Introduction to Theoretical Particle Physics

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Exercise Sheet 7

Issue: 22.11. – Submission: 29.11. @ 12:00 Uhr – Discussion: 3.12. and 4.12

Exercise 16: Source free electromagnetism — 5 points

Consider the Lagrangian for source free electromagnetism:

$$\mathcal{L}_{\mathrm{EM}}(A, \partial A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{16.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Show that the Euler-Lagrange equations are $\partial_\mu F^{\mu\nu} = 0$. In order to perform the variation, treat each component of A_μ as an independent scalar field:

$$\frac{\delta}{\delta(\partial_\alpha A_\beta(y))} \partial_\mu A_\nu(x) = \delta^\alpha_\mu \delta^\beta_\nu \delta^{(4)}(x-y). \tag{16.2}$$

(b) Rewrite the Euler-Lagrange equations in terms of the electric and magnetic fields: $E^i = -F^{0i}$ and $B^i = -(1/2)\epsilon^{ijk} F^{jk}$. Show that by doing this, you recover two of Maxwell's equations in the vacuum:

$$\vec{\nabla} \cdot \vec{E} = 0 \,, \qquad \partial_t \vec{E} = \vec{\nabla} \times \vec{B} \,. \tag{16.3}$$

Hint: Remember that $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

(c) Rewrite $\mathcal{L}_{\mathrm{EM}}$ in terms of $\vec{E}$ and $\vec{B}$. *Hint:* Remember also that $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$.

The energy-momentum tensor $T^{\mu\nu}$ is given by

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu}\mathcal{L} = -F^{\mu\rho}\partial^\nu A_\rho + g^{\mu\nu}\left(\frac{1}{4} F_{\rho\sigma}F^{\rho\sigma}\right). \tag{16.4}$$

It is conserved, i.e. $\partial_\mu T^{\mu\nu} = 0$. Note that it is not a symmetric tensor, $T^{\mu\nu} \neq T^{\nu\mu}$. Define now a new energy-momentum tensor

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{16.5}$$

where the tensor K is antisymmetric on its first two indices: $K^{\lambda\mu\nu} = -K^{\mu\lambda\nu}$.

(d) Show that $\hat{T}^{\mu\nu}$ is also conserved ($\partial_\mu \hat{T}^{\mu\nu} = 0$). Show furthermore that for the choice $K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$, $\hat{T}^{\mu\nu}$ is a symmetric tensor (you can use the Euler-Lagrange equations to do so).

(e) For the choice of $\hat{T}^{\mu\nu}$ in the previous subquestion, show that the energy and momentum of the field are the familiar expressions:

$$\begin{aligned} P^0 &= \int \mathrm{d}^3\vec{x}\ \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2) \\ \vec{P} &= \int \mathrm{d}^3\vec{x}\ (\vec{E} \times \vec{B}). \end{aligned} \tag{16.6}$$

Exercise 17: Gauge invariance and geometry $\boxed{\textbf{3 points}}$

In a recent lecture local gauge invariance was discussed. We considered a local $U(1)$ gauge transformation of the complex scalar field

$$\phi(x) \to e^{i\alpha(x)}\phi(x), \tag{17.1}$$

where $\alpha(x)$ is a real scalar function. The Lagrangian for a free scalar field of mass m exhibits invariance under this transformation only after replacing the ordinary derivative with the covariant derivative

$$D_\mu = \partial_\mu - igA_\mu, \tag{17.2}$$

where $A_\mu = A_\mu(x)$ is a vector field with the gauge transformation rule,

$$A_\mu(x) \to A_\mu(x) + \frac{1}{g}\partial_\mu\alpha(x). \tag{17.3}$$

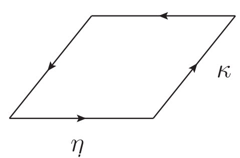
Unlike the transformation of the scalar (17.1) the transformation of the gauge field (17.3) is inhomogeneous. One might worry that this new vector field can be transformed to zero everywhere and is therefore without physical relevance.

(a) To show that the gauge field has physical relevance, we consider transporting the scalar field around a closed loop in space-time using the covariant derivative. We perform the transportation by considering an infinitesimal displacement (parametrised by $\epsilon \ll 1$) of the scalar field in a direction, η^μ,

$$\phi(x + \epsilon\eta) = \phi(x) + \epsilon\eta^\mu D_\mu\phi(x) + \frac{1}{2}\epsilon^2\eta^\mu\eta^\nu D_\mu D_\nu\phi(x) + \mathcal{O}(\epsilon^3). \tag{17.4}$$

Show that transporting the scalar field around the closed loop in the illustration below, we obtain

$$\phi^\square(x) = (1 - \epsilon^2\eta^\mu\kappa^\nu[D_\mu, D_\nu])\phi(x). \tag{17.5}$$



(b) Verify that the change in the field after transportation, i.e. the commutator, is gauge-invariant and proportional to the field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{17.6}$$

Knowing that the change in the scalar field under transportation around a closed loop is gauge-invariant and non-zero, what can you conclude about the physical significance of the gauge field, A_μ?

Exercise 18: Charge conjugation $\boxed{\textbf{5 points}}$

In this exercise we consider a transformation known as charge conjugation. For the complex scalar field encountered in a lecture this transformation is defined through a unitary operator, ζ,

$$\phi(x) \to \zeta\phi(x)\zeta = \eta_c \phi^\dagger(x), \tag{18.1}$$

where η_c is a phase factor. Note that $\zeta = \zeta^{-1}$.

(a) Show that the Lagrangian for a complex field (suppressing space-time dependence)

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi), \tag{18.2}$$

is invariant under charge conjugation. How does the conserved charge,

$$Q = -i \int d^3\vec{x} \, \left[\phi \partial_0 \phi^\dagger - \phi^\dagger \partial_0 \phi\right], \tag{18.3}$$

transform?

(b) Invert the field expansions

$$\phi(t, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega_k}} \left(a_{\vec{k}} e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} + b^\dagger_{\vec{k}} e^{i\omega_k t - i\vec{k}\cdot\vec{x}} \right), \tag{18.4}$$

$$\phi^\dagger(t, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega_k}} \left(a^\dagger_{\vec{k}} e^{i\omega_k t - i\vec{k}\cdot\vec{x}} + b_{\vec{k}} e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} \right), \tag{18.5}$$

to obtain expressions for the creation and annihilation operators in terms of linear combinations of the fields and their time derivatives.

(c) Derive

$$\zeta a_{\vec{k}} \zeta = \eta_c b_{\vec{k}}, \qquad \zeta b_{\vec{k}} \zeta = \eta_c^* a_{\vec{k}}, \tag{18.6}$$

and show that

$$\zeta |\vec{k}\rangle_a = \eta_c^* |\vec{k}\rangle_b, \qquad \zeta |\vec{k}\rangle_b = \eta_c |\vec{k}\rangle_a. \tag{18.7}$$

Note that charge conjugation leaves the vacuum invariant. You can either use the result of b) or directly consider the transformation properties of Eqs. (18.4) and (18.5).

(d) In a recent lecture we encountered the photon. Its interaction with the complex scalar field is given by the gauge-invariant Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi), \tag{18.8}$$

where $D_\mu = \partial_\mu - igA_\mu(x)$. Assuming that the Lagrangian is invariant under charge conjugation, show that

$$\zeta A^\mu(x) \zeta = -A^\mu(x). \tag{18.9}$$