0 - \Sigma(p)} \\ &= \frac{1}{[M + \omega - \Sigma_1(\omega)]\,\not{v} - M + \delta M - \Sigma_0(\omega) + \Sigma_1(\omega)} \end{aligned}$$

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$$S(p) = \frac{1}{\rlap{/}p - M_0 - \Sigma(p)}$$
$$= \frac{1}{[M + \omega - \Sigma_1(\omega)]\,\rlap{/}v - M + \delta M - \Sigma_0(\omega) + \Sigma_1(\omega)}$$

The denomunator

$$[M + \omega - \Sigma_1(\omega)]^2 - [M - \delta M + \Sigma_0(\omega) - \Sigma_1(\omega)]^2$$

should vanish at $\omega = 0$:

$$\delta M = \Sigma_0(0)$$

Electron propagator near the mass shell

$$S(p) = \frac{1}{[M+\omega-\Sigma_1(\omega)]\,\not{v} - M - \Sigma_0(\omega) + \Sigma_0(0) + \Sigma_1(\omega)}$$
$$= \frac{[M+\omega-\Sigma_1(\omega)]\,\not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M+\omega-\Sigma_1(\omega)]^2 - [M+\Sigma_0(\omega)-\Sigma_0(0)-\Sigma_1(\omega)]^2}$$

Electron propagator near the mass shell

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The denominator at $\omega \to 0$

$$[M - \Sigma_1(0) + \omega - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$- [M - \Sigma_1(0) + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$\approx 2\,(M - \Sigma_1(0))\,[\omega - \Sigma_0(\omega) + \Sigma_0(0)]$$

Electron propagator near the mass shell

$$S(p) = \frac{1}{[M+\omega-\Sigma_1(\omega)]\,\not{v} - M - \Sigma_0(\omega) + \Sigma_0(0) + \Sigma_1(\omega)}$$
$$= \frac{[M+\omega-\Sigma_1(\omega)]\,\not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M+\omega-\Sigma_1(\omega)]^2 - [M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)]^2}$$

The denominator at $\omega \to 0$

$$[M - \Sigma_1(0) + \omega - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$- [M - \Sigma_1(0) + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$\approx 2\,(M - \Sigma_1(0))\,[\omega - \Sigma_0(\omega) + \Sigma_0(0)]$$

The numerator at $\omega \to 0$

$$(M - \Sigma_1(0))\,(\not{v} + 1)$$

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The denominator at $\omega \to 0$

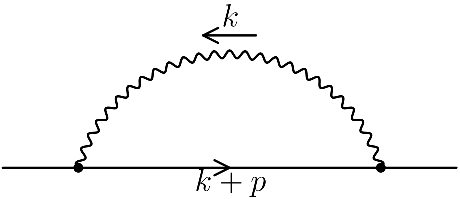
$$[M - \Sigma_1(0) + \omega - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$- [M - \Sigma_1(0) + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$\approx 2\,(M - \Sigma_1(0))\,[\omega - \Sigma_0(\omega) + \Sigma_0(0)]$$

The numerator at $\omega \to 0$

$$(M - \Sigma_1(0))\,(\not{v} + 1)$$

$$S(p) \approx \frac{\not{v} + 1}{2} \frac{1}{\omega - \Sigma_0(\omega) + \Sigma_0(0)}$$

Electron self-energy



$$p = (M+\omega)v$$
$$D_1 = M^2 - (k+p)^2$$
$$D_2 = -k^2$$

$$\Sigma_0(\omega) = \frac{1}{4}\operatorname{Tr}(\not{v}+1)\Sigma(p) = -ie_0^2 \int \frac{d^d k}{(2\pi)^d}\frac{1}{2D_1 D_2}$$
$$\left[(d+2)M - (d-2)\omega - (d-2)\frac{D_2+M^2}{M+\omega}\right.$$
$$\left. + \frac{\xi\omega^2}{D_2}\frac{D_2+4M\omega+\omega^2}{M+\omega}\right]$$

Hard contribution $k \sim M$

$$D_1 = D_h - (D_2 - D_h + 2M^2)\frac{\omega}{M} - \omega^2$$
$$D_h = M^2 - (k + Mv)^2$$

$D_h \sim M^2$, $D_2 \sim M^2$; Taylor series in ω; single scale M

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$$\Sigma_h(\omega) = \frac{e_0^2 M^{1-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{d-1}{d-3} \left(1 - \frac{\omega}{M} + \cdots\right)$$

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On-shell mass renormalization (gauge invariant)

$$\delta M = M\left[\frac{e_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{d-1}{d-3} + \cdots\right]$$

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On-shell wave-function renormalization
(gauge invariant in QED)

$$Z_\psi^{\text{os}} = \frac{1}{1 - \Sigma_0'(0)} = 1 - \frac{e_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{d-1}{d-3} + \cdots$$

Soft contribution $k \sim \omega$

$$D_1 = M D_s - (k + \omega v)^2 \qquad D_s = -2(k \cdot v + \omega)$$

$D_s \sim \omega$, $D_2 \sim \omega^2$; Taylor series in $1/M$; single scale ω

Soft contribution $k \sim \omega$

$$D_1 = M D_s - (k + \omega v)^2 \qquad D_s = -2(k \cdot v + \omega)$$

$D_s \sim \omega$, $D_2 \sim \omega^2$; Taylor series in $1/M$; single scale ω

$$\Sigma_s(\omega) = \Sigma(\omega) \left(1 + \mathcal{O}\left(\frac{\omega}{M}\right)\right)$$

$$\Sigma(\omega) = \frac{e_0^2 (-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)}{d-4} \left(\xi + \frac{2}{d-3}\right)$$

$\Sigma(\omega)$ — HEET self-energy

Electron propagator in QED and HEET

$$S(p) = \frac{1+\not{v}}{2} \frac{1}{\omega - \Sigma'_h(0)\omega - \Sigma_s(\omega)} = z_0 S(\omega)$$

$$z_0 = Z_\psi^{\text{os}} = \frac{1}{1 - \Sigma'_h(0)}$$

$$S(\omega) = \frac{1+\not{v}}{2} \frac{1}{\omega - \Sigma(\omega)}$$

$S(\omega)$ — HEET propagator

Electron propagator in QED and HEET

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$$S(\omega) = \frac{1+\rlap/v}{2} \frac{1}{\omega - \Sigma(\omega)}$$

$S(\omega)$ — HEET propagator

- Higher terms in $\Sigma_h \Rightarrow$ corrections to ψ_0 via h_{v0}
- Higher terms in $\Sigma_s \Rightarrow$ corrections to $S(\omega)$ due to $1/M$ terms in the HEET Lagrangian

Power counting

Small parameter (p — residual momentum)

$$\lambda \sim \frac{p}{M}$$

Soft fields: $\partial \sim \lambda$, $A \sim \lambda$, $D \sim \lambda$

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$$\varphi \sim \lambda^{3/2}$$

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$$\varphi \sim \lambda^{3/2}$$

$$\varphi^{+} i D_0 \varphi \sim \lambda^4$$

$$\varphi^{+} \vec{D}^{\,2} \varphi \sim \lambda^5 \qquad \varphi^{+} \vec{B} \cdot \vec{\sigma} \varphi \sim \lambda^5$$

Action: main ~ 1, corrections $\sim \lambda$

$1/M$ corrections: spin 0

Kinetic energy

$$L = L_0 + \frac{C_k^0}{2M} O_k^0$$

$$O_k^0 = \varphi_0^+ \vec{D}_0^2 \varphi_0 = -\varphi_0^+ D_\perp^2 \varphi_0$$

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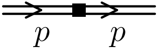
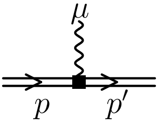
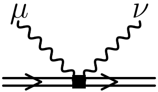
New vertices ($g_\perp^{\mu\nu} = g^{\mu\nu} - v^\mu v^\nu$)



$$= i \frac{C_k^0}{2M} p_\perp^2$$



$$= i \frac{C_k^0}{2M} e_0 (p + p')_\perp^\mu$$



$$= i \frac{C_k^0}{M} e_0^2 g_\perp^{\mu\nu}$$

$1/M$ corrections: spin 0

$$S(p) = \frac{1}{\omega - \Sigma(\omega) - \frac{C_k^0}{2M}\left(\vec{p}^{\,2} + \Sigma_k(\omega,\vec{p})\right)}$$

Mass shell

$$\omega = C_k^0 \frac{\vec{p}^{\,2}}{2M}$$

$$C_k^0 = 1$$

$1/M$ corrections: spin 0

$$S(p) = \frac{1}{\omega - \Sigma(\omega) - \frac{C_k^0}{2M}\left(\vec{p}^{\,2} + \Sigma_k(\omega, \vec{p})\right)}$$

Mass shell

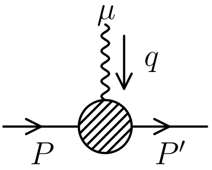
$$\omega = C_k^0 \frac{\vec{p}^{\,2}}{2M}$$

$$C_k^0 = 1$$

$$C_k^0 O_k^0 = C_k(\mu) O_k(\mu) \Rightarrow C_k^0 = Z_k(\alpha(\mu)) C_k(\mu)$$

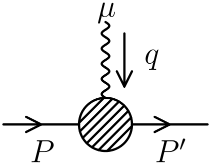
$$Z_k = 1 \qquad \gamma_k = 0 \qquad C_k(\mu) = 1$$

Scattering in external field: spin 0



$$F(q^2)(P + P')^\mu \qquad F(0) = 1$$

Scattering in external field: spin 0

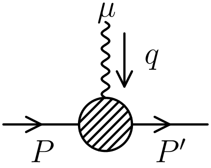


$$F(q^2)(P+P')^\mu \qquad F(0)=1$$

$$j^\mu = 2P^\mu|\Phi|^2,\ j^0 = 1 \Rightarrow |\Phi| = 1/\sqrt{2E}$$
$$P^\mu = Mv^\mu + p^\mu,\ E = M + p^0$$

$$F(q^2)\frac{(P+P')^\mu}{2\sqrt{EE'}} = v^\mu + \frac{(p+p')^\mu_\perp}{2M} + \mathcal{O}\left(\frac{1}{M^2}\right)$$

Scattering in external field: spin 0



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HEET

$$v^\mu + C_k^0\frac{(p+p')^\mu_\perp}{2M} + \mathcal{O}\left(\frac{1}{M^2}\right)$$

$1/M$ corrections: spin 1/2

Magnetic interaction

$$L = L_0 + \frac{C_k^0}{2M} O_k^0 + \frac{C_m^0}{2M} O_m^0$$
$$O_m^0 = -e_0 h_0^+ \vec{B}_0 \cdot \vec{\sigma} h_0 = \frac{e_0}{2} \bar{h}_{v0} F_{0\mu\nu} \sigma^{\mu\nu} h_{v0}$$

$1/M$ corrections: spin 1/2

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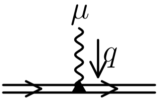
New vertex



$$= \frac{C_m^0}{2M} e_0 \sigma^{\nu\mu} q_\nu$$

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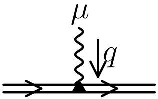
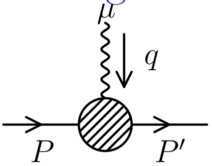


$$= \frac{C_m^0}{2M} e_0 \sigma^{\nu\mu} q_\nu$$

Scattering in external field loop corrections vanish

$$\bar{u}_v(p') \left[v^\mu + C_k^0 \frac{(p+p')_\perp^\mu}{2M} + C_m^0 \frac{[\not{q}, \gamma^\mu]}{4M} \right] u_v(p)$$

Scattering in external field in QED

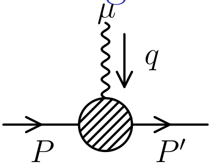


$$\bar{u}(P')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{[\not{q},\gamma^\mu]}{4M}\right]u(P) =$$

$$\bar{u}(P')\left[\left(F_1(q^2)+F_2(q^2)\right)\gamma^\mu - F_2(q^2)\frac{(P+P')^\mu}{2M}\right]u(P) =$$

$$\bar{u}(P')\left[F_1(q^2)\frac{(P+P')^\mu}{2M} + \left(F_1(q^2)+F_2(q^2)\right)\frac{[\not{q},\gamma^\mu]}{4M}\right]u(P)$$

Scattering in external field in QED



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$$\bar{u}(P')\left[F_1(q^2)\frac{(P+P')^\mu}{2M} + \left(F_1(q^2)+F_2(q^2)\right)\frac{[\not{q},\gamma^\mu]}{4M}\right]u(P)$$

$$F_1(q^2) = 1 + F_1'(0)\frac{q^2}{M^2} + \cdots$$

$$F_m(q^2) = F_1(q^2) + F_2(q^2) = F_m(0) + \cdots$$

Foldy–Wouthuysen transformation

$P = Mv + p$

$$u(P) = \left[1 + \frac{\not{p}}{2M} + \cdots\right] u_v(p)$$

$\not{v} u_v = u_v$

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$\not{v} u_v = u_v$

Scattering amplitude

$$\bar{u}_v(k') \left[v^\mu + \frac{(p+p')^\mu_\perp}{2M} + F_m(0)\frac{[\not{q}, \gamma^\mu]}{4M}\right] u_v(k)$$

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$$C_m = F_m(0) = 1 + F_2(0) = 1 + \frac{\alpha}{2\pi} + \cdots$$

$1/M^2$ corrections

Darwin term and spin–orbit interaction

$$L = L_0 + \frac{C_k}{2M} O_k + \frac{C_m}{2M} O_m + \frac{C_d}{8M^2} O_d + \frac{C_s}{8M^2} O_s + \cdots$$

$$O_d = -eh^+ \left(\vec{D} \cdot \vec{E} - \vec{E} \cdot \vec{D} \right) h = -e\bar{h} v^\mu [D_\perp^\nu, F_{\mu\nu}] h$$

$$O_s = -ieh^+ \left(\vec{D} \times \vec{E} - \vec{E} \times \vec{D} \right) \cdot \vec{\sigma} h = ie\bar{h} [D_\perp^\mu, F^{\lambda\nu}]_+ v_\lambda \sigma_{\mu\nu} h$$

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Scattering amplitude (loops vanish)

$$\bar{u}_v(p') \left[v^\mu + C_k \frac{(p+p')^\mu}{2M} + C_m \frac{[\not{q}, \gamma^\mu]}{4M} \right.$$
$$\left. + C_d \frac{q^2}{8M^2} v^\mu + C_s \frac{[\not{p}, \not{q}]}{8M^2} v^\mu + \cdots \right] u_v(p)$$

$1/M^2$ corrections

Scattering in full QED

$$\bar{u}_v(p')\left[F_1(q^2)\left(v^\mu + \frac{(k+k')^\mu}{2M} - \frac{q^2+[\not{p},\not{q}]}{8M^2}v^\mu + \cdots\right)\right.$$
$$\left. + F_m(q^2)\left(\frac{[\not{q},\gamma^\mu]}{4M} + \frac{q^2+[\not{p},\not{q}]}{4M^2}v^\mu + \cdots\right)\right]u_v(p)$$

$1/M^2$ corrections

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$$\left. + F_m(q^2)\left(\frac{[\not{q},\gamma^\mu]}{4M} + \frac{q^2+[\not{p},\not{q}]}{4M^2}v^\mu + \cdots\right)\right]u_v(p)$$

$$C_k = 1 \qquad C_m = F_m(0)$$
$$C_d = 8F_1'(0) + 2F_m(0) - 1 \qquad C_s = 2F_m(0) - 1$$

$1/M^2$ corrections

Scattering in full QED

$$\bar{u}_v(p')\Bigg[F_1(q^2)\left(v^\mu + \frac{(k+k')^\mu}{2M} - \frac{q^2 + [\not{p},\not{q}]}{8M^2}v^\mu + \cdots\right) + F_m(q^2)\left(\frac{[\not{q},\gamma^\mu]}{4M} + \frac{q^2 + [\not{p},\not{q}]}{4M^2}v^\mu + \cdots\right)\Bigg]u_v(p)$$

$$C_k = 1 \qquad C_m = F_m(0)$$

$$C_d = 8F_1'(0) + 2F_m(0) - 1 \qquad C_s = 2F_m(0) - 1$$

Reparametrization invariance $v \to v + \delta v$, $\delta v \sim p/M$

$$C_k = 1 \qquad C_s = 2C_m - 1$$

Qualitative explanation

Spin–orbit interaction: a moving electron in an electric field

$$C_s = 2C_m - 1$$

- In the electron rest frame there is magnetic field (Lorentz transformation), and the electron magnetic moment C_m interacts with it.
- Kinematical effect — Thomas precession, no radiative corrections. If we neglect corrections to C_m, it compensates $1/2$ of the first term.

Another derivation at tree level

$$\psi(x) = e^{-iMv\cdot x}\left(h_v(x) + H_v(x)\right)$$

$$h_v(x) = e^{iMv\cdot x}\frac{1+\not{v}}{2}\psi(x) \qquad H_v(x) = e^{iMv\cdot x}\frac{1-\not{v}}{2}\psi(x)$$

$$\not{v}h_v(x) = h_v(x) \qquad \not{v}H_v(x) = -H_v(x)$$

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$$\begin{aligned} L &= \bar{\psi}\left(i\not{D} - M\right)\psi \\ &= \bar{h}_v iv\cdot D h_v + \bar{H}_v\left(-iv\cdot D - 2M\right)H_v + \bar{h}_v i\not{D}_\perp H_v + \bar{H}_v i\not{D}_\perp h_v \end{aligned}$$

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$$\psi(x) = e^{-iMv\cdot x}\left(h_v(x) + H_v(x)\right)$$

$$h_v(x) = e^{iMv\cdot x}\frac{1+\not{v}}{2}\psi(x) \qquad H_v(x) = e^{iMv\cdot x}\frac{1-\not{v}}{2}\psi(x)$$

$$\not{v}h_v(x) = h_v(x) \qquad \not{v}H_v(x) = -H_v(x)$$

$$\begin{aligned} L &= \bar{\psi}\left(i\not{D} - M\right)\psi \\ &= \bar{h}_v iv\cdot Dh_v + \bar{H}_v\left(-iv\cdot D - 2M\right)H_v + \bar{h}_v i\not{D}_\perp H_v + \bar{H}_v i\not{D}_\perp h_v \end{aligned}$$

Solution of the equation of motion

$$H_v = \frac{1}{2M + iv\cdot D}\not{D}_\perp h_v = \frac{1}{2M}i\not{D}_\perp h_v - \frac{iv\cdot D}{(2M)^2}i\not{D}_\perp h_v + \cdots$$

Another derivation at tree level

$$\psi(x) = e^{-iMv\cdot x}\left(h_v(x) + H_v(x)\right)$$

$$h_v(x) = e^{iMv\cdot x}\frac{1+\not{v}}{2}\psi(x) \qquad H_v(x) = e^{iMv\cdot x}\frac{1-\not{v}}{2}\psi(x)$$

$$\not{v}h_v(x) = h_v(x) \qquad \not{v}H_v(x) = -H_v(x)$$

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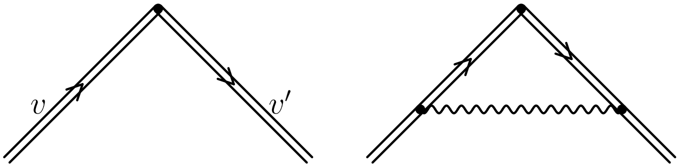
Solution of the equation of motion

$$H_v = \frac{1}{2M + iv\cdot D}\not{D}_\perp h_v = \frac{1}{2M}i\not{D}_\perp h_v - \frac{iv\cdot D}{(2M)^2}i\not{D}_\perp h_v + \cdots$$

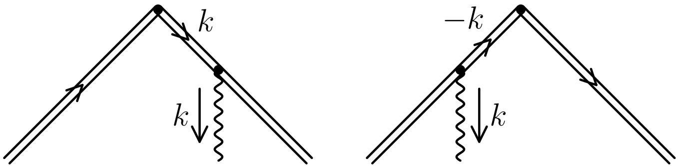
$$L = \bar{h}_v\left[iv\cdot D - \frac{D_\perp^2}{2M} + \frac{eF_{\mu\nu}\sigma^{\mu\nu}}{4M} + \cdots\right]h_v$$

Heavy–heavy current

$$J_0 = \varphi^*_{v'0}\varphi_{v0} \qquad J(\mu) = Z_J^{-1}(\vartheta) J_0 \qquad \cosh\vartheta = v\cdot v'$$

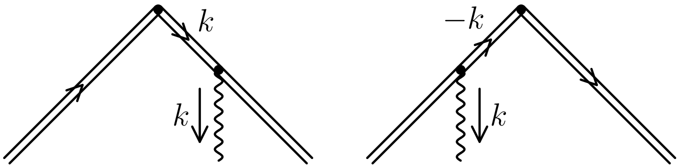


Real radiation



$$M^\mu = e\left(\frac{v^\mu}{k\cdot v} - \frac{v'^\mu}{k\cdot v'}\right)$$

Real radiation



$$M^\mu = e\left(\frac{v^\mu}{k\cdot v} - \frac{v'^\mu}{k\cdot v'}\right)$$

$$F(\omega) = -e^2 \int \frac{d^dk}{(2\pi)^d}\, 2\pi\delta(k^2)\,\delta(k\cdot v-\omega)\left(\frac{v}{k\cdot v} - \frac{v'}{k\cdot v'}\right)^2$$

Real radiation

$$F(\omega) = -\frac{2}{\Gamma(1-\varepsilon)}\frac{e^2}{(4\pi)^{d/2}}\frac{1}{\omega^{1+2\varepsilon}} \times \int_{-1}^{+1} dc \left[1 + \frac{2\coth\vartheta}{c-\coth\vartheta} + \frac{1}{\sinh^2\vartheta}\frac{1}{(c-\coth\vartheta)^2}\right]$$

Real radiation

$$
\begin{aligned}
F(\omega) = &- \frac{2}{\Gamma(1-\varepsilon)} \frac{e^2}{(4\pi)^{d/2}} \frac{1}{\omega^{1+2\varepsilon}} \\
&\times \int_{-1}^{+1} dc \left[1 + \frac{2\coth\vartheta}{c - \coth\vartheta} + \frac{1}{\sinh^2\vartheta} \frac{1}{(c-\coth\vartheta)^2} \right] \\
= &\frac{8}{\Gamma(1-\varepsilon)} \frac{e^2}{(4\pi)^{d/2}} \frac{\vartheta\coth\vartheta - 1}{\omega^{1+2\varepsilon}}
\end{aligned}
$$

Soft radiation function in classical electrodynamics

Bjorken sum rule

ξ — amplitude *not* to emit a photon

$$\xi^2 + \int_0^\infty F(\omega)\, d\omega = 1$$

Bjorken sum rule

ξ — amplitude *not* to emit a photon

$$\xi^2 + \int_0^\infty F(\omega)\,d\omega = 1$$

IR regularization, UV $1/\varepsilon$ only

$$\xi = 1 - \frac{1}{2}\int_\lambda^\infty F(\omega)\,d\omega = 1 - 2\frac{\alpha}{4\pi\varepsilon}(\vartheta\coth\vartheta - 1)$$

Cusp anomalous dimension

$$Z_J = 1 - 2\frac{\alpha}{4\pi\varepsilon}(\vartheta\coth\vartheta - 1)$$

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$$\Gamma(\vartheta) = (\vartheta \coth \vartheta - 1)\frac{\alpha}{\pi}$$

given by the classical soft radiation function

Cusp anomalous dimension

$$Z_J = 1 - 2\frac{\alpha}{4\pi\varepsilon}(\vartheta \coth \vartheta - 1)$$

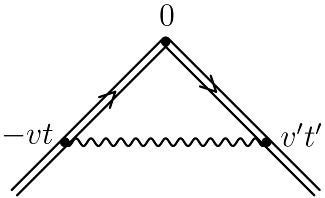
$$\Gamma(\vartheta) = (\vartheta \coth \vartheta - 1)\frac{\alpha}{\pi}$$

given by the classical soft radiation function

The Guinness book of records

The anomalous dimension known for the longest time (definitely > 100 years)

Coordinate space



$$\Lambda(t,t';\vartheta) = ie^2 D^0_{\mu\nu}(x) v^\mu v'^\nu \theta(t)\theta(t')$$

$$= -\frac{e^2}{8\pi^{d/2}}\Gamma(1-\varepsilon)\theta(t)\theta(t')$$

$$\times \frac{(1+a)x^2\cosh\vartheta + (d-2)(1-a)(t+t'\cosh\vartheta)(t'+t\cosh\vartheta)}{(-x^2)^{d/2}}$$

$$x = vt + v't'$$

$$\Lambda(\omega,\omega';\vartheta) = \int dt\,dt'\,e^{i\omega t+i\omega' t'}\Lambda(t,t';\vartheta)$$

$$t = \tau\frac{1+\xi}{2} \qquad t' = \tau\frac{1-\xi}{2}$$

$$\Lambda(0,0;\vartheta) = -\frac{e^2}{16\pi^{d/2}}\Gamma(1-\varepsilon)\int_0^T \frac{d\tau}{\tau^{1-2\varepsilon}}\int_{-1}^{+1} d\xi$$

$$\times\,\frac{(1+a)\cosh\vartheta(c^2-s^2\xi^2)+(d-2)(1-a)(c^4-s^4\xi^2)}{(-c^2+s^2\xi^2)^{d/2}}$$

$$c = \cosh(\vartheta/2) \qquad s = \sinh(\vartheta/2)$$

$$t = \tau \frac{1+\xi}{2} \qquad t' = \tau \frac{1-\xi}{2}$$

$$\Lambda(0,0;\vartheta) = -\frac{e^2}{16\pi^{d/2}} \Gamma(1-\varepsilon) \int_0^T \frac{d\tau}{\tau^{1-2\varepsilon}} \int_{-1}^{+1} d\xi$$

$$\times \frac{(1+a)\cosh\vartheta (c^2 - s^2\xi^2) + (d-2)(1-a)(c^4 - s^4\xi^2)}{(-c^2 + s^2\xi^2)^{d/2}}$$

$$c = \cosh(\vartheta/2) \qquad s = \sinh(\vartheta/2)$$

$$Z_\Gamma(\vartheta) = 1 - \frac{\alpha}{4\pi\varepsilon} \int_{-1}^{+1} d\xi \left[(1+a) \frac{\cosh\vartheta}{2\cosh^2(\vartheta/2)} \frac{1}{1 - \xi^2 \tanh^2(\vartheta/2)} \right.$$

$$\left. + (1-a) \frac{1 - \xi^2 \tanh^4(\vartheta/2)}{[1 - \xi^2 \tanh^2(\vartheta/2)]^2} \right]$$

$$\xi = \frac{\tanh\psi}{\tanh(\vartheta/2)}$$

$$\begin{aligned} Z_\Gamma(\vartheta) &= 1 - \frac{\alpha}{4\pi\varepsilon}\int_{-\vartheta/2}^{+\vartheta/2} d\psi \, \left[2\coth\vartheta + \frac{1-a}{\sinh\vartheta}\cosh 2\psi\right] \\ &= 1 - \frac{\alpha}{4\pi\varepsilon}(2\vartheta\coth\vartheta + 1 - a) \end{aligned}$$

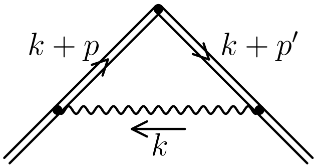
$$\xi = \frac{\tanh\psi}{\tanh(\vartheta/2)}$$

$$Z_\Gamma(\vartheta) = 1 - \frac{\alpha}{4\pi\varepsilon}\int_{-\vartheta/2}^{+\vartheta/2} d\psi \left[2\coth\vartheta + \frac{1-a}{\sinh\vartheta}\cosh 2\psi\right]$$

$$= 1 - \frac{\alpha}{4\pi\varepsilon}(2\vartheta\coth\vartheta + 1 - a)$$

$$Z_J(\vartheta) = Z_\Gamma(\vartheta) Z_h = 1 - 2\frac{\alpha}{4\pi\varepsilon}(\vartheta\coth\vartheta - 1)$$

Momentum space



$$\Lambda(\omega,\omega';\vartheta) = 8ie^2\cosh\vartheta$$

$$\int \frac{d^dk}{(2\pi)^d}\frac{dy\,dy'}{[-k^2-2(k\cdot v+\omega)y-2(k\cdot v'+\omega')y']^3}$$

$$= -4\Gamma(1+\varepsilon)\frac{e^2}{(4\pi)^{d/2}}\cosh\vartheta$$

$$\int \frac{dy\,dy'}{[y^2+y'^2+2yy'\cosh\vartheta-2(\omega y+\omega' y')]^{1+\varepsilon}}$$

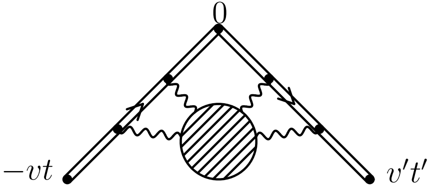
$y = zx$, $y' = z(1-x)$, integrate in z

$$\Lambda(\omega, \omega'; \vartheta) = -4\Gamma(2\varepsilon)\Gamma(1-\varepsilon)\frac{e^2}{(4\pi)^{d/2}}\cosh\vartheta$$

$$\int_0^1 \frac{dx}{[1-(1-e^{\vartheta})x]^{1-\varepsilon}[1-(1-e^{-\vartheta})x]^{1-\varepsilon}[-2\omega x - 2\omega'(1-x)]^{2\varepsilon}}$$

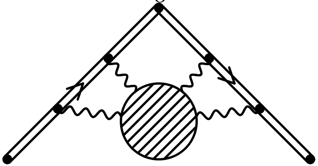
$$= 2\frac{\alpha}{4\pi\varepsilon}\vartheta\coth\vartheta + \mathcal{O}(1)$$

Exponentiation



$$G(t,t';\vartheta) = \theta(t)\theta(t')\exp\left[\frac{e^2}{(4\pi)^{d/2}} F(t,t';\vartheta)\right]$$

Exponentiation



$$G(t,t';\vartheta) = \theta(t)\theta(t')\exp\left[\frac{e^2}{(4\pi)^{d/2}}F(t,t';\vartheta)\right]$$

Divide by $G(t,t';\vartheta=0) = iS(t+t')\theta(t)\theta(t')$

$$\begin{aligned}\mathcal{G}(t,t';\vartheta) &= \frac{G(t,t';\vartheta)}{iS(t+t')} \\ &= \exp\left[\frac{e^2}{(4\pi)^{d/2}}\left(\mathcal{F}(t,t';\vartheta) - \mathcal{F}(t,t';\vartheta=0)\right)\right]\end{aligned}$$

Exponentiation

Should be $Z_J(\vartheta)\mathcal{G}_r(t,t';\vartheta)$:

$$Z_J(\vartheta) = \exp\left[\frac{\alpha}{4\pi\varepsilon}\left(f(\vartheta) - f(\vartheta = 0)\right)\right]$$
$$\varepsilon e^{\gamma\varepsilon}\mathcal{F}(t,t';\vartheta) = f(\vartheta) + \mathcal{O}(\varepsilon)$$

Exponentiation

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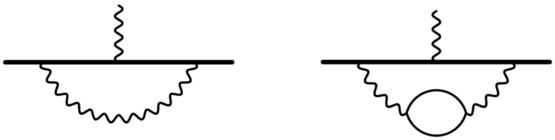
$$Z_J(\vartheta) = \exp\left[\frac{\alpha}{4\pi\varepsilon}\left(f(\vartheta) - f(\vartheta=0)\right)\right]$$
$$\varepsilon e^{\gamma\varepsilon}\mathcal{F}(t,t';\vartheta) = f(\vartheta) + \mathcal{O}(\varepsilon)$$

$$\Gamma(\vartheta) = (\vartheta\coth\vartheta - 1)\frac{\alpha}{\pi}$$

exactly

Muon magnetic moment



Form factors

$$F_1(q^2) = \frac{Z_\psi^{\rm os}}{2(d-2)(1+t)^2} \times \frac{1}{4}\,\mathrm{Tr}\Big[(d-1)v_\mu - (1+t)\gamma_\mu\Big](\not{v}'+1)\Gamma^\mu(\not{v}+1)$$

$$F_2(q^2) = \frac{Z_\psi^{\rm os}}{2(d-2)t(1+t)^2} \times \frac{1}{4}\,\mathrm{Tr}\Big[\big(1-(d-2)t\big)v_\mu - (1+t)\gamma_\mu\Big](\not{v}'+1)\Gamma^\mu(\not{v}+1)$$

$$t = -\frac{q^2}{4M^2}$$

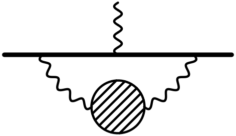
Form factors

$$F_1(0) = 1$$

$$F_2(0) = \frac{Z_\psi^{\text{os}}}{d-2} \left[\frac{1}{4} \operatorname{Tr}(\gamma_\mu - d v_\mu) \Gamma_0^\mu (\not{v} + 1) + \frac{2}{d-1} \frac{1}{4} \operatorname{Tr}\left(\gamma_\mu \gamma_\nu + \gamma_\mu v_\nu - \gamma_\nu v_\mu - v_\mu v_\nu \right) \Gamma_1^{\mu\nu} (\not{v} + 1) \right]$$

$$\Gamma^\mu(Mv, Mv + q) = \Gamma_0^\mu + \Gamma_1^{\mu\nu} \frac{q_\nu}{M} + \cdots$$

Muon magnetic moment

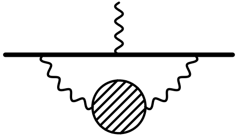


$$D_1 = M^2 - (Mv + k)^2$$

$$D_2 = -k^2$$

$$\begin{aligned}\mu = & -i\frac{e_0^2}{d-1}\int\frac{d^dk}{(2\pi)^d}\Pi(k^2)\Bigg[\frac{16}{d-2}\frac{M^2}{D_1^3} - 4\frac{d-3}{d-2}\frac{D_2}{D_1^3} - \frac{D_2^2}{M^2D_1^3}\\ & - 2\frac{2d^2-9d+13}{(d-2)D_1^2} - \frac{(d+2)(d-3)D_2}{2M^2D_1^2}\\ & + 2\frac{d^2-4d+5}{(d-2)D_1D_2} + \frac{d^2-d-3}{M^2D_1} - \frac{(d+1)(d-2)}{2M^2D_2}\Bigg]\end{aligned}$$

Muon magnetic moment



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$$\mu = -i\frac{e_0^2}{d-1}\int\frac{d^dk}{(2\pi)^d}\Pi(k^2)\left[\frac{16}{d-2}\frac{M^2}{D_1^3} - 4\frac{d-3}{d-2}\frac{D_2}{D_1^3} - \frac{D_2^2}{M^2D_1^3}\right.$$
$$- 2\frac{2d^2-9d+13}{(d-2)D_1^2} - \frac{(d+2)(d-3)D_2}{2M^2D_1^2}$$
$$\left. + 2\frac{d^2-4d+5}{(d-2)D_1D_2} + \frac{d^2-d-3}{M^2D_1} - \frac{(d+1)(d-2)}{2M^2D_2}\right]$$

$\Pi(k^2) = 1 \Rightarrow 1$ loop

$$\mu_0 = -2\frac{d-5}{d-3}\frac{e_0^2M^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(1+\varepsilon)$$

Hard contribution $k \sim M$

$D_1 \sim D_2 \sim M^2$; Taylor series in m^2; single scale M

$$\Pi(k^2) = -2\frac{d-2}{d-1}\frac{e_0^2(-k^2)^{-\varepsilon}}{(4\pi)^{d/2}} G_1 \left[1 + \mathcal{O}\left(\frac{m^2}{k^2}\right)\right]$$

$$G_1 = -\frac{2}{(d-3)(d-4)} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

Hard contribution $k \sim M$

$D_1 \sim D_2 \sim M^2$; Taylor series in m^2; single scale M

$$\Pi(k^2) = -2\frac{d-2}{d-1}\frac{e_0^2(-k^2)^{-\varepsilon}}{(4\pi)^{d/2}}G_1\left[1+\mathcal{O}\left(\frac{m^2}{k^2}\right)\right]$$

$$G_1 = -\frac{2}{(d-3)(d-4)}\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$\frac{\mu_h}{\mu_0} = 32\frac{(d-2)(d^2-7d+11)}{(d-1)(d-4)(d-5)(3d-8)(3d-10)}\frac{e_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(1+\varepsilon)R$$

$$R = \frac{\Gamma(1+2\varepsilon)\Gamma^2(1-\varepsilon)\Gamma(1-4\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1-3\varepsilon)} = 1+\mathcal{O}(\varepsilon^2)$$

Hard contribution $k \sim M$

Re-expressing via renormalized $\alpha(\mu)$:

$$\mu_0 + \mu_h = \frac{\alpha(M)}{2\pi}\left[1 - \frac{25}{18}\frac{\alpha}{\pi}\right]$$

$$\alpha(M) = \alpha(m)\left(1 + \frac{2}{3}\frac{\alpha}{\pi}\log\frac{M}{m}\right)$$

$$\mu_0 + \mu_h = \frac{\alpha}{2\pi}\left[1 + \frac{2}{3}\frac{\alpha}{\pi}\left(\log\frac{M}{m} - \frac{25}{12}\right)\right]$$

Soft contribution $k \sim m$

$$D_1 = MD_s \qquad D_s = -2k \cdot v$$

$D_s \sim m$, $D_2 \sim m^2$; Taylor series in $1/M$; single scale m

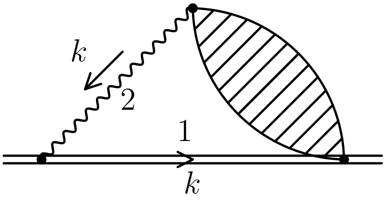
Soft contribution $k \sim m$

$$D_1 = M D_s \qquad D_s = -2k \cdot v$$

$D_s \sim m$, $D_2 \sim m^2$; Taylor series in $1/M$; single scale m

$$\mu_s = \frac{-2ie_0^2}{(d-1)(d-2)M} \int \frac{d^d k}{(2\pi)^d} \Pi(k^2) \left[\frac{8}{D_s^3} + \frac{d^2 - 4d + 5}{D_s D_2} \right]$$

On-shell HQET diagrams with mass

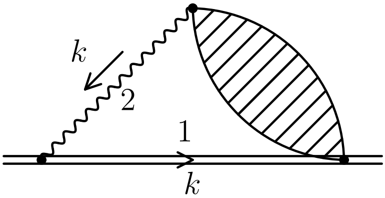


$$F(n_1, n_2) = \int \frac{\Pi(k^2)\, d^d k}{D_1^{n_1} D_2^{n_2}}$$

$$D_1 = -2k \cdot v - i0$$

$$D_2 = -k^2 - i0$$

On-shell HQET diagrams with mass



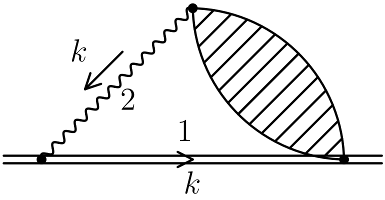
$$F(n_1, n_2) = \int \frac{\Pi(k^2)\, d^d k}{D_1^{n_1} D_2^{n_2}}$$

$$D_1 = -2k \cdot v - i0$$

$$D_2 = -k^2 - i0$$

$$\frac{\partial}{\partial k} \cdot \left(k - 2\frac{D_2}{D_1} v \right) \frac{\Pi(k^2)}{D_1^{n_1} D_2^{n_2}} = \left[d - n_1 - 2 - 4(n_1 + 1)\frac{D_2}{D_1^2} \right] \frac{\Pi(k^2)}{D_1^{n_1} D_2^{n_2}}$$

On-shell HQET diagrams with mass



$$F(n_1, n_2) = \int \frac{\Pi(k^2)\, d^d k}{D_1^{n_1} D_2^{n_2}}$$

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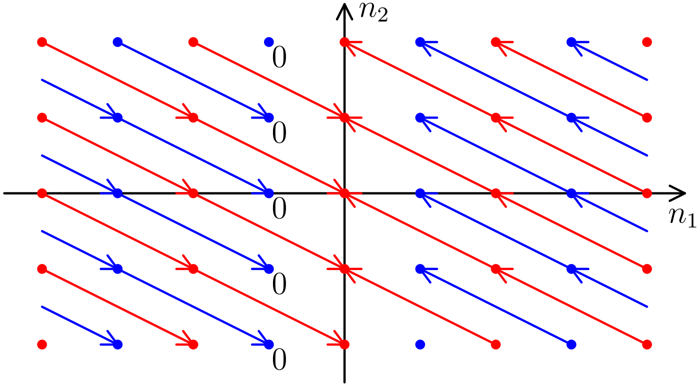
$$\frac{\partial}{\partial k} \cdot \left(k - 2\frac{D_2}{D_1} v \right) \frac{\Pi(k^2)}{D_1^{n_1} D_2^{n_2}} = \left[d - n_1 - 2 - 4(n_1 + 1)\frac{D_2}{D_1^2} \right] \frac{\Pi(k^2)}{D_1^{n_1} D_2^{n_2}}$$

$$(d - n_1 - 2) F(n_1, n_2) = 4(n_1 + 1) \mathbf{1}^{++} \mathbf{2}^{-} F(n_1, n_2)$$

On-shell HQET diagrams with mass



On-shell HQET diagrams with mass

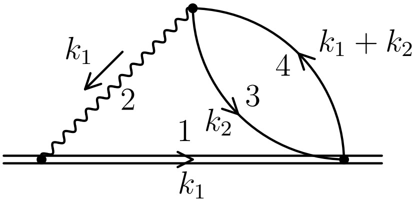
$$F(n_1,n_2) = \begin{cases} (-4)^{-n_1/2}\dfrac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-n_1}{2}\right)}\dfrac{\Gamma\left(\frac{1-n_1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}F\left(0,n_2+\dfrac{n_1}{2}\right) & \text{even } n_1 \\ 2^{1-n_1}\dfrac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{n_1+1}{2}\right)\Gamma\left(\frac{d-n_1}{2}\right)}F\left(1,n_2+\dfrac{n_1-1}{2}\right) & \text{odd } n_1>0 \\ 0 & \text{odd } n_1<0 \end{cases}$$

On-shell HQET diagrams with mass

$$F(n_1,n_2) = \begin{cases} (-4)^{-n_1/2} \dfrac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-n_1}{2}\right)} \dfrac{\Gamma\left(\frac{1-n_1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} F\left(0, n_2 + \dfrac{n_1}{2}\right) & \text{even } n_1 \\ 2^{1-n_1} \dfrac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{n_1+1}{2}\right)\Gamma\left(\frac{d-n_1}{2}\right)} F\left(1, n_2 + \dfrac{n_1-1}{2}\right) & \text{odd } n_1 > 0 \\ 0 & \text{odd } n_1 < 0 \end{cases}$$

$n_1 < 0$: $i0 \Rightarrow 0$ in $D_1^{-n_1}$, averaging over k directions

2 loops



Symmetry $3 \leftrightarrow 4$

0 if $n_3 \leq 0$ or $n_4 \leq 0$

$$F(n_1, n_2, n_3, n_4) = \frac{1}{(i\pi^{d/2})^2} \int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4}}$$

$$D_1 = -2k_1 \cdot v - i0 \qquad D_2 = -k_1^2 - i0$$

$$D_3 = 1 - k_2^2 - i0 \qquad D_4 = 1 - (k_1 + k_2)^2 - i0$$

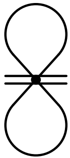
$$F(n_1,n_2,n_3,n_4)=\frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{d-n_1}{2}-n_2\right)}{2\Gamma(n_1)\Gamma(n_3)\Gamma(n_4)}\times$$

$$\frac{\Gamma\left(\frac{n_1-d}{2}+n_2+n_3\right)\Gamma\left(\frac{n_1-d}{2}+n_2+n_4\right)\Gamma\left(\frac{n_1}{2}+n_2+n_3+n_4-d\right)}{\Gamma\left(\frac{d-n_1}{2}\right)\Gamma(n_1+2n_2+n_3+n_4-d)}$$

$$F(n_1,n_2,n_3,n_4) = \frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{d-n_1}{2}-n_2\right)}{2\Gamma(n_1)\Gamma(n_3)\Gamma(n_4)} \times$$

$$\frac{\Gamma\left(\frac{n_1-d}{2}+n_2+n_3\right)\Gamma\left(\frac{n_1-d}{2}+n_2+n_4\right)\Gamma\left(\frac{n_1}{2}+n_2+n_3+n_4-d\right)}{\Gamma\left(\frac{d-n_1}{2}\right)\Gamma(n_1+2n_2+n_3+n_4-d)}$$

Apparently even $\Rightarrow F(0, n_2 + n_1/2, n_3, n_4)$ (vacuum)

$I_0^2 =$ 

$$F(n_1,n_2,n_3,n_4) = \frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{d-n_1}{2}-n_2\right)}{2\Gamma(n_1)\Gamma(n_3)\Gamma(n_4)} \times$$

$$\frac{\Gamma\left(\frac{n_1-d}{2}+n_2+n_3\right)\Gamma\left(\frac{n_1-d}{2}+n_2+n_4\right)\Gamma\left(\frac{n_1}{2}+n_2+n_3+n_4-d\right)}{\Gamma\left(\frac{d-n_1}{2}\right)\Gamma(n_1+2n_2+n_3+n_4-d)}$$

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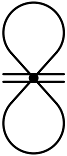
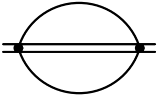
$$I_0^2 =$$ 

Apparently odd

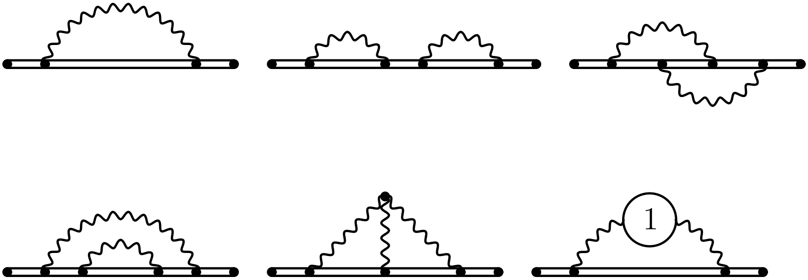
- $n_1 \leq 0 \Rightarrow 0$
- $n_1 > 0 \Rightarrow F(1, n_2+(n_1-1)/2, n_3, n_4)$

$J_0 =$  $= 2^{4d-9}\pi^2 \dfrac{\Gamma(5-2d)}{\Gamma^2(2-d/2)}$

Soft contribution $k \sim m$

$$\mu_s = \frac{-2ie_0^2}{M} \int \frac{d^d k}{(2\pi)^d} \frac{\Pi(k^2)}{D_s D_2} = \frac{\alpha^2}{4} \frac{m}{M}$$

HQET propagator



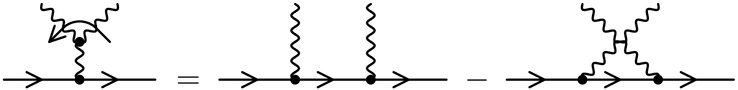
Non-abelian exponentiation

Unlike the abelian case

$$
\begin{aligned}
S(t) = -i\theta(t)\exp\Bigg[& C_F \frac{g_0^2}{(4\pi)^{d/2}} \left(\frac{it}{2}\right)^{2\varepsilon} S_F \\
& + C_F \frac{g_0^4}{(4\pi)^d} \left(\frac{it}{2}\right)^{4\varepsilon} (C_A S_{FA} + T_F n_l S_{Fl}) \\
& + C_F \frac{g_0^6}{(4\pi)^{3d/2}} \left(\frac{it}{2}\right)^{6\varepsilon} \Big(C_A^2 S_{FAA} \\
& \qquad + C_F T_F n_l S_{FFl} + C_A T_F n_l S_{FAl} + (T_F n_l)^2 S_{Fll}\Big) \\
& + \cdots \Bigg]
\end{aligned}
$$

Non-abelian exponentiation

If the colour factors of 3 diagrams were the same as that of the one-particle-reducible diagram, i. e. equal to the square of the colour factor C_F of the one-loop diagram (as in the abelian case), then the sum of these diagrams would be equal to $\frac{1}{2}$ of the square of the one-loop correction S_F. However, the colour factor of 1 diagram differs from C_F^2 by $-C_F C_A/2$, which is the colour factor the diagram with a 3-gluon vertex:

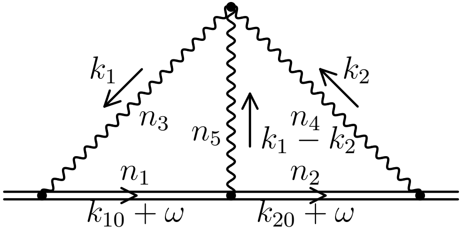


$$[t^a, t^b] = i f^{abc} t^c$$

Non-abelian exponentiation

We should include this contribution with $-C_F C_A/2$ instead of its full colour factor into the term S_{FA} (maximally non-abelian or colour-connected part). Of course, the diagram with 3-gluon vertex also contributes to S_{FA}. The diagrams with the one-loop gluon self-energy contribute to S_{Fl} (quark loop) and S_{FA} (gluon and ghost loops).

Diagram 1



Symmetry $1 \leftrightarrow 2$, $3 \leftrightarrow 4$

0 if two adjacent indices ≤ 0

$$\frac{1}{(i\pi^{d/2})^2} \int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} =$$

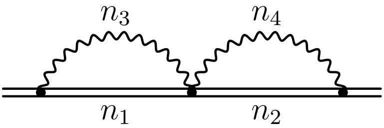
$$I(n_1, n_2, n_3, n_4, n_5)(-2\omega)^{2d-n_1-n_2-2(n_3+n_4+n_5)}$$

$$D_1 = -2(k_{10} + \omega) \qquad D_2 = -2(k_{20} + \omega)$$

$$D_3 = -k_1^2 \qquad D_4 = -k_2^2 \qquad D_5 = -(k_1 - k_2)^2$$

Trivial case $n_5 = 0$

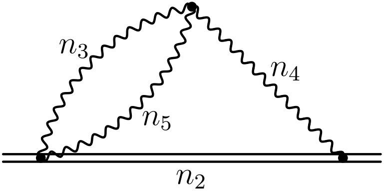
$$I(n_1, n_2, n_3, n_4, 0) =$$

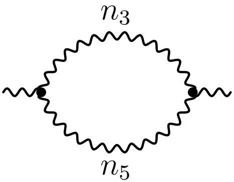


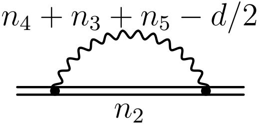
$$= I(n_1, n_3)I(n_2, n_4)$$

Trivial case $n_1 = 0$

Inner loop $G(n_3, n_5)(-p^2)^{d/2-n_3-n_5}$

$I(0, n_2, n_3, n_4, n_5) =$ 

$=$ 

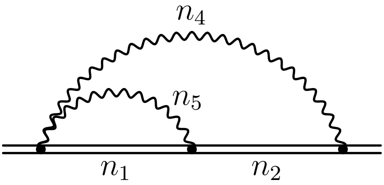
$\times$ 

$= G(n_3, n_5) I(n_2, n_4 + n_3 + n_5 - d/2)$

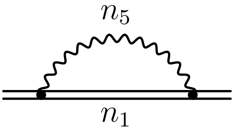
Trivial case $n_3 = 0$

Inner loop $I(n_1, n_5)(-2\omega)^{d-n_1-2n_5}$

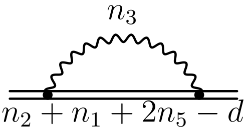
$I(n_1, n_2, 0, n_4, n_5) =$



$=$



$\times$



$= I(n_1, n_5)I(n_2 + n_1 + 2n_5 - d, n_4)$

Integration by parts

When applied to the integrand

$$\frac{\partial}{\partial k_2} \to \frac{n_2}{D_2} 2v + \frac{n_4}{D_4} 2k_2 + \frac{n_5}{D_5} 2(k_2 - k_1)$$

Integration by parts

When applied to the integrand

$$\frac{\partial}{\partial k_2} \to \frac{n_2}{D_2} 2v + \frac{n_4}{D_4} 2k_2 + \frac{n_5}{D_5} 2(k_2 - k_1)$$

Applying $(\partial/\partial k_2) \cdot k_2$, $(\partial/\partial k_2) \cdot (k_2 - k_1)$ and using $2k_2 \cdot v = -D_2 - 2\omega$, $2(k_2 - k_1) \cdot k_2 = D_3 - D_4 - D_5$:

$$d - n_2 - n_5 - 2n_4 - 2\omega \frac{n_2}{D_2} + \frac{n_5}{D_5}(D_3 - D_4)$$

$$d - n_2 - n_4 - 2n_5 + \frac{n_2}{D_2} D_1 + \frac{n_4}{D_4}(D_3 - D_5)$$

Integration by parts

$$\left[d - n_2 - n_5 - 2n_4 + n_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{3}^- - \mathbf{4}^-)\right] I = 0$$

$$\left[d - n_2 - n_4 - 2n_5 + n_2\mathbf{2}^+\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-)\right] I = 0$$

Integration by parts

$$\left[d - n_2 - n_5 - 2n_4 + n_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{3}^- - \mathbf{4}^-)\right] I = 0$$

$$\left[d - n_2 - n_4 - 2n_5 + n_2\mathbf{2}^+\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-)\right] I = 0$$

Applying $(\partial/\partial k_2) \cdot v$

$$\left[-2n_2\mathbf{2}^+ + n_4\mathbf{4}^+(\mathbf{2}^- - 1) + n_5\mathbf{5}^+(\mathbf{2}^- - \mathbf{1}^-)\right] I = 0$$

Homogeneity

Applying $\omega(d/d\omega)$ to the integral

$$\left[2(d - n_3 - n_4 - n_5) - n_1 - n_2 + n_1\mathbf{1}^+ + n_2\mathbf{2}^+\right] I = 0$$

This is the sum of the $(\partial/\partial k_2) \cdot k_2$ relation
and the symmetric $(\partial/\partial k_1) \cdot k_1$ one

Homogeneity

Applying $\omega(d/d\omega)$ to the integral

$$\left[2(d - n_3 - n_4 - n_5) - n_1 - n_2 + n_1 \mathbf{1}^+ + n_2 \mathbf{2}^+\right] I = 0$$

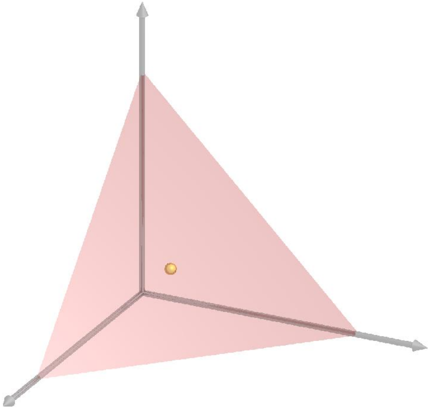
This is the sum of the $(\partial/\partial k_2) \cdot k_2$ relation
and the symmetric $(\partial/\partial k_1) \cdot k_1$ one
The $(\partial/\partial k_2) \cdot (k_2 - k_1)$ relation minus $\mathbf{1}^-$ times the homogeneity relation:

$$\begin{aligned}&\Big[d - n_1 - n_2 - n_4 - 2n_5 + 1 \\ &\quad - \big(2(d - n_3 - n_4 - n_5) - n_1 - n_2 + 1\big)\mathbf{1}^- \\ &\quad + n_4 \mathbf{4}^+ (\mathbf{3}^- - \mathbf{5}^-)\Big] I = 0\end{aligned}$$

Integration by parts

$$I = \frac{(2(d - n_3 - n_4 - n_5) - n_1 - n_2 + 1)\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{3}^-)}{d - n_1 - n_2 - n_4 - 2n_5 + 1} I$$

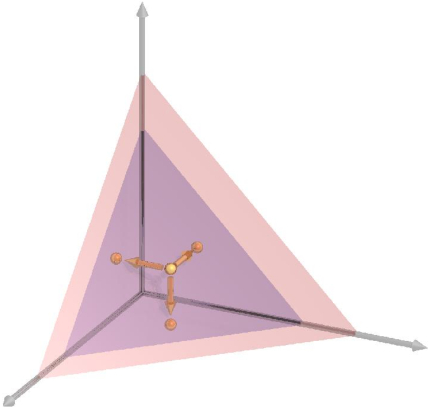
$n_1 + n_3 + n_5$ reduces by 1



Integration by parts

$$I = \frac{(2(d - n_3 - n_4 - n_5) - n_1 - n_2 + 1)\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{3}^-)}{d - n_1 - n_2 - n_4 - 2n_5 + 1} I$$

$n_1 + n_3 + n_5$ reduces by 1



Integration by parts

$$I = \frac{(2(d - n_3 - n_4 - n_5) - n_1 - n_2 + 1)\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{3}^-)}{d - n_1 - n_2 - n_4 - 2n_5 + 1} I$$

$n_1 + n_3 + n_5$ reduces by 1

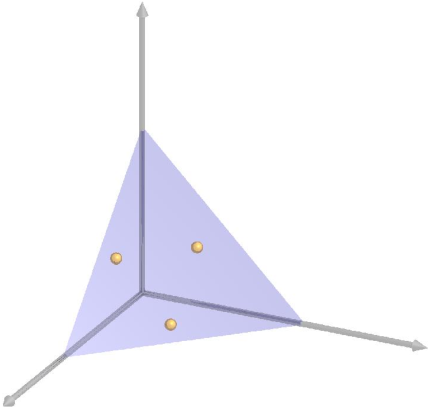
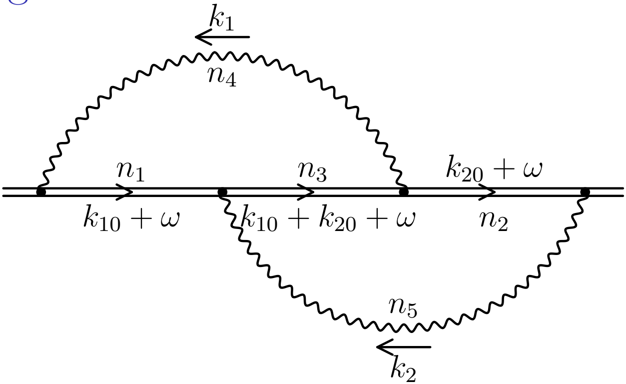


Diagram 2



Symmetry

$1 \leftrightarrow 2,\ 4 \leftrightarrow 5$

0 if $n_4 \le 0$

or $n_5 \le 0$

or 2 adjacent

$n_{1\ldots 3} \le 0$

$$\frac{1}{(i\pi^{d/2})^2} \int \frac{d^d k_1\, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} =$$

$$J(n_1, n_2, n_3, n_4, n_5)(-2\omega)^{2d-n_1-n_2-n_3-2(n_4+n_5)}$$

$$D_1 = -2(k_{10}+\omega) \qquad D_2 = -2(k_{20}+\omega)$$

$$D_3 = -2(k_{10}+k_{20}+\omega) \qquad D_4 = -k_1^2 \qquad D_5 = -k_2^2$$

Partial fractioning

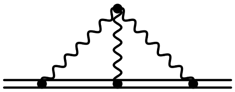
Trivial cases: $n_3 = 0$, $n_{1,2} = 0$
Denominators are linearly dependent:

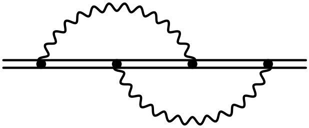
$$D_1 + D_2 - D_3 = -2\omega$$
$$J = (\mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-)J$$

$n_1 + n_2 + n_3$ reduces by 1
Numerator $(k_1 \cdot k_2)^n$ — not a problem

2 loops: summary

2 generic topologies (for all integer n_i)





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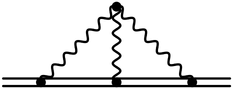
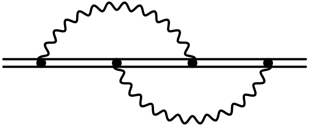
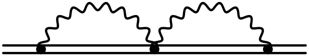
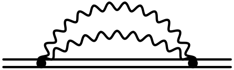




Basis (all $n_i = 1$)



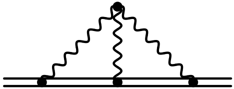
$= I_1^2$

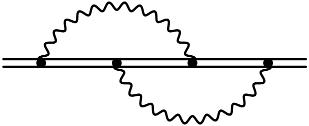


$= I_2$

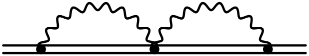
2 loops: summary

2 generic topologies (for all integer n_i)

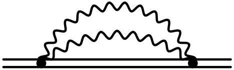




Basis (all $n_i = 1$)

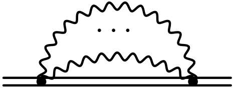


$= I_1^2$



$= I_2$

Sunset



$$= I_n = \frac{\Gamma(1+2n\varepsilon)\Gamma^n(1-\varepsilon)}{(1-n(d-2))_{2n}}$$

HQET self-energy

$$
\begin{aligned}
\Sigma(\omega) = {} & -C_F \frac{g_0^2(-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}} (d-3) I_1 A \\
& + C_F \frac{g_0^4(-2\omega)^{1-4\varepsilon}}{(4\pi)^d} \Biggl\{ -16 \frac{(d-2)(2d-5)}{(d-4)(d-6)} I_2 P \\
& \quad - C_F \frac{4(d-3)^2(2d-5)}{d-4} I_2 A^2 \\
& \quad + \left(C_F - \frac{C_A}{2} \right) 2(d-3) \left[(d-3) I_1^2 - 2(2d-5) I_2 \right] A^2 \\
& \quad - C_A (d-3) \left[(d-3) I_1^2 + 2 \frac{2d-5}{d-4} I_2 \right] A (1 - a_0) \Biggr\}
\end{aligned}
$$

HQET self-energy

$$A = a_0 - 1 - \frac{2}{d-3} \qquad \xi = 1 - a_0$$

$$P = T_F n_l - \frac{3d - 2 + (d-1)(2d-7)\xi - \frac{1}{4}(d-1)(d-4)\xi^2}{4(d-2)} C_A$$

HQET propagator

$$
\begin{aligned}
\omega S(\omega) &= 1 + C_F \frac{g_0^2(-2\omega)^{-2\varepsilon}}{(4\pi)^{d/2}} 2(d-3) I_1 A \\
&+ C_F \frac{g_0^4(-2\omega)^{-4\varepsilon}}{(4\pi)^d} \Bigg\{ 32 \frac{(d-2)(2d-5)}{(d-4)(d-6)} T_F n_l I_2 \\
&\quad + 8 \frac{(d-3)(2d-5)(2d-7)}{d-4} A^2 C_F I_2 \\
&\quad - 4(d-3) A C_A I_1^2 \\
&\quad + 8 \frac{(2d-5)(2d-7)}{(d-3)(d-4)(d-6)} \left[\frac{(d-2)^2(d-5)}{(d-3)(2d-7)} \right. \\
&\qquad \left. + (d^2-4d+5)A - \frac{1}{4}(d-3)(d^2-9d+16)A^2 \right] C_A I_2 \Bigg\}
\end{aligned}
$$

HQET propagator

$$
\begin{aligned}
S(t) = -i\theta(t) \exp\Bigg\{ & C_F \frac{g_0^2}{(4\pi)^{d/2}} \left(\frac{it}{2}\right)^{2\varepsilon} \Gamma(-\varepsilon) A \\
& + C_F \frac{g_0^4}{(4\pi)^d} \left(\frac{it}{2}\right)^{4\varepsilon} \Gamma^2(-\varepsilon) \Bigg[2 \frac{d-2}{(d-3)(d-6)(2d-7)} T_F n_l \\
& \quad + \frac{1}{2(d-3)^2(d-6)} \left(\frac{(d-2)^2(d-5)}{(d-3)(2d-7)} \right. \\
& \qquad \left. + (d^2-4d+5)A - \frac{1}{4}(d-3)(d^2-9d+16)A^2 \right) C_A \\
& \quad - \frac{A}{d-3} \frac{\Gamma^2(1+2\varepsilon)}{\Gamma(1+4\varepsilon)} C_A \Bigg] \Bigg\}
\end{aligned}
$$

Scattering in an external field

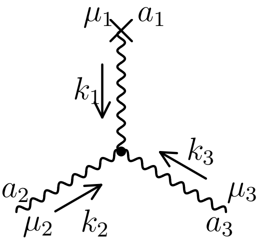
Background field $A_0^\mu \to \bar{A}_0^\mu + A_0^\mu$

$$L = \sum_i \bar{q}_{i0}(i\not{D}_0 - m_{i0})q_{i0} - \frac{1}{4}G^a_{0\mu\nu}G_0^{a\mu\nu}$$
$$- \frac{1}{2a_0}\left(\bar{D}_\mu A_0^\mu\right)^2 + (\bar{D}_\mu \bar{c}_0^a)(D_0^\mu c_0^a)$$

Scattering in an external field

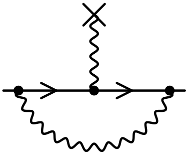
Background field $A_0^\mu \to \bar{A}_0^\mu + A_0^\mu$

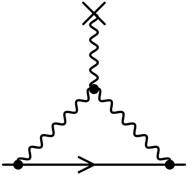
$$L = \sum_i \bar{q}_{i0}(i\bar{\not{D}}_0 - m_{i0})q_{i0} - \frac{1}{4}G^a_{0\mu\nu}G_0^{a\mu\nu}$$
$$- \frac{1}{2a_0}\left(\bar{D}_\mu A_0^\mu\right)^2 + (\bar{D}_\mu \bar{c}_0^a)(D_0^\mu c_0^a)$$



$$g_0 f^{a_1a_2a_3}\Bigg[(k_2 - k_3)^{\mu_1}\, g^{\mu_2\mu_3}$$
$$+ \left(k_3 - k_1 + \frac{1}{a_0}k_2\right)^{\mu_2} g^{\mu_3\mu_1}$$
$$+ \left(k_1 - k_2 - \frac{1}{a_0}k_3\right)^{\mu_3} g^{\mu_1\mu_2}\Bigg]$$

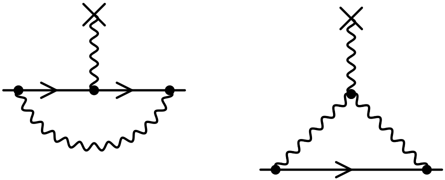
Chromomagnetic interaction





$$F_2(0) = \frac{g_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(\varepsilon)}{2(d-3)} \left[2(d-4)(d-5)C_F - (d^2 - 8d + 14)C_A\right]$$

Chromomagnetic interaction



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$$C_m^0 = 1 + F_2(0)$$

$$C_m(\mu) Z_m^{-1}(\alpha_s(\mu)) = 1 + \frac{\alpha_s(\mu)}{4\pi} e^{-2L\varepsilon} \left[2C_F + \left(\frac{1}{\varepsilon} + 2\right) C_A\right]$$

$$L = \log \frac{M}{\mu}$$

Chromomagnetic interaction

$$\gamma_m = 2C_A \frac{\alpha_s}{4\pi} + \frac{4}{9} C_A \left(17C_A - 13T_F n_l\right) \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

Chromomagnetic interaction

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$$M_{B^*} - M_B = \frac{2}{3M_b} C_m(\mu) \mu_G^2(\mu) + \mathcal{O}\left(\frac{1}{M_b^2}\right)$$

$$\mu_G^2(\mu) = \hat{\mu}_G^2 \alpha_s(\mu)^{\gamma_{m0}/(2\beta_0)} \left[1 + \mathcal{O}(\alpha_s)\right]$$

Chromomagnetic interaction

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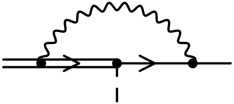
$$\frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} = \left(\frac{\alpha_s(M_b)}{\alpha_s(M_c)}\right)^{\gamma_{m0}/(2\beta_0)} \left[1 + \mathcal{O}\left(\alpha_s, \frac{\Lambda_{\mathrm{QCD}}}{M_{c,b}}\right)\right]$$

HQET heavy–light current

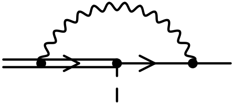


$$\tilde{\jmath}_0 = \bar{q}_0 \varphi_0 = \tilde{Z}_j \tilde{\jmath}(\mu)$$

$$\tilde{\Lambda} = -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu \not{k} v^\nu \left[g_{\mu\nu} - (1-a_0) k_\mu k_\nu / k^2\right]}{(k^2)^2 k_0}$$

$$= -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_0 \not{k} - (1-a_0) k_0}{(k^2)^2 k_0}$$

HQET heavy–light current



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$$= -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_0 \not{k} - (1-a_0) k_0}{(k^2)^2 k_0}$$

$$\not{k} = k_0 \gamma_0 - \vec{k} \cdot \vec{\gamma}$$

$$\tilde{\Lambda} = -iC_F g_0^2 a_0 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2}$$

$$\tilde{Z}_\Gamma = 1 + C_F a \frac{\alpha_s}{4\pi\varepsilon} \qquad \tilde{Z}_j = Z_q^{1/2} Z_\varphi^{1/2} \tilde{Z}_\Gamma = 1 + \frac{3}{2} C_F \frac{\alpha_s}{4\pi\varepsilon}$$

HQET heavy–light current

$$\tilde{\gamma}_j = -3C_F \frac{\alpha_s}{4\pi}$$
$$+ C_F \left[\left(-\frac{8}{3}\pi^2 + \frac{5}{2} \right) C_F + \left(\frac{2}{3}\pi^2 - \frac{49}{6} \right) C_A + \frac{10}{3} T_F n_l \right] \left(\frac{\alpha_s}{4\pi} \right)^2$$
$$+ \cdots$$

QCD/HQET matching

$$j(\mu) = C_\Gamma(\mu,\mu')\tilde{j}(\mu') + \frac{1}{2M}\sum_i B_i^\Gamma(\mu,\mu')\tilde{O}_i(\mu') + \mathcal{O}\left(\frac{1}{M^2}\right)$$

Matrix elements of $j(\mu)$, in situations amenable to HQET treatment, after expansion to a given order in $1/M$, coincide with the corresponding matrix elements of the right-hand side of this equation.

QCD/HQET matching

Let's consider the decay of a heavy quark into a light quark with energy $\omega \ll M$ via a heavy–light weak current. The matrix element in QCD depends on two widely separated large scales $M \gg \omega$ and the renormalization scale μ (if the current has a non-zero anomalous dimension). For no choice of μ can we get rid of large logarithmic corrections. When we go to HQET, all M-dependence is isolated in the matching coefficient of the heavy–light current C_Γ. The HQET matrix element knows nothing about M, and depends only on ω and μ', where the μ'-dependence is determined by the anomalous dimension of the HQET heavy–light current. If μ' is chosen to be of the order of ω, then there are no large logarithmic corrections.

QCD/HQET matching

On-shell matrix element of $j(\mu)$

$$M(P, p', \mu) = (Z_q^{\text{os}})^{1/2} (Z_Q^{\text{os}})^{1/2} Z_j^{-1}(\mu) \Gamma(P, p')$$

should be equal to

$$C_\Gamma(\mu, \mu') \tilde{M}(p, p', \mu') + \mathcal{O}((p, p')/M)$$

where

$$\tilde{M}(p, p', \mu') = (Z_q'^{\text{os}})^{1/2} (\tilde{Z}_Q^{\text{os}})^{1/2} \tilde{Z}_j^{-1}(\mu') \tilde{\Gamma}(p, p')$$

Both matrix elements are UV-finite; their IR divergences coincide, because HQET coincides with QCD in the IR region.

QCD/HQET matching

On-shell matrix element of $j(\mu)$

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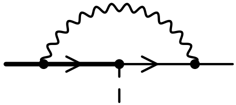
Both matrix elements are UV-finite; their IR divergences coincide, because HQET coincides with QCD in the IR region.

$$C_\Gamma(\mu,\mu') = \left(\frac{Z_q^{\text{os}}}{Z_q'^{\text{os}}}\right)^{1/2}\left(\frac{Z_Q^{\text{os}}}{\tilde{Z}_Q^{\text{os}}}\right)^{1/2}\frac{\tilde{Z}_j(\mu')}{Z_j(\mu)}\frac{\Gamma(P,p')}{\tilde{\Gamma}(p,p')} + \mathcal{O}\left(\frac{p,p'}{m}\right)$$

$p = p' = 0$

QCD/HQET matching



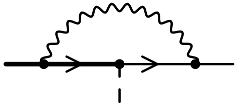
$$\Lambda = -C_F \frac{g_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{(1-h)(d-2+2h)}{(d-2)(d-3)}$$

$$\not{v}\Gamma = \sigma\Gamma\not{v} \qquad \sigma = \pm 1$$

$$\gamma^\mu \Gamma \gamma_\mu = 2\sigma h(d)\Gamma$$

$$h(d) = \eta \left(n - \frac{d}{2} \right) \qquad \eta = (-1)^{n+1}\sigma$$

QCD/HQET matching



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$$h(d) = \eta\left(n - \frac{d}{2}\right) \qquad \eta = (-1)^{n+1}\sigma$$

$$C_\Gamma(M, M) = 1 + C_F \frac{\alpha_s(M)}{4\pi}\left[3(n-2)^2 + (2-\eta)(n-2) - 4\right]$$

f_B

$$<0|\bar{b}\gamma^\mu\gamma_5^{\mathrm{AC}}q|B(P)> = if_B P^\mu$$

$$<B(P')|B(P)> = 2P^0(2\pi)^3\delta\left(\vec{P}' - \vec{P}\right)$$

$${}_{\mathrm{nr}}<B(P')|B(P)>_{\mathrm{nr}} = \delta\left(\vec{P}' - \vec{P}\right)$$

$$<0|\bar{b}\gamma^0\gamma_5^{\mathrm{AC}}q|B(Mv)>_{\mathrm{nr}} = iF = i\frac{M_B}{\sqrt{2M_B}}f_B$$

$$f_B = \sqrt{\frac{2}{M_B}}F(M_b)$$

$$\frac{f_B}{f_D} = \sqrt{\frac{M_c}{M_b}}\left(\frac{\alpha_s(M_b)}{\alpha_s(M_c)}\right)^{\tilde{\gamma}_{j0}/(2\beta_0)}\left[1 + \mathcal{O}\left(\alpha_s, \frac{\Lambda_{\mathrm{QCD}}}{M_{c,b}}\right)\right]$$

f_B

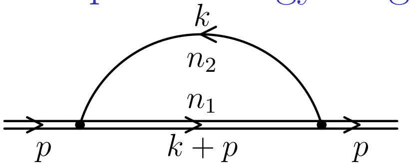
$$
\frac{f_{B^*}}{f_B} = \frac{C_{\gamma^1}(M,M)}{C_{\gamma^0}(M,M)} + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{M_b}\right)
$$

$$
= 1 - 2C_F\frac{\alpha_s(M)}{4\pi} + \mathcal{O}\left(\alpha_s^2, \frac{\Lambda_{\mathrm{QCD}}}{M_b}\right)
$$

Heavy–heavy currents

$$\Gamma(\vartheta) = (\vartheta\coth\vartheta - 1)C_F\frac{\alpha_s}{\pi} + \Bigg\{-\frac{5}{9}(\vartheta\coth\vartheta - 1)T_F n_l$$
$$+\Bigg[-\frac{1}{2}\coth^2\vartheta\Big(\zeta_3 - \zeta_2\vartheta - \vartheta\,\mathrm{Li}_2\left(e^{-2\vartheta}\right) - \mathrm{Li}_3\left(e^{-2\vartheta}\right)\Big)$$
$$-\frac{1}{2}\coth\vartheta\Bigg(\zeta_2 + \left(2\zeta_2 - \frac{67}{18}\right)\vartheta + \vartheta^2 + \frac{\vartheta^3}{3}$$
$$+ 2\vartheta\log\left(1 - e^{-2\vartheta}\right) - \mathrm{Li}_2\left(e^{-2\vartheta}\right)\Bigg)$$
$$+\zeta_2 - \frac{49}{36} + \frac{\vartheta^2}{2}\Bigg]C_A\Bigg\}C_F\left(\frac{\alpha_s}{\pi}\right)^2 + \cdots$$

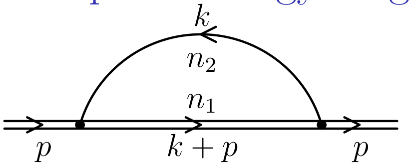
One-loop self-energy diagram with mass



$$I_{n_1 n_2}(m, p_0) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D_1^{n_1} D_2^{n_2}}$$

$$D_1 = -2(k+p)_0 - i0 \qquad D_2 = m^2 - k^2 - i0$$

One-loop self-energy diagram with mass



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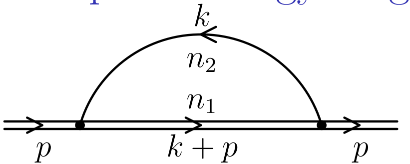
Cut from the threshold $\omega = m$ to $+\infty$
Integer $n_2 \leq 0$: vanishes (HEET loop)
Integer $n_1 \leq 0$: vacuum diagram, e. g,

$$I_{0n}(m, \omega) = m^{d-2n} V(n)$$

One-loop self-energy diagram with mass



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Integer $n_1 \le 0$: vacuum diagram, e. g,

$$I_{0n}(m, \omega) = m^{d-2n} V(n)$$

$$\lim_{m \to 0} I_{n_1 n_2}(m, \omega) = I(n_1, n_2)(-2\omega)^{d-2n_1-n_2} \qquad \text{if} \qquad n_2 < \frac{d}{2}$$

HQET Feynman parametrization

$$I_{n_1 n_2}(m,\omega) = \frac{\Gamma\left(n_1 + n_2 - \frac{d}{2}\right)}{\Gamma(n_1)\Gamma(n_2)} \int\limits_0^\infty y^{n_1-1}(y^2-2\omega y+m^2)^{d/2-n_1-n_2}\, dy$$

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$$I_{n_1 n_2}(m,0) = I_0(n_1,n_2) m^{d-n_1-2n_2}$$

$$I_0(n_1,n_2) = \frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_1-d}{2}+n_2\right)}{2\Gamma(n_1)\Gamma(n_2)} = \frac{\pi^{1/2}}{2^{n_1}} \frac{\Gamma\left(\frac{n_1-d}{2}+n_2\right)}{\Gamma\left(\frac{n_1+1}{2}\right)\Gamma(n_2)}$$

vanishes at odd negative integer n_1 — odd in k

Result

$\omega < 0$

$$I_{n_1 n_2}(m,\omega) = m^{d-n_1-2n_2} \frac{\Gamma\left(n_1+n_2-\frac{d}{2}\right)\Gamma(n_1+2n_2-d)}{\Gamma(n_2)\Gamma(2(n_1+n_2)-d)} \times {}_2F_1\left(\left.\begin{array}{c}\frac{n_1}{2}, \frac{n_1-d}{2}+n_2 \\ n_1+n_2-\frac{d-1}{2}\end{array}\right| 1-\frac{\omega^2}{m^2}\right)$$

The point $\omega = 0$ is regular; when we go from a small $\omega < 0$ to $\omega > 0$ along some path in the complex plane, we make a full cycle around the branch point of the hypergeometric function, and arrive at another Riemann sheet.

$\omega \ll m$

One integration region $k \sim m$
Expand $D_1^{-n_1}$ in ω

$$I_{n_1 n_2}(m,\omega) = m^{d-n_1-2n_2} \sum_{n=0}^{\infty} I_0(n_1+n,n_2)\frac{(n_1)_n}{n!}\left(\frac{2\omega}{m}\right)^n$$

$$= m^{d-n_1-2n_2} I_0(n_1,n_2)\left[{}_2F_1\left(\left.\begin{array}{c}\frac{n_1}{2}, \frac{n_1-d}{2}+n_2\\ \frac{1}{2}\end{array}\right| \frac{\omega^2}{m^2}\right)\right.$$

$$\left.+\frac{\Gamma\left(\frac{n_1+1}{2}\right)\Gamma\left(\frac{n_1-d+1}{2}+n_2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_1-d}{2}+n_2\right)}\frac{2\omega}{m}\,{}_2F_1\left(\left.\begin{array}{c}\frac{n_1+1}{2}, \frac{n_1-d+1}{2}+n_2\\ \frac{3}{2}\end{array}\right| \frac{\omega^2}{m^2}\right)\right]$$

Regular Taylor series in ω,
power of m — dimension counting

$\omega \gg m$

OPE

- hard $k \sim \omega$ — the (1-loop) coefficient function of the unit operator
- soft $k \sim m$ — the series of perturbative (1-loop) vacuum averages of local operators (with $2n$ derivatives) accompanied by their tree-level coefficient functions

$$I_{n_1 n_2}(m, \omega) = I_h + I_s$$

Hard $k \sim \omega$

Expand $D_2^{-n_2}$ in m^2

$$I_h = (-2\omega)^{d-n_1-2n_2} \sum_{n=0}^{\infty} I(n_1, n_2+n) \frac{(n_2)_n}{n!} \left(-\frac{m^2}{4\omega^2}\right)^n$$
$$= (-2\omega)^{d-n_1-2n_2} I(n_1, n_2)\, {}_2F_1\left(\left. \begin{array}{c} \frac{n_1-d}{2}+n_2, \frac{n_1-d+1}{2}+n_2 \\ n_2+1-\frac{d}{2} \end{array} \right| \frac{m^2}{\omega^2} \right)$$

Regular Taylor series in m^2,
powers of -2ω — dimension counting

Soft $k \sim m$

Expand $D_1^{-n_1}$ in k (all odd terms vanish)

$$
\begin{aligned}
I_s &= m^{d-2n_2}(-2\omega)^{-n_1} \sum_{n=0}^{\infty} I_0(-2n, n_2) \frac{(n_1)_{2n}}{(2n)!} \left(\frac{m^2}{4\omega^2}\right)^n \\
&= m^{d-2n_2}(-2\omega)^{-n_1} V(n_2)\, {}_2F_1\left(\left. \begin{array}{c} \frac{n_1}{2}, \frac{n_1+1}{2} \\ \frac{d}{2} - n_2 + 1 \end{array} \right| \frac{m^2}{\omega^2} \right)
\end{aligned}
$$

Regular Taylor series in ω (after extracting $(-2\omega)^{-n_1}$), powers of m — dimension counting

Soft $k \sim m$

Expand $D_1^{-n_1}$ in k (all odd terms vanish)

$$I_s = m^{d-2n_2}(-2\omega)^{-n_1} \sum_{n=0}^{\infty} I_0(-2n, n_2) \frac{(n_1)_{2n}}{(2n)!} \left(\frac{m^2}{4\omega^2} \right)^n$$
$$= m^{d-2n_2}(-2\omega)^{-n_1} V(n_2) \, {}_2F_1 \left(\begin{array}{c} \frac{n_1}{2}, \frac{n_1+1}{2} \\ \frac{d}{2} - n_2 + 1 \end{array} \middle| \frac{m^2}{\omega^2} \right)$$

Regular Taylor series in ω (after extracting $(-2\omega)^{-n_1}$), powers of m — dimension counting
The leading term in I_h dominates over the leading term in I_s at $m \to 0$ if $n_2 < d/2$

Mellin–Barnes

$$\frac{1}{(a+b)^n} = \frac{a^{-n}}{\Gamma(n)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz\, \Gamma(-z)\Gamma(n+z) \left(\frac{b}{a}\right)^z$$

- all poles of $\Gamma(\cdots + z)$ are to the left of the contour
- all poles of $\Gamma(\cdots - z)$ are to the right of it

Mellin–Barnes

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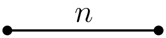
- all poles of $\Gamma(\cdots + z)$ are to the left of the contour
- all poles of $\Gamma(\cdots - z)$ are to the right of it

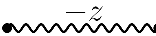
- closing the contour to the right — the expansion in b/a
- closing it to the left — the expansion in a/b

Massive propagator via massless one

$$\frac{1}{(m^2-p^2)^n} = \frac{m^{-2n}}{\Gamma(n)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(-z)\Gamma(n+z)\left(\frac{-p^2}{m^2}\right)^z$$



$$= \frac{m^{-2n}}{\Gamma(n)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(-z)\Gamma(n+z)m^{-2z}$$



Massive diagram via massless one

$$I_{n_1 n_2}(m,\omega) = \frac{m^{-2n_2}(-2\omega)^{d-n_1}}{\Gamma(n_1)\Gamma(n_2)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz$$

$$\Gamma(n_1 - d - 2z)\Gamma\left(\tfrac{d}{2} + z\right)\Gamma(n_2 + z)\left(\frac{-2\omega}{m}\right)^{2z}$$

Massive diagram via massless one

$$I_{n_1 n_2}(m,\omega) = \frac{m^{-2n_2}(-2\omega)^{d-n_1}}{\Gamma(n_1)\Gamma(n_2)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz$$
$$\Gamma(n_1 - d - 2z)\Gamma\left(\tfrac{d}{2} + z\right)\Gamma(n_2 + z)\left(\frac{-2\omega}{m}\right)^{2z}$$

- Close the contour to the right: the sum over residues of the right poles — the expansion in ω/m. 1 series of right poles $z_n = (n + n_1 - d)/2$ ($n = 0$, 1, 2…) — the small ω result
- Close the contour to the left: the sum over residues of the left poles — the expansion in m/ω (analytical continuation to large ω) 2 series of left poles: $z_n^h = -n - n_2$ and $z_n^s = -n - \frac{d}{2}$ — I_h and I_s

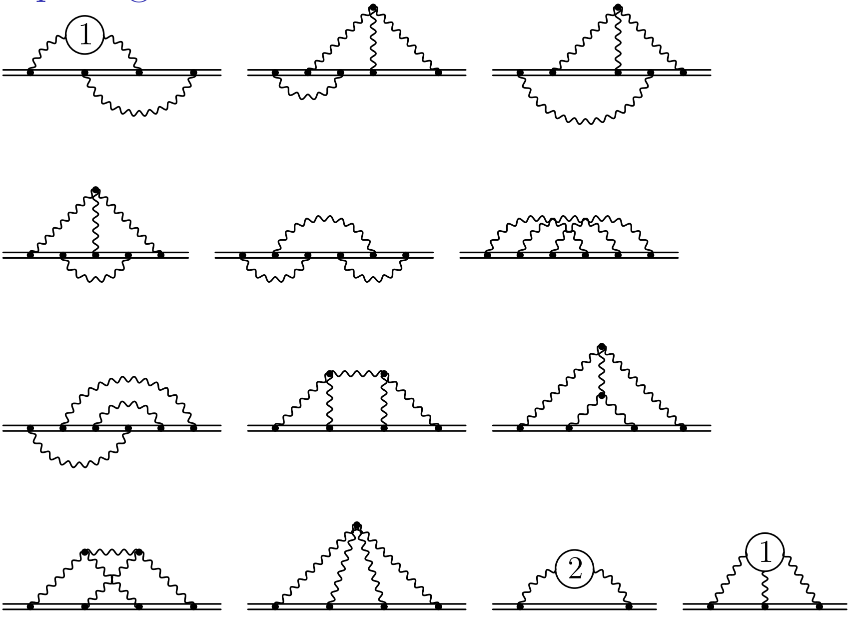
3 loops: 2 c-webs

Let's multiply the 1-loop correction and the last diagram. We can imagine that this set is obtained from the one-particle-reducible diagram by allowing the gluon – heavy-quark vertices to slide along the heavy-quark line, crossing each other. These diagrams are said to contain two connected webs. Everything is already accounted for by the product of the one-loop correction and the part of) two-loop correction in the expansion of the exponent, except the contribution of the first diagram (and its mirror-symmetric), taken with the maximally non-abelian part of its colour factor. It contributes to the three-loop correction in the exponent.

3 loops: 2 c-webs

Similarly, out of all the diagrams with two connected webs, only 3 diagrams contribute to S_{FAA}, with the maximally non-abelian part of their colour factors. This part appears, in the first case, for example, when we commute t^a matrices to obtain the colour structure of the reducible diagram; it is identical to the colour factor of the ladder diagram equal to $C_F C_A^2/4$.

3-loop diagrams



3 loops: 3 c-webs

We move the vertices along the heavy-quark line in such a way as to disentangle those c-webs. While doing so, we get extra terms from the commutators, having colour structures of the corresponding diagrams with the three-gluon vertex. These diagrams have fewer c-webs, which are more complicated. Finally, each colour factor can be expressed as a linear combination of three ones:

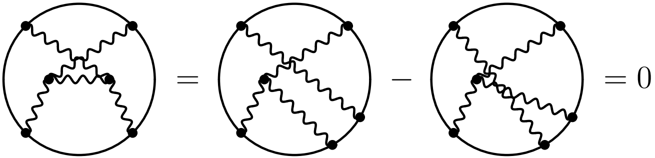
- C_F^3 (3 1-loop c-webs)
- $-C_F^2 C_A/2$ (2 c-webs: 1-loop and with 3-gluon vertex)
- $C_F C_A^2/4$ (1 c-web: the ladder diagram)

3 loops: 3 c-webs

- Occurs with the unit coefficient in all 15 colour factors. The sum of the corresponding contributions is just the term with the cube of the 1-loop correction in the expansion of the exponent.
- Occurs in the diagrams obtained by multiplying the 1-loop correction and the third diagram, the sum of the corresponding contributions is contained in the product of the one-loop term and the two-loop one in the expansion of the exponent.
- We are left with the colour-connected parts of the colour factors (a single c-web contributions). They are present in 3 diagrams, and contribute to S_{FAA}.

3 loops: 1 c-web

- 2 diagrams have equal colour factors (just close the quark line), they contribute to S_{FAA}.
- 1 diagram has 0 colour factor:



- The diagram with the four-gluon vertex can be decomposed into three terms, with colour factors of the previous 3 ones.

3 loops: 1 c-web

- 2-loop gluon self-energy corrections, including one-particle-reducible ones; it contributes to S_{FAA}, S_{FFl}, S_{FAl}, S_{Fll}.
- 1-loop corrections to the three-gluon vertex, including one-particle-reducible ones (i. e., one-loop self-energy corrections to each gluon propagator); it contributes to S_{FAA}, S_{FAl}.

Larin factors

$$Z_P(\alpha_s(\mu)) = \frac{C_{\gamma_5^{\mathrm{AC}}}(\mu,\mu')}{C_{\gamma_5^{\mathrm{HV}}}(\mu,\mu')} = \frac{C_1(\mu,\mu')}{C_{\gamma^0\gamma^1\gamma^2\gamma^3}(\mu,\mu')}$$

$$Z_A(\alpha_s(\mu)) = \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^0}(\mu,\mu')}{C_{\gamma_5^{\mathrm{HV}}\gamma^0}(\mu,\mu')} = \frac{C_{\gamma^0}(\mu,\mu')}{C_{\gamma^1\gamma^2\gamma^3}(\mu,\mu')}$$

$$= \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^3}(\mu,\mu')}{C_{\gamma_5^{\mathrm{HV}}\gamma^3}(\mu,\mu')} = \frac{C_{\gamma^3}(\mu,\mu')}{C_{\gamma^0\gamma^1\gamma^2}(\mu,\mu')}$$

$$Z_T(\alpha_s(\mu)) = \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^0\gamma^1}(\mu,\mu')}{C_{\gamma_5^{\mathrm{HV}}\gamma^0\gamma^1}(\mu,\mu')} = \frac{C_{\gamma^0\gamma^1}(\mu,\mu')}{C_{\gamma^1\gamma^2}(\mu,\mu')}$$

$$= \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^2\gamma^3}(\mu,\mu')}{C_{\gamma_5^{\mathrm{HV}}\gamma^2\gamma^3}(\mu,\mu')} = \frac{C_{\gamma^2\gamma^3}(\mu,\mu')}{C_{\gamma^0\gamma^1}(\mu,\mu')} = 1$$

Equation of motion

$$i\partial_\alpha j_0^\alpha = i\partial_\alpha j^\alpha = M_0 j_0 = M(\mu) j(\mu)$$

Matrix element from $Q(Mv)$ to $q(0)$

$$MC_{\gamma^0}(\mu,\mu') = M(\mu)C_1(\mu,\mu')$$