Exercise 21: Relations of gamma matrices

In this exercise we are going to prove some relations regarding gamma matrices, $\gamma^\mu$, without referring to a specific representation. Recall from the lecture that we have the following relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{1},$$

where $\mathbb{1}$ is the $4 \times 4$ identity matrix. We also introduce the matrix $\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$.

(a) Using the relation in Eq. (21.1), verify that $\gamma_5$ anti-commutes with $\gamma^\mu$ for any index $\mu = 0, 1, 2, 3$, i.e. show that

$$\gamma^\mu \gamma_5 + \gamma_5 \gamma^\mu = 0.$$  

(b) Show that $\gamma_5$ can also be written as

$$\gamma_5 = -\frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma,$$

where $\epsilon_{\mu\nu\rho\sigma}$ is a totally anti-symmetric tensor with $\epsilon_{0123} = -1$. Hint: Think about what values $\mu, \nu, \rho$ and $\sigma$ can take, and how you can capture this in a single term.

(c) Products of multiple gamma matrices can often be simplified to a shorter form. Using symmetry we can write the following relations

(i). $\gamma^\mu \gamma_\mu = A \mathbb{1}$,
(ii). $\gamma^\mu \gamma_5 \gamma_\mu = B \gamma_5$,
(iii). $\gamma^\mu \gamma^\nu \gamma_\mu = C \gamma^\nu$.

Find the values of the proportionality constants $A$, $B$, and $C$. Hint: Remember that $g^{\mu\nu} g_{\mu\nu} = 4$.

(d) Simplify

(i). $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu$,
(ii). $\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu$.
Exercise 22: Majorana Fermions

You have seen in the lecture how fermions are described in terms of four component spinors $\psi$ which satisfy the Dirac equation

$$(i\partial_{\mu}\gamma^{\mu} - m)\psi(x) = 0,$$  

(22.1)

for a particular representation of the Dirac matrices $\gamma^{\mu}$. Let us introduce the so-called Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$  

(22.2)

where $\sigma^i$ are the Pauli sigma matrices. These are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$  

(22.3)

and they satisfy the relation

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}1,$$  

(22.4)

where $\{.,.\}$ denotes the anti-commutator. We further introduce the four-vector of matrices $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. Then we can write the compact form:

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}.$$  

(22.5)

In this representation, solutions of the Dirac equation take the form

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix},$$  

(22.6)

where $\psi_L(x)$ and $\psi_R(x)$ are two-component left- and right-handed spinors. They distinguish themselves through their different transformation behaviour under Lorentz transformations. One can then show that the object $-i\sigma^2\psi_L^*(x)$ transforms as a right-handed spinor, such that the four-component spinor

$$\psi_M(x) = \begin{pmatrix} \psi_L(x) \\ -i\sigma^2\psi_L^*(x) \end{pmatrix}$$  

(22.7)

is a valid spinor in the sense that it transforms in the correct way under Lorentz transformations. It can be shown that if a four-component spinor $\psi$ satisfies the Dirac equation in the presence of the electromagnetic field, then the spinor $i\gamma^2\psi^*$ satisfies the Dirac equation with the opposite charge.

(a) Show that

$$i\gamma^2\psi^*_M = \psi_M.$$  

(22.8)

In this sense, $\psi_M$ describes a neutral fermion which is its own antiparticle. We call $\psi_M$ a Majorana spinor. In this exercise you will construct an explicit representation of the Majorana spinor using plane wave solutions.
(b) Show that $\sigma^\mu\sigma^2 = \sigma^2\sigma^\mu$. Use this to show that the Dirac equation for $\psi_M(x)$ is equivalent to the equation

$$\bar{\sigma}^\mu\partial_\mu \psi_L(x) + m\sigma^2 \psi_L^*(x) = 0$$

(22.9)

for the left-handed spinor $\psi_L(x)$.

(c) Since every solution of the Dirac equation is also a solution of the Klein-Gordon equation, we can write the spinor $\psi_L(x)$ as a linear combination of plane waves $e^{ip_\mu x^\mu}$. To start, show the relation

$$\bar{\sigma}^\mu\partial_\mu e^{\pm ip_\mu x^\mu} = \pm i(E + \bar{\sigma} \cdot \vec{p})e^{\pm ip_\mu x^\mu}.$$  

(22.10)

(d) Another property of spinors is their helicity, which is the projection of the spin of the fermion along its momentum. Helicity eigenvectors are defined as

$$\bar{\sigma} \cdot \vec{p} |_{\vec{p}} \xi_{\pm} (p) = \pm \xi_{\pm} (p).$$  

(22.11)

For massless fermions the spinors $\psi_L$ and $\psi_R$ defined above are eigenstates of helicity. However, for massive fermions, which we consider here, helicity is not Lorentz invariant. We thus write the Majorana spinor $\psi_L$ in terms of eigenstates $\zeta_r(p)$ and $\eta_r(p)$ of both helicities,

$$\psi_L(x) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} (a_r(p)\zeta_r(p)e^{-ip_\mu x^\mu} + a_r^*(p)\eta_r(p)e^{ip_\mu x^\mu}).$$  

(22.12)

Use this in Eq. (22.9) to derive the relations

$$\eta_r(p) = \frac{E - \bar{\sigma} \cdot \vec{p}}{m} i\sigma^2 \zeta_r^*(p),$$  

(22.13)

$$\zeta_r(p) = -\frac{E - \bar{\sigma} \cdot \vec{p}}{m} i\sigma^2 \eta_r^*(p),$$  

(22.14)

by comparing the coefficients of $a_r(p)$ and $a_r^*(p)$. You will find the relation $(\bar{\sigma} \cdot \vec{p})(\bar{\sigma} \cdot \vec{q}) = (\vec{p} \cdot \vec{q})\hat{1} + i(\vec{p} \times \vec{q}) \cdot \bar{\sigma}$ useful. Be aware that $\zeta_r$, $\eta_r$ and $\sigma^i$ are vectors respectively a matrix in the two-dimensional spin space.

(e) From here, any linear independent choice of the $\zeta_r$’s is a solution of the two-component Majorana equation. As an example, use,

$$\zeta_1 = \xi_-, \quad \zeta_2 = \xi_-, $$

(22.15)

where $\xi_{\pm}$ are helicity eigenvectors satisfying Eq. (22.11), and fulfil the relation

$$\xi_- = i\sigma^2 \xi^*_+, $$

(22.16)

to show that the Fourier expansion of $\psi_L(x)$ can be written as

$$\psi_L(x) = \int \frac{d^3p}{(2\pi)^3} \left( a_-(p)\xi_-(p)e^{-ip_\mu x^\mu} + \frac{m}{E + |p|}a_+(p)\xi_+(p)e^{-ip_\mu x^\mu} ight.$$

$$+ a_+^*(p)\xi_-(p)e^{ip_\mu x^\mu} - \frac{m}{E + |p|}a_-^*(p)\xi^*_+(p)e^{ip_\mu x^\mu} \right).$$  

(22.17)