Exercise 23: Representations of the Lorentz group

The (proper orthochronous) Lorentz group is generated by boosts and rotations. There are different representations – one of which is the vector representation that you encounter when first learning about Special Relativity. In the lectures you have now also seen spinors, which transform in a different representation.

(a) Given the matrix \( R^x(\theta) \) for a rotation around the \( x \)-axis by an angle \( \theta \) and the matrix \( \Lambda^x(\eta) \) for a boost along the \( x \)-axis in the vector representation,

\[
R^x(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta \\
0 & 0 & -\sin \theta & \cos \theta
\end{pmatrix}, \quad \Lambda^x(\eta) = \begin{pmatrix}
cosh \eta & \sinh \eta & 0 & 0 \\
\sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(23.1)

calculate the infinitesimal generators of rotations \( (J^x) \) and boosts \( (K^x) \). The infinitesimal generators are related to the finite transformations via

\[
R^x(\theta) = e^{i\theta J^x}, \quad \Lambda^x(\eta) = e^{i\eta K^x}.
\]

(23.2)

Similar expressions can be derived for the generators of boosts and rotations around the other axes. Those can then be used to verify the commutation relations of the generators

\[
[J^i, J^j] = i\epsilon^{ijk} J^k, \quad [J^i, K^j] = i\epsilon^{ijk} K^k, \quad [K^i, K^j] = -i\epsilon^{ijk} J^k.
\]

(23.3)

This is the algebra of the generators of the Lorentz group and it is valid for all representations.

Note that the algebra of the \( J^i \) alone closes and corresponds to the Lie algebra \( \mathfrak{so}(3) \equiv \mathfrak{su}(2) \), as expected for rotations in three-dimensional space. However, the commutator of two boost generators is proportional to a rotation generator, i.e. the algebra of the boosts alone does not close.

(b) Consider the linear combinations of generators

\[
J^i_+ = \frac{J^i + iK^i}{2}, \quad J^i_- = \frac{J^i - iK^i}{2}.
\]

(23.4)

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Show that the $J^i$ and $K^i$ can be written in terms of the $J^i_\pm$. Use the commutation relations in Eq. (23.3) to show that the algebra of the $J^i_\pm$ is given by

$$ [J^i_+, J^j_+] = i\epsilon^{ijk} J^k_+, \quad [J^i_-, J^j_-] = i\epsilon^{ijk} J^k_-, \quad [J^i_+, J^j_-] = 0. \quad (23.5) $$

In this form it is obvious that the algebra of the generators of the Lorentz group is isomorphic to the direct product of two $su(2)$ algebras – one generated by the $J^i_-$ and one generated by the $J^i_+$. The representations of $SU(2)$ were discussed in the Quantum Mechanics lecture in the context of angular momentum and spin. Recall that these representations can be classified according to their spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Representations of the Lorentz group can now be classified according to their decompositions into representations of the two $SU(2)$. The trivial case is the $(0,0)$ representation which is a Lorentz scalar and transforms as a singlet under both $SU(2)$. The first non-trivial cases are the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations.

(c) Recall that the spin-$\frac{1}{2}$ representation consists of two-component spinors and is generated by $\tau^i = \frac{\sigma^i}{2}$, where the $\sigma^i$ are the Pauli matrices. Thus, for the $(\frac{1}{2},0)$ representation use

$$ J^i_- = \tau^i, \quad J^i_+ = 0. \quad (23.6) $$

What are the generators $J^i$ and $K^i$? Work out the explicit form of the matrices $R^x$ and $\Lambda^x$ in this representation. Repeat the same for the $(0,\frac{1}{2})$ representation, where

$$ J^i_- = 0, \quad J^i_+ = \tau^i. \quad (23.7) $$

Show that $R^{x,\dagger}_{(\frac{1}{2},0)}(\theta) = R^{x,-1}_{(0,\frac{1}{2})}(\theta)$ and $\Lambda^{x,\dagger}_{(\frac{1}{2},0)}(\eta) = \Lambda^{x,-1}_{(0,\frac{1}{2})}(\eta)$. Note that these two-component representations correspond to the left- and right-handed spinors that you encountered in the lecture. We will return to this point in the next problem.

(d) Consider a two-component spinor $\psi = (\psi_1, \psi_2)$ transforming in the $(\frac{1}{2},0)$ representation. Apply a rotation around the $x$-axis by an angle of $\theta = 2\pi$ and by an angle of $\theta = 4\pi$. Compare the results to the original spinor.

Exercise 24: Solutions of the Dirac equation

In the lecture, you saw the Dirac matrices $\gamma^\mu$ in the Dirac representation. Other representations are possible. One useful representation is the Weyl representation, which is defined by

$$ \gamma^\mu = \begin{pmatrix} 0_{2\times2} & \sigma^\mu \\ \bar{\sigma}^\mu & 0_{2\times2} \end{pmatrix} \quad (24.1) $$

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where $\sigma^\mu = (1_{2\times2}, \vec{0})$ and $\bar{\sigma}^\mu = (1_{2\times2}, -\vec{0})$ and $\vec{0}$ are the Pauli matrices. Since the Pauli matrices fulfill the anticommutation relation $\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2g^{\mu\nu}1_{2\times2}$, the Dirac matrices also fulfill the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (24.2)$$

(a) Calculate the fifth Dirac matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in the Weyl representation. Work out the left- and right-handed projectors $P_L = \frac{1}{2}(1 - \gamma_5)$ and $P_R = \frac{1}{2}(1 + \gamma_5)$. Given a four-component spinor $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$, apply the projectors to calculate $\psi_L = P_L\psi$ and $\psi_R = P_R\psi$. Verify that $\psi = \psi_L + \psi_R$. In the Weyl representation it is manifest that $\psi_L$ transforms in the $(\frac{1}{2}, 0)$ representation of the Lorentz group, while $\psi_R$ transforms according to the $(0, \frac{1}{2})$ representation.

(b) Solve the Dirac equation in momentum space,

$$\left(\hat{p} - m\right)u(p), \quad \left(\hat{p} + m\right)v(p), \quad (24.3)$$

for a massive fermion with mass $m$ at rest in its rest frame, i.e. with momentum $p^\mu = (m, 0, 0, 0)$.

(c) As mentioned above, the left- and right-handed spinors in the Weyl representation are in one-to-one correspondence to the two-component spinors whose Lorentz transformations you calculated in the previous problem. In particular, the boost in $x$-direction is

$$\Lambda^x_{(\frac{1}{2}, 0)}(\eta) = \begin{pmatrix} \cosh(\eta/2) & -\sinh(\eta/2) \\ -\sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix}. \quad (24.4)$$

and $\Lambda^x_{(\frac{1}{2}, 0)}(\eta) = \Lambda^x_{(0, \frac{1}{2})}(\eta)$.

Apply a boost along the $x$-axis to the solution of the Dirac equation from the previous subproblem to obtain the solution of the Dirac equation for a fermion moving along the $x$-axis. Verify that the boosted solution still fulfills the Dirac equation.