Introduction to Theoretical Particle Physics

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Exercise Sheet 9

Issue: 06.12. – Submission: 13.12. @ 12:00 Uhr – Discussion: 17.12. and 18.12

Exercise 21: Relations of gamma matrices — 4 points

In this exercise we are going to prove some relations regarding gamma matrices, γ^μ, without referring to a specific representation. Recall from the lecture that we have the following relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}\,, \tag{21.1}$$

where $\mathbb{1}$ is the 4×4 identity matrix. We also introduce the matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

(a) Using the relation in Eq. (21.1), verify that γ_5 anti-commutes with γ^μ for any index $\mu = 0, 1, 2, 3$, i.e. show that

$$\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0. \tag{21.2}$$

(b) Show that γ_5 can also be written as

$$\gamma_5 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\,, \tag{21.3}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is a totally anti-symmetric tensor with $\epsilon_{0123} = -1$. *Hint:* Think about what values μ, ν, ρ and σ can take, and how you can capture this in a single term.

(c) Products of multiple gamma matrices can often be simplified to a shorter form. Using symmetry we can write the following relations

(i). $\gamma^\mu\gamma_\mu = A\,\mathbb{1}$,

(ii). $\gamma^\mu\gamma_5\gamma_\mu = B\gamma_5$,

(iii). $\gamma^\mu\gamma^\nu\gamma_\mu = C\gamma^\nu$.

Find the values of the proportionality constants A, B, and C. *Hint:* Remember that $g^{\mu\nu}g_{\mu\nu} = 4$.

(d) Simplify

(i). $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu$,

(ii). $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu$.

Exercise 22: Majorana Fermions

7 points

You have seen in the lecture how fermions are described in terms of four component spinors ψ which satisfy the Dirac equation

$$(i\partial_\mu \gamma^\mu - m)\psi(x) = 0 \ , \tag{22.1}$$

for a particular representation of the Dirac matrices γ^μ. Let us introduce the so-called Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \tag{22.2}$$

where σ^i are the Pauli sigma matrices. These are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{22.3}$$

and they satisfy the relation

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}\mathbb{1} \ , \tag{22.4}$$

where $\{.,.\}$ denotes the anti-commutator. We further introduce the four-vector of matrices $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$. Then we can write the compact form:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{22.5}$$

In this representation, solutions of the Dirac equation take the form

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} , \tag{22.6}$$

where $\psi_L(x)$ and $\psi_R(x)$ are two-component left- and right-handed spinors. They distinguish themselves through their different transformation behaviour under Lorentz transformations. One can then show that the object $-i\sigma^2\psi_L^*(x)$ transforms as a right-handed spinor, such that the four-component spinor

$$\psi_M(x) = \begin{pmatrix} \psi_L(x) \\ -i\sigma^2\psi_L^*(x) \end{pmatrix} \tag{22.7}$$

is a valid spinor in the sense that it transforms in the correct way under Lorentz transformations.

It can be shown that if a four-component spinor ψ satisfies the Dirac equation in the presence of the electromagnetic field, then the spinor $i\gamma^2\psi^*$ satisfies the Dirac equation with the opposite charge.

(a) Show that

$$i\gamma^2\psi_M^* = \psi_M \ . \tag{22.8}$$

In this sense, ψ_M describes a neutral fermion which is its own antiparticle. We call ψ_M a Majorana spinor. In this exercise you will construct an explicit representation of the Majorana spinor using plane wave solutions.

(b) Show that $\sigma^\mu \sigma^2 = \sigma^2 \bar{\sigma}^{*\mu}$. Use this to show that the Dirac equation for $\psi_M(x)$ is equivalent to the equation

$$\bar{\sigma}^\mu \partial_\mu \psi_L(x) + m\sigma^2 \psi_L^*(x) = 0 \tag{22.9}$$

for the left-handed spinor $\psi_L(x)$.

(c) Since every solution of the Dirac equation is also a solution of the Klein-Gordon equation, we can write the spinor $\psi_L(x)$ as a linear combination of plane waves $e^{ip_\mu x^\mu}$. To start, show the relation

$$\bar{\sigma}^\mu \partial_\mu e^{\pm i p_\mu x^\mu} = \pm i (E + \vec{\sigma} \cdot \vec{p}) e^{\pm i p_\mu x^\mu} \ . \tag{22.10}$$

(d) Another property of spinors is their *helicity*, which is the projection of the spin of the fermion along its momentum. Helicity eigenvectors are defined as

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \xi_\pm(p) = \pm \xi_\pm(p) \ . \tag{22.11}$$

For massless fermions the spinors ψ_L and ψ_R defined above are eigenstates of helicity. However, for massive fermions, which we consider here, helicity is not Lorentz invariant. We thus write the Majorana spinor ψ_L in terms of eigenstates $\zeta_r(p)$ and $\eta_r(p)$ of both helicities,

$$\psi_L(x) = \sum_{r=1,2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(a_r(p) \zeta_r(p) e^{-ip_\mu x^\mu} + a_r^*(p) \eta_r(p) e^{ip_\mu x^\mu} \right) . \tag{22.12}$$

Use this in Eq. (22.9) to derive the relations

$$\eta_r(p) = \frac{E - \vec{\sigma} \cdot \vec{p}}{m} i\sigma^2 \zeta_r^*(p) \ , \tag{22.13}$$

$$\zeta_r(p) = -\frac{E - \vec{\sigma} \cdot \vec{p}}{m} i\sigma^2 \eta_r^*(p) \ , \tag{22.14}$$

by comparing the coefficients of $a_r(p)$ and $a_r^*(p)$. You will find the relation $(\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{q}) = (\vec{p} \cdot \vec{q})\mathbb{1} + i(\vec{p} \times \vec{q}) \cdot \vec{\sigma}$ useful. Be aware that ζ_r, η_r and σ^i are vectors respectively a matrix in the two-dimensional spin space.

(e) From here, any linear independent choice of the ζ_r's is a solution of the two-component Majorana equation. As an example, use,

$$\zeta_1 = \xi_- \ , \qquad \eta_2 = \xi_- \ , \tag{22.15}$$

where $\xi_\pm$ are helicity eigenvectors satisfying Eq. (22.11), and fulfil the relation

$$\xi_- = i\sigma^2 \xi_+^* \ , \tag{22.16}$$

to show that the Fourier expansion of $\psi_L(x)$ can be written as

$$\begin{aligned}\psi_L(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \Bigg(& a_-(p) \xi_-(p) e^{-ip_\mu x^\mu} + \frac{m}{E + |p|} a_+(p) \xi_+(p) e^{-ip_\mu x^\mu} \\ & + a_+^*(p) \xi_-(p) e^{ip_\mu x^\mu} - \frac{m}{E + |p|} a_-^*(p) \xi_+(p) e^{ip_\mu x^\mu} \Bigg) \ .\end{aligned} \tag{22.17}$$