Introduction to Theoretical Particle Physics

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Exercise Sheet 8

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Exercise 19: Spontaneous symmetry breaking — 7 points

Consider a theory with N real scalar fields governed by a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^T(\partial^\mu \Phi) - V(\Phi^T\Phi)\,, \tag{19.1}$$

where Φ is a vector, $\Phi = (\phi_1, \dots, \phi_N)$. Let the potential be

$$V(\Phi^T\Phi) = -\frac{\mu^2}{2}(\Phi^T\Phi) + \frac{\lambda}{4}(\Phi^T\Phi)^2\,, \tag{19.2}$$

where μ and λ are positive constants. This potential is manifestly symmetric under the $SO(N)$ group.

(a) Let $R = (\Phi^T\Phi)$. Find the minimum, R_{vac}, of the potential in Eq. (19.2).

Consider a possible pattern of symmetry breaking where one of the scalar fields obtains a non-zero expectation value. In this case one can write

$$\phi_1 = v + \chi_1\,, \qquad \phi_i = \chi_i \text{ for } i \in \{2, \dots, N\}\,. \tag{19.3}$$

where v is a constant and χ_i describe small excitations around the minimum of the potential.

(b) Express v in terms of R_{vac}.

(c) Rewrite the Lagrangian in terms of the fields χ_i. What is the symmetry of the Lagrangian after the symmetry breaking? How many Goldstone bosons do you expect? How many massive scalar fields are present?

Now think of a different pattern of symmetry breaking, where two fields obtain a non-zero expectation value. In this case one can write

$$\phi_{1,2} = v_{1,2} + \chi_{1,2}\,, \qquad \phi_i = \chi_i \text{ for } i \in \{3, \dots, N\} \tag{19.4}$$

where v_1 and v_2 are constants and, again, χ_i describe small excitations around the minimum of the potential.

(d) Express v_1 and v_2 in terms of R_{vac} and a mixing angle θ.

(e) Explain why, despite the fact that two fields obtain a non-zero expectation value, the above symmetry breaking pattern is equivalent to the previous one. Confirm this by rewriting the Lagrangian in terms of the fields χ_i and by diagonalising the mass matrix.

Exercise 20: Non-abelian gauge transformation **6 points**

In lecture 7 it was discussed why the gauge field $\hat{A}_\mu$ belongs to the Lie algebra for the case of an SU(2) gauge theory. This will be generalised in this exercise.

Consider a non-abelian gauge theory with a gauge group G. Under a non-abelian gauge transformation, the gauge field transforms as

$$\hat{A}_\mu(x) \to \hat{A}'_\mu(x) = U(x)\hat{A}_\mu(x)U^{-1}(x) + \frac{1}{ig}(\partial_\mu U^{-1}(x))U(x)\,. \tag{20.1}$$

Recall that the gauge field is defined as $\hat{A}_\mu(x) = \sum_a A^a_\mu(x)t^a$, where the t^a are the generators of the Lie algebra $\mathfrak{g}$ associated to the gauge group. In particular, this means that the gauge field belongs to the algebra, $\hat{A}_\mu \in \mathfrak{g}$. Now one has to show that also the gauge transformed field belongs to the algebra, $\hat{A}'_\mu \in \mathfrak{g}$.

The matrices t^a fulfil the commutation relations $[t^a, t^b] = F^{abc}t^c$ for some structure constants F^{abc}. Thus, the commutator of tw,o generators is again a linear combination of generators, i.e. $[t^a, t^b] \in \mathfrak{g}$. The gauge transformations can be written as $U(x) = \exp(i\sum_a \theta^a(x)t^a)$. To make the notation more compact, we define the following shorthands for nested commutators:

$$\tilde{\Delta}_0(X,Y) = Y\,, \qquad \Delta_0(Y,X) = Y\,, \tag{20.2}$$

$$\tilde{\Delta}_n(X,Y) = [X, \tilde{\Delta}_{n-1}(X,Y)]\,, \qquad \Delta_n(Y,X) = [\Delta_{n-1}(Y,X), X]\,. \tag{20.3}$$

(a) It will be shown in the next subquestions that

$$\begin{aligned} e^X Y e^{-X} &= \sum_{n=0}^{\infty} \frac{\tilde{\Delta}_n(X,Y)}{n!} \\ &= Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \ldots \end{aligned} \tag{20.4}$$

and that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}e^{X(t)} &= e^{X(t)}\left(\sum_{n=0}^{\infty} \frac{\Delta_n(\frac{\mathrm{d}X}{\mathrm{d}t}, X)}{(n+1)!}\right) \\ &= e^{X(t)}\left(\frac{\mathrm{d}X}{\mathrm{d}t} + \frac{1}{2!}\left[\frac{\mathrm{d}X}{\mathrm{d}t}, X\right] + \frac{1}{3!}\left[\left[\frac{\mathrm{d}X}{\mathrm{d}t}, X\right], X\right] + \ldots\right)\,. \end{aligned} \tag{20.5}$$

For now, use Eqs. (20.4) and (20.5) to show that $\hat{A}'_\mu \in \mathfrak{g}$.

(b) Prove Eq. (20.4). To that end, it is useful to define an auxiliary function $F(z) = e^{zX}Ye^{-zX}$. For $z = 1$ this becomes the left-hand side of Eq. (20.4). Make a power series ansatz for $F(z)$,

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{n!}F_n z^n\,, \tag{20.6}$$

and use the derivative $\frac{\mathrm{d}F(z)}{\mathrm{d}z}$ to derive a recurrence relation for the coefficients F_n. Use this to show Eq. (20.4).

Now, prove Eq. (20.5). Use the following steps:

(c) Show that the left-hand side of Eq. (20.5) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{X(t)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^{n} X^{n-k}(t) \left(\frac{\mathrm{d}X(t)}{\mathrm{d}t} \right) X^{k}(t) \,. \tag{20.7}$$

(d) **Optional** Show that

$$[Y, X^n] = \sum_{k=0}^{n-1} \binom{n}{k} X^k \, \Delta_{n-k}(Y, X) \,, \tag{20.8}$$

and conclude from it that

$$Y X^n = \sum_{k=0}^{n} \binom{n}{k} X^k \, \Delta_{n-k}(Y, X). \tag{20.9}$$

Proceed by induction over n. You may find the identity $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ to be useful.

(e) Use the previous result to further reexpress Eq. (20.7) as

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{X(t)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} X^{n-j}(t) \, \Delta_j\left(\tfrac{\mathrm{d}X(t)}{\mathrm{d}t}, X(t)\right) . \tag{20.10}$$

(f) Show that the right-hand side of Eq. (20.5) can be transformed into

$$e^{X(t)} \left(\sum_{n=0}^{\infty} \frac{\Delta_n(\frac{\mathrm{d}X}{\mathrm{d}t}, X)}{(n+1)!} \right) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{j=0}^{n} \binom{n+1}{j+1} X^{n-j}(t) \, \Delta_j\left(\tfrac{\mathrm{d}X}{\mathrm{d}t}, X\right) . \tag{20.11}$$

Use for this the Cauchy product of two series,

$$\left(\sum_{n=0}^{\infty} a_n \right) \cdot \left(\sum_{m=0}^{\infty} b_m \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} \,. \tag{20.12}$$

(g) **Optional** Prove that for arbitrary coefficients $c_{n,j}$

$$\sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} c_{n,j} = \sum_{j=0}^{n} \binom{n+1}{j+1} c_{n,j} \,. \tag{20.13}$$

Change the order of summation on the left-hand side and use the "hockey-stick identity" $\sum_{k=j}^{n} \binom{k}{j} = \binom{n+1}{j+1}$.

(h) Use this result to show that the right-hand sides of Eqs. (20.10) and (20.11) are equal and, therefore, that Eq. (20.5) holds.