Introduction to Theoretical Particle Physics

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Exercise Sheet 10

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Exercise 23: Representations of the Lorentz group — 6 points

The (proper orthochronous) Lorentz group is generated by boosts and rotations. There are different representations – one of which is the vector representation that you encounter when first learning about Special Relativity. In the lectures you have now also seen spinors, which transform in a different representation.

(a) Given the matrix $R^x(\theta)$ for a rotation around the x-axis by an angle θ and the matrix $\Lambda^x(\eta)$ for a boost along the x-axis in the vector representation,

$$R^x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad \Lambda^x(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{23.1}$$

calculate the infinitesimal generators of rotations (J^x) and boosts (K^x). The infinitesimal generators are related to the finite transformations via

$$R^x(\theta) = e^{i\theta J^x}, \qquad \Lambda^x(\eta) = e^{i\eta K^x}. \tag{23.2}$$

Similar expressions can be derived for the generators of boosts and rotations around the other axes. Those can then be used to verify the commutation relations of the generators

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \qquad [J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k. \tag{23.3}$$

This is the algebra of the generators of the Lorentz group and it is valid for all representations.
Note that the algebra of the J^i alone closes and corresponds to the Lie algebra $\mathfrak{so}(3) \equiv \mathfrak{su}(2)$, as expected for rotations in three-dimensional space. However, the commutator of two boost generators is proportional to a rotation generator, i.e. the algebra of the boosts alone does not close.

(b) Consider the linear combinations of generators

$$J^i_+ = \frac{J^i + iK^i}{2}, \qquad J^i_- = \frac{J^i - iK^i}{2}. \tag{23.4}$$

Show that the J^i and K^i can be written in terms of the $J^i_\pm$. Use the commutation relations in Eq. (23.3) to show that the algebra of the $J^i_\pm$ is given by

$$[J^i_+, J^j_+] = i\epsilon^{ijk} J^k_+ \,, \qquad [J^i_-, J^j_-] = i\epsilon^{ijk} J^k_- \,, \qquad [J^i_+, J^j_-] = 0 \,. \tag{23.5}$$

In this form it is obvious that the algebra of the generators of the Lorentz group is isomorphic to the direct product of two $\mathfrak{su}(2)$ algebras – one generated by the J^i_- and one generated by the J^i_+.

The representations of SU(2) were discussed in the Quantum Mechanics lecture in the context of angular momentum and spin. Recall that these representations can be classified according to their spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Representations of the Lorentz group can now be classified according to their decompositions into representations of the two SU(2). The trivial case is the $(0, 0)$ representation which is a Lorentz scalar and transforms as a singlet under both SU(2). The first non-trivial cases are the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations.

(c) Recall that the spin-$\frac{1}{2}$ representation consists of two-component spinors and is generated by $\tau^i = \frac{\sigma^i}{2}$, where the σ^i are the Pauli matrices. Thus, for the $(\frac{1}{2}, 0)$ representation use

$$J^i_- = \tau^i \,, \qquad\qquad J^i_+ = 0 \,. \tag{23.6}$$

What are the generators J^i and K^i? Work out the explicit form of the matrices R^x and Λ^x in this representation. Repeat the same for the $(0, \frac{1}{2})$ representation, where

$$J^i_- = 0 \,, \qquad\qquad J^i_+ = \tau^i \,. \tag{23.7}$$

Show that $R^{x,\dagger}_{(\frac{1}{2},0)}(\theta) = R^{x,-1}_{(0,\frac{1}{2})}(\theta)$ and $\Lambda^{x,\dagger}_{(\frac{1}{2},0)}(\eta) = \Lambda^{x,-1}_{(0,\frac{1}{2})}(\eta)$. Note that these two-component representations correspond to the left- and right-handed spinors that you encountered in the lecture. We will return to this point in the next problem.

(d) Consider a two-component spinor $\psi = (\psi_1, \psi_2)$ transforming in the $(\frac{1}{2}, 0)$ representation. Apply a rotation around the x-axis by an angle of $\theta = 2\pi$ and by an angle of $\theta = 4\pi$. Compare the results to the original spinor.

Exercise 24: Solutions of the Dirac equation — 5 points

In the lecture, you saw the Dirac matrices γ^μ in the Dirac representation. Other representations are possible. One useful representation is the Weyl representation, which is defined by

$$\gamma^\mu = \begin{pmatrix} 0_{2\times 2} & \sigma^\mu \\ \bar{\sigma}^\mu & 0_{2\times 2} \end{pmatrix} \tag{24.1}$$

where $\sigma^\mu = (\mathbb{1}_{2\times2}, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}_{2\times2}, -\vec{\sigma})$ and $\vec{\sigma}$ are the Pauli matrices. Since the Pauli matrices fulfil the anticommutation relation $\{\sigma^\mu, \bar{\sigma}^\nu\} = 2g^{\mu\nu}$, the Dirac matrices also fulfil the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \,. \tag{24.2}$$

(a) Calculate the fifth Dirac matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in the Weyl representation. Work out the left- and right-handed projectors $P_L = \frac{1}{2}(\mathbb{1} - \gamma_5)$ and $P_R = \frac{1}{2}(\mathbb{1} + \gamma_5)$. Given a four-component spinor $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$, apply the projectors to calculate $\psi_L = P_L\psi$ and $\psi_R = P_R\psi$. Verify that $\psi = \psi_L + \psi_R$. In the Weyl representation it is manifest that ψ_L transforms in the $(\frac{1}{2}, 0)$ representation of the Lorentz group, while ψ_R transforms according to the $(0, \frac{1}{2})$ representation.

(b) Solve the Dirac equation in momentum space,

$$(\hat{p} - m)u(p) \,, \qquad (\hat{p} + m)v(p) \,, \tag{24.3}$$

for a massive fermion with mass m at rest in its rest frame, i.e. with momentum $p^\mu = (m, 0, 0, 0)$.

(c) As mentioned above, the left- and right-handed spinors in the Weyl representation are in one-to-one correspondence to the two-component spinors whose Lorentz transformations you calculated in the previous problem. In particular, the boost in x-direction is

$$\Lambda^x_{(\frac{1}{2},0)}(\eta) = \begin{pmatrix} \cosh(\eta/2) & -\sinh(\eta/2) \\ -\sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix} \,. \tag{24.4}$$

and $\Lambda^{x,\dagger}_{(\frac{1}{2},0)}(\eta) = \Lambda^{x,-1}_{(0,\frac{1}{2})}(\eta)$.
Apply a boost along the x-axis to the solution of the Dirac equation from the previous subproblem to obtain the solution of the Dirac equation for a fermion moving along the x-axis. Verify that the boosted solution still fulfils the Dirac equation.