# Introduction to Theoretical Particle Physics

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## Exercise Sheet 13

Issue: 17.01. – Submission: 24.01. @ 12:00 Uhr – Discussion: 28.01. and 29.01

### Exercise 26: Massless propagator — 5 points

In the lecture the Feynman propagator for a scalar particle has been introduced:

$$D_F(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{ie^{-ip_\mu x^\mu}}{p^2 - m^2 + i\epsilon}\,, \tag{26.1}$$

where $m$ is the mass of the particle. In this exercise we will consider the simpler propagator with zero mass,

$$D_F(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{ie^{-ip_\mu x^\mu}}{p^2 + i\epsilon}\,, \tag{26.2}$$

and evaluate it explicitly. You can do this by taking the following steps.

(a) Write the time and space components in Eq. (26.2) explicitly, and perform the integration over $p_0$. The computation is very similar to exercise 7f on sheet 3:

- Factorise the denominator in Eq. (26.2) to determine its single poles. Remember that you can neglect $\mathcal{O}(\epsilon^2)$ terms, and that you are free to rescale $\epsilon$ as long as you do not change its sign. Draw the location of the poles on the complex $p_0$ plane.
- Consider a closed integration contour which contains the real axis. Distinguish the cases $x^0 > 0$ and $x^0 < 0$, and close the contour appropriately in the upper or lower complex half-plane to ensure the convergence of the integral.
- Compute the integral using the residue theorem. The integral over $p_0$ should be proportional to

$$\frac{ie^{-i(|p|-i\epsilon)|x^0|}}{2(|p|-i\epsilon)}\,. \tag{26.3}$$

At this point you may drop the $i\epsilon$. How would this result change if the the sign of $i\epsilon$ in eq. 26.2 changes?

(b) Use spherical coordinates to perform the remaining integral over the spatial components $\vec{p}$. The final integral over $p = |\vec{p}|$ needs to be regularised by adding a small imaginary part:

$$\int_0^\infty \mathrm{d}p e^{-ipa} \to \int_0^\infty \mathrm{d}p e^{-ip(a-i\delta)} \tag{26.4}$$

where $\delta$ is an infinitesimal quantity. If you haven't dropped the $i\epsilon$ from the previous subquestion, you can verify that it cannot be used to regulate this integral. Evaluate the integral and show that

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i e^{-ip_\mu x^\mu}}{p^2 + i\epsilon} \propto \frac{1}{x_\mu x^\mu - i\delta}. \tag{26.5}$$

**Exercise 27: Green's function** **6 points**

A function $G(x-y)$ is a Green's function for the Klein-Gordon equation if it satisfies the following equation

$$(\partial^2 + m^2) G(x-y) = -i\delta^{(4)}(x-y)\,. \tag{27.1}$$

(a) Show that the Feynman propagator

$$\begin{aligned} D_F(x) &= \langle 0 | T\{\varphi(x)\varphi(0)\} | 0 \rangle \\ &= \Theta(+t)\langle 0|\varphi(x)\varphi(0)|0\rangle + \Theta(-t)\langle 0|\varphi(0)\varphi(x)|0\rangle \end{aligned} \tag{27.2}$$

satisfies Eq. (27.1).

One of the ingredients used in this exercise is the correlation function $D(x-y) = \langle 0|\varphi(x)\varphi(y)|0\rangle$. Note that the time-ordering operator was dropped in this case, contrary to the Feynman propagator.

(b) Show that

$$D(x-y) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\omega_k(x^0-y^0) + i\vec{k}(\vec{x}-\vec{y})}\,, \tag{27.3}$$

with $\omega_k = \sqrt{\vec{k}^2 + m^2}$.

During the lecture we defined the Fourier transform of the Feynman propagator,

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \tilde{D}_F(p) e^{-ip_\mu(x^\mu - y^\mu)}\,, \tag{27.4}$$

with

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}. \tag{27.5}$$

Let us now consider two other choices for Green's functions, which differ from the above by the small imaginary part in the denominator.

(c) Consider the function

$$D_R(x-y) = \int \frac{d^4 p}{(2\pi)^4} \tilde{D}_R(p) e^{-ip_\mu(x^\mu - y^\mu)}\,, \tag{27.6}$$

with

$$\tilde{D}_R(p) = \frac{i}{p^2 - m^2 + 2p_0 i\epsilon}\,. \tag{27.7}$$

Using the momentum representation defined in Eqs. (27.6) and (27.7), show that it is a Green's function of the Klein-Gordon equation, i.e. it satisfies Eq. (27.1).
Show that in contrast to $\tilde{D}_F(p)$, $\tilde{D}_R(p)$ as a function of $p_0$ has two poles in the lower complex $p_0$ plane. Perform the $p_0$ integral using the residue theorem and show that

$$D_R(x-y) = \Theta(x^0 - y^0)\big(D(x-y) - D(y-x)\big)\,. \tag{27.8}$$

Note that the function vanishes for $x^0 < y^0$, hence it is called a *retarded* Green's function.

(d) Consider the function

$$D_A(x-y) = \int \frac{d^4p}{(2\pi)^4}\tilde{D}_A(p)e^{-ip_\mu(x^\mu - y^\mu)}\,, \tag{27.9}$$

with

$$\tilde{D}_A(p) = \frac{i}{p^2 - m^2 - 2p_0 i\epsilon}\,. \tag{27.10}$$

Similarly to the previous question, show that it is a Green's function of the Klein-Gordon equation and that

$$D_A(x-y) = \Theta(y^0 - x^0)\big(D(y-x) - D(x-y)\big)\,. \tag{27.11}$$

In doing so, show that $\tilde{D}_A(p)$ has two poles in the upper complex $p_0$ plane. Note that the function vanishes for $x^0 < y^0$, hence it is called an *advanced* Green's function.