Introduction to Theoretical Particle Physics

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Exercise Sheet 11

Issue: 20.12. – Submission: 10.01. @ 12:00 Uhr – Discussion: 14.01. and 15.01

Exercise 25: Structure of the Standard Model — 11 points

Consider part of the Standard Model (SM) Lagrangian

$$L_{\text{SM}} = L_{\text{gauge}} + L_{\text{Higgs}} + L_{\text{lepton}} + \cdots \tag{25.1}$$

where L_{gauge} contains kinetic terms for the $U(1)$ and $SU(2)$ gauge fields, B_μ and W^i_μ, as well as self-interaction terms for the latter. L_{Higgs} is the Lagrangian describing the Higgs physics and L_{lepton} is the leptonic part of the SM. The gauge sector reads

$$L_{\text{gauge}} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}\sum_{i=1}^{3} W^i_{\mu\nu}W^{\mu\nu,i}\,, \tag{25.2}$$

with

$$\begin{aligned} W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon^{ijk}W^j_\mu W^k_\nu\,, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu\,. \end{aligned} \tag{25.3}$$

The Higgs boson Lagrangian has the form

$$L_{\text{Higgs}} = \big((D_H)_\mu\phi\big)^\dagger (D_H)^\mu\phi - \frac{\lambda}{4}\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2\,, \tag{25.4}$$

where the covariant derivative D_μ is given by

$$D_\mu = \partial_\mu - ig\sum_{j=1}^{3}\frac{\sigma^j}{2}W^j_\mu - ig'\frac{Y_H}{2}B_\mu\,, \tag{25.5}$$

where g and g' are coupling constants related to the $SU(2)$ and $U(1)$ groups, respectively; σ^j are the Pauli matrices and Y_H is the hypercharge of the scalar field, ϕ. This field is an $SU(2)$ doublet which undergoes spontaneous symmetry breaking (SSB), i.e.

$$\phi(x) = \begin{pmatrix} \phi_1(x) + i\phi_2(x) \\ \phi_3(x) + i\phi_4(x) \end{pmatrix} \xrightarrow{SSB} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}\,, \tag{25.6}$$

where v stands for the vacuum expectation value of the field $\phi(x)$ and $h(x)$ is the Higgs boson field (a real scalar).

Finally, the leptonic part of the SM can be written as

$$L_{\text{lepton}} = \sum_{f=\{e,\mu,\tau\}} \overline{\psi}^f_L (i\hat{D}_L)\psi^f_L + \overline{\psi}^f_R (i\hat{D}_R)\psi^f_R + y_f(\overline{\psi}^f_L \phi \psi^f_R + \overline{\psi}^f_R \phi^\dagger \psi^f_L)\,, \qquad (25.7)$$

where the index f denotes the three fermion families (e = electron, μ = muon and τ = tau) and we use the notation

$$\hat{D} = \gamma^\mu D_\mu\,. \qquad (25.8)$$

The covariant derivatives for left- and right-handed fields read

$$\begin{aligned}(D_L)_\mu &= \partial_\mu - ig\sum_j W^j_\mu \frac{\sigma^j}{2} - ig' B_\mu \frac{Y_L}{2}\,,\\ (D_R)_\mu &= \partial_\mu - ig' B_\mu \frac{Y_R}{2}\,,\end{aligned} \qquad (25.9)$$

where Y_L and Y_R are the hypercharges of the left- and right-handed leptons, respectively.
The left- and right-handed electron fields are

$$\psi^f_L = \begin{pmatrix}\nu_e\\ e_L\end{pmatrix} \qquad \text{and} \qquad \psi^f_R = e_R\,. \qquad (25.10)$$

A similar construction follows for the muon and the tau. The last term in Eq. (25.7) describes the Yukawa interaction between the scalar field, ϕ, and leptonic fields; y_f denotes the Yukawa coupling constants.
In the following, use the SM Lagrangian after the symmetry breaking and set the hypercharges to

$$Y_H = -1\,, \qquad Y_L = +1\,, \qquad Y_R = +2\,. \qquad (25.11)$$

(a) Following the ideas outlined during the lecture, rewrite the derivatives $(D_L)_\mu$ and $(D_R)_\mu$ of Eq. (25.9) in terms of the photon field, A_μ, and the massive fields W^+_μ, W^-_μ and Z_μ. In addition to definitions given in the lecture, you will need that

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^2_\mu). \qquad (25.12)$$

Defining $\sigma^\pm = \frac{1}{\sqrt{2}}(\sigma^1 \pm i\sigma^2)$, you should find that

$$\begin{aligned}(D_L)_\mu &= \partial_\mu - \frac{ig}{2}\sigma^+ W^+_\mu - \frac{ig}{2}\sigma^- W^-_\mu\\ &\quad - \frac{i}{2}(g\cos\theta\sigma^3 - g'\sin\theta)Z_\mu - \frac{i}{2}(g\sin\theta\sigma^3 + g'\cos\theta)A_\mu, \qquad (25.13)\\ (D_R)_\mu &= \partial_\mu + ig'\sin\theta Z_\mu - ig'\cos\theta A_\mu. \qquad (25.14)\end{aligned}$$

(b) Using the formulae derived in the lecture, express the coupling constants, g and g', in terms of the QED coupling constant q_e and the mixing angle θ.

(c) Consider the lepton Lagrangian of Eq. (25.7). Using the results of the previous questions, derive the couplings of the electroweak gauge bosons ($W^{\pm}$ and Z) to electrons and neutrinos. Do the electroweak gauge bosons couple to left- and right-handed fields with equal strength? *Note:* A coupling constant should be considered as the prefactor of the interaction term, without fields or σ- matrices. Coupling constants should be written in terms of the electroweak parameters, q_e and θ, masses of gauge bosons, M_W and M_Z, and the vacuum expectation value, v. Not all of these will appear in the coupling constants.

(d) The Z and W boson masses are measured to be

$$M_Z = 91.188 \text{ GeV}\,, \qquad M_W = 80.379 \text{ GeV}\,. \tag{25.15}$$

The fine structure constant is measured to be

$$\alpha = \frac{q_e^2}{4\pi} \approx \frac{1}{137.036}\,. \tag{25.16}$$

Using these input values and formulae from the lecture, calculate the mixing angle θ and the vacuum expectation value v.

We know that the quantum mechanical amplitude for a process that involves an initial state $|I\rangle$ and a final state $|F\rangle$ can be written as

$$\mathcal{A}(I \to F) \sim \langle F|H_{\text{int}}|I\rangle\,, \tag{25.17}$$

where H_{int} describes the Hamiltonian that gives rise to an $I \to F$ transition. Initial and final states can be constructed using creation and annihilation operators for the particles present in the initial/final state, eg.

$$|I\rangle = |h(\vec{k})\rangle = \sqrt{2\omega_k}\, a_h^\dagger(\vec{k})|0\rangle \tag{25.18}$$

describes an initial state that contains a Higgs boson with momentum $\vec{k}$. Similarly, fermions and antifermions are introduced using creation operators $a_\psi^\dagger(\vec{k}, s)$ and $b_\psi^\dagger(\vec{k}, s)$, respectively, where the index s denotes the spin of the fermion. Finally, we introduce vector bosons with creation operators $a_V^\dagger(\vec{k}, \lambda)$ where λ is the polarisation state of the vector boson.

(e) Identify terms in the SM Lagrangian responsible for the Z boson splitting into an electron-positron pair. What are the initial and final states of such a process in terms of creation/annihilation operators? To which of left- and right-handed electrons does the Z boson couple strongest to? *Hint:* You are not required to evaluate the amplitude, but only to read off the coupling constants from the interaction terms.

(f) Identify terms in the SM Lagrangian responsible for the decay of the Higgs boson into an electron-positron pair. Write the initial and final states of this process in terms of the creation/annihilation operators and identify the coupling constant of this process. Compare the numerical value of the $h \to e^+e^-$ coupling constant with the coupling constant of $Z \to e^+e^-$ calculated in the previous question.

Hint: To find the value of the electron Yukawa coupling, use the electron mass, $m_e = 0.511$ MeV, and the vacuum expectation value (the mass of the electron appears due to the spontaneous symmetry breaking and the Yukawa term in the leptonic Lagrangian in Eq. (25.7)).

(g) Similarly to the previous question, identify terms in the SM Lagrangian responsible for the $h \to W^+W^-$ splitting and identify the relevant coupling constant.