Introduction to Theoretical Particle Physics

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Exercise Sheet 14

Issue: 24.01. – Submission: 31.01. @ 12:00 Uhr – Discussion: 04.02 and 05.02.

Exercise 28: Polarisation sum

4 points

The outer product of the polarisation states of vector particles is a quantity which appears in many calculations. For a massive on-shell vector particle, the polarisation vectors $\epsilon_i^\mu(p)$ span the space transverse to the momentum of the particle ($p_\mu \epsilon_i^\mu = 0$) and are conventionally normalised as $\epsilon_i^2 = -1$. For a particle in its rest frame, a simple choice is

$$\begin{aligned} \epsilon_1^\mu &= (0,1,0,0) \\ \epsilon_2^\mu &= (0,0,1,0) \\ \epsilon_3^\mu &= (0,0,0,1)\ . \end{aligned} \tag{28.1}$$

(a) Verify that the polarisation vectors given above satisfy the identity

$$\sum_{\lambda=1}^{3} \epsilon_\lambda^\mu \epsilon_\lambda^\nu = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\ , \tag{28.2}$$

where $p^\mu = (m,0,0,0)$.

(b) Argue from the general properties of a Lorentz boost that the form of the polarisation sum, Eq. (28.2), should be the same in all reference frames. Do the polarisation vectors satisfy all required properties in a boosted frame?

(c) Verify the statement of the previous subquestion explicitly by considering the vector particle of subquestion a) boosted in the z-direction, such that its momentum is given by $p^\mu = (E,0,0,p_z)$. Write down suitable polarisation vectors and verify Eq. (28.2) again.

Exercise 29: Generating functional

8 points

In the lecture you have seen how Green's functions for a scalar field $\varphi(x)$ can be obtained in terms of functional derivatives of a functional $Z[J]$, defined as

$$Z[J] = \frac{\int \mathcal{D}\varphi e^{iS[\varphi,J]}}{\int \mathcal{D}\varphi e^{iS[\varphi,0]}}\ , \tag{29.1}$$

where $S[\varphi, J]$ is the action including a coupling to some external source $J(x)$,

$$S[\varphi, J] = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{2}\varphi^2 - V(\varphi) + J\varphi\right] . \tag{29.2}$$

You have also seen how the field $\varphi(x)$ can be integrated out such that a functional of the sources only remains:

$$Z[J] = e^{iW[J]} = e^{i\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y J(x)D(x,y)J(y)}, \tag{29.3}$$

where $D(x, y)$ is such that

$$(\partial_x^2 + m^2)D(x, y) = \delta^{(4)}(x - y) , \tag{29.4}$$

and is being identified with the Feynman propagator, $D(x, y) = iD_F(x - y)$.

In this exercise, you will repeat the steps in the lecture to obtain a similar expression for the generating functional of a vector field $A_\mu(x)$. The action in the presence of an external current $J^\mu(x)$ is given as:

$$S[A, J] = \int \mathrm{d}^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu + J_\mu A^\mu\right] . \tag{29.5}$$

We assume that the current J^μ is conserved, $\partial_\mu J^\mu$=0. Proceed along the following steps. Remember that the field A_μ and its derivatives are assumed to vanish at infinite space and time, such that boundary terms can be dropped when performing integration-by-parts.

(a) Show that the action $S[A, J]$ can be written as

$$S[A, J] = \int \mathrm{d}^4x \left[A_\mu \mathcal{O}^{\mu\nu} A_\nu + J_\mu A^\mu\right] , \tag{29.6}$$

and determine the operator $\mathcal{O}^{\mu\nu}$.

(b) We now shift the field,

$$A_\mu(x) = \bar{A}_\mu(x) + \chi_\mu(x) , \tag{29.7}$$

in order to make the action quadratic in the field. Perform the shift, Eq. (29.7) in the action $S[A, J]$. Collect all terms linear in the field $\bar{A}_\mu$ and write down a condition for χ_μ such that those terms vanish.

(c) Perform a Fourier transform and solve the condition you obtained in the previous subquestion in momentum space. You can do so by writing an ansatz

$$\tilde{\chi}_\mu(p) = (A(p)g_{\mu\nu} + B(p)p_\mu p_\nu)\tilde{J}^\nu(p) , \tag{29.8}$$

and determining the coefficients A and B. Fourier transform back into position space and write the solution as

$$\chi_\mu(x) = \int \mathrm{d}^4y D_{\mu\nu}(x, y)J^\nu(y) . \tag{29.9}$$

The pole in $D_{\mu\nu}(x,y)$ can be regularised in a similar fashion to the Feynman propagator. Show that

$$D^{\mu\nu}(x,y) = i(-g^{\mu\nu})D_F(x-y) \ , \tag{29.10}$$

where the Feynman propagator is given by

$$D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{ie^{ip(x-y)}}{p^2-m^2+i0} \ . \tag{29.11}$$

(d) We define the generating functional

$$Z[J] = \frac{\int \mathcal{D}A_\mu e^{iS[A,J]}}{\int \mathcal{D}A_\mu e^{iS[A,0]}} \ . \tag{29.12}$$

Use the invariance of the measure $\mathcal{D}A_\mu$ under the shift, Eq. (29.7), and the properties of the solution $\chi_\mu(x)$ to show that the generating functional can be written as

$$Z[J] = e^{\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y J_\mu(x)D_F(x-y)J^\mu(y)} \ . \tag{29.13}$$

Note the different sign with respect to Eq. (29.3), which as explained in the lecture leads to a repulsive force between same charges.

In the lecture we saw how we could obtain time-ordered n-point functions in a scalar theory as functional derivatives of the generating functional with respect to the source, $J(x)$. In the vector field case the source carries a Lorentz index and hence the formula becomes

$$\frac{\delta}{i\delta J_{\mu_1}(x_1)} \frac{\delta}{i\delta J_{\mu_2}(x_2)} \cdots \frac{\delta Z[J]}{i\delta J_{\mu_n}(x_n)}\bigg|_{J=0} = \langle 0|TA^{\mu_1}(x_1)A^{\mu_2}(x_2)....A^{\mu_n}(x_n)|0\rangle \ . \tag{29.14}$$

The functional derivative is defined by

$$\frac{\delta J^\mu(x)}{\delta J^\nu(y)} = \delta^\mu_\nu \delta^{(4)}(x-y). \tag{29.15}$$

(e) By explicit calculation, show that for $n=2$ the functional derivative in Eq. (29.14) yields the two-point function, $-g^{\mu_1\mu_2}D_F(x_1-x_2)$, when applied to the generating functional in Eq. (29.13).