Introduction to Theoretical Particle Physics

Lecture: Prof. Dr. K. Melnikov
Exercises: Dr. C. Brønnum-Hansen, Dr. M. Jaquier

Exercise Sheet 6

Issue: 15.11. – Submission: 22.11. @ 12:00 Uhr – Discussion: 26.11. and 27.11

Exercise 14: Commutators and generators — 5 points

In lecture 3 we encountered the conserved current $T^{\mu\nu}$, $\partial_\mu T^{\mu\nu} = 0$. In particular the components for $\nu = 0$ gave us

$$\begin{aligned} H &= \int d^3\vec{x}\; T^{00}, \\ P^i &= \int d^3\vec{x}\; T^{i0}. \end{aligned} \tag{14.1}$$

In this exercise we will work with a free scalar field for which the Hamiltonian reads

$$H = \frac{1}{2}\int d^3\vec{x}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right), \tag{14.2}$$

where $\pi = \dot{\phi}$.

(a) Consider the unitary operator

$$U = e^{i\alpha G}, \tag{14.3}$$

where α is a continuous parameter and G is a Hermitian operator. Recall that the transformation rule for an operator (without explicit time-dependence), O, is given by

$$O \to O' = UOU^{-1}. \tag{14.4}$$

For an infinitesimal transformation, $\alpha \ll 1$, show that the change in the operator is

$$\delta O = O' - O = i\alpha[G, O] + \mathcal{O}(\alpha^2). \tag{14.5}$$

G is called the generator of the transformation.

(b) Consider the infinitesimal time translation

$$\phi(t+\alpha, \vec{x}) = \phi(t, \vec{x}) + \alpha\dot{\phi}(t, \vec{x}) + \mathcal{O}(\alpha^2). \tag{14.6}$$

Show by explicit evaluation of the commutator $[H, \phi]$ using the commutation relation

$$[\pi(t, \vec{x}), \phi(t, \vec{y})] = -i\delta^{(3)}(\vec{x} - \vec{y}), \tag{14.7}$$

that H is the generator of time translation.

(c) The momentum operator in terms of annihilation and creation operators was derived in the lecture,

$$P^i = \int \frac{d^3\vec{k}}{(2\pi)^3} k^i a_{\vec{k}} a^\dagger_{\vec{k}}. \tag{14.8}$$

Using the expansion of the field

$$\phi(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \left(a_{\vec{k}} e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}} e^{i\omega_k t - i\vec{k}\cdot\vec{x}} \right), \tag{14.9}$$

and the commutation relations

$$[a_{\vec{k}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{q}), \quad [a_{\vec{k}}, a_{\vec{q}}] = 0, \quad [a^\dagger_{\vec{k}}, a^\dagger_{\vec{q}}] = 0, \tag{14.10}$$

calculate the commutator $[P^i, \phi]$. What is P^i the generator of?

(d) Consider the Lagrangian density for three real scalar fields with equal masses

$$\mathcal{L} = \frac{1}{2}\left(\sum_{i=1}^{3} (\partial_\mu \phi_i)^2 - m^2 \phi_i^2 \right). \tag{14.11}$$

We consider the three conserved charges,

$$Q_1 = \int d^3\vec{x} \left(\phi_2 \dot{\phi}_3 - \phi_3 \dot{\phi}_2 \right), \tag{14.12}$$

$$Q_2 = \int d^3\vec{x} \left(\phi_3 \dot{\phi}_1 - \phi_1 \dot{\phi}_3 \right), \tag{14.13}$$

$$Q_3 = \int d^3\vec{x} \left(\phi_1 \dot{\phi}_2 - \phi_2 \dot{\phi}_1 \right). \tag{14.14}$$

Calculate the commutator $[Q_i, \phi_j]$. What symmetry do the charges generate? Use that

$$\begin{aligned} &[\phi_i(t,\vec{x}), \pi_j(t,\vec{y})] = i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}), \\ &[\phi_i(t,\vec{x}), \phi_j(t,\vec{y})] = [\pi_i(t,\vec{x}), \pi_j(t,\vec{y})] = 0. \end{aligned} \tag{14.15}$$

(e) Show that the commutation relations of the charges have the structure of angular momentum,

$$[Q_i, Q_j] = i\epsilon_{ijk} Q_k. \tag{14.16}$$

Hint: Make use of $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.

Exercise 15: Symmetries **7 points**

Consider the Lagrangian density for three real scalar fields with equal masses:

$$\mathcal{L} = \frac{1}{2}\left(\sum_{i=1}^{3} (\partial_\mu \phi_i)^2 - m^2 \phi_i^2 \right) - V(\phi_1, \phi_2, \phi_3), \tag{15.1}$$

where

$$V(\phi_1, \phi_2, \phi_3) = V\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right) . \tag{15.2}$$

Due to the particular shape of the potential, we can write this Lagrangian in terms of a vector of fields, $\vec{\phi} = (\phi_1, \phi_2, \phi_3)^{\mathrm{T}}$. Here we investigate transformations in this abstract field space and the associated internal symmetries of the Lagrangian density.

(a) Show that the Lagrangian density, Eq. 15.1 , is invariant under the transformation parametrised by an angle α,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \to \begin{pmatrix} \phi_1' \\ \phi_2' \\ \phi_3' \end{pmatrix} = \begin{pmatrix} \cos(\alpha)\phi_1 + \sin(\alpha)\phi_2 \\ -\sin(\alpha)\phi_1 + \cos(\alpha)\phi_2 \\ \phi_3 \end{pmatrix} . \tag{15.3}$$

Determine the change $\Delta\phi_i = \phi_i' - \phi_i$ of the fields for small values of α and write down the operator T_{ij} which generates the small change, $\Delta\phi_i = \alpha T_{ij}\phi_j$.

(b) Find the remaining two internal symmetries of the Lagrangian density Eq. 15.1 and write down the generators of the corresponding transformations.

(c) In the lecture it has been shown that for every such symmetry there is a conserved current,

$$J^\mu = \sum_i \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \Delta\phi_i \ . \tag{15.4}$$

Use this to show that the conserved charges associated with those symmetries are those given in part d) of the previous exercise, up to an overall constant.

Consider now a different Lagrangian with two *complex* scalar fields:

$$\mathcal{L} = (\partial_\mu \phi_1^\dagger)(\partial^\mu \phi_1) + (\partial_\mu \phi_2^\dagger)(\partial^\mu \phi_2) - m^2 \phi_1^\dagger \phi_1 - m^2 \phi_2^\dagger \phi_2. \tag{15.5}$$

(d) Write the above Lagrangian density in terms of a complex vector of the two fields and show that it is invariant under transformations which belong to the group $U(2)$.

(e) It can be shown that the group $U(2)$ is locally isomorph to $U(1) \otimes SU(2)$. Consider an additional potential term in the Lagrangian density, $V = (\phi_1^\dagger)^2\phi_1^2 + (\phi_2^\dagger)^2\phi_2^2$. Give an argument why this potential is invariant under $U(1)$ but not under the full $U(2)$ group. What term has to be added to the potential such that it is invariant under the full $U(2)$ group?

(f) Determine the four conserved charges associated to the symmetry transformations you found in part d) of this exercise in terms of the fields and their conjugate momenta. The generators of the group $SU(2)$ are

$$i\frac{\sigma^1}{2},\ i\frac{\sigma^2}{2},\ i\frac{\sigma^3}{2}\ , \tag{15.6}$$

where σ_i are the Pauli matrices, which are 2×2 Hermitian matrices. You do not need their explicit expression for this exercise.