Introduction to Theoretical Particle Physics

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Exercise Sheet 5

Issue: 08.11. – Submission: 15.11. @ 12:00 Uhr – Discussion: 19.11. and 20.11

Exercise 10: Matrix identity — 2 points

Let $\mathbf{M}$ be an arbitrary $n \times n$ matrix of complex numbers with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

(a) Show that the determinant of the exponential of $\mathbf{M}$ can be expressed in terms of the exponential of its trace, i.e. $\det(\exp(\mathbf{M})) = \exp(\mathrm{Tr}(\mathbf{M}))$.

(b) Using the above identity, prove the identity you have used in the lecture to derive the Noether theorem,

$$\det(\mathbb{1} + \epsilon\mathbf{M}) = 1 + \epsilon\,\mathrm{Tr}(\mathbf{M}) + \mathcal{O}(\epsilon^2)\,, \tag{10.1}$$

where ϵ is a small parameter.

Exercise 11: Relativistic normalisation — 3 points

Under a boost of velocity v in the z-direction we have

$$p'_z = \gamma(p_z + \beta E)$$
$$E' = \gamma(E + \beta p_z),$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

(a) Using the delta function identity

$$\delta(f(x) - f(x_0)) = \frac{1}{|\frac{df}{dx}(x_0)|}\delta(x - x_0), \tag{11.1}$$

show that

$$E\,\delta^{(3)}(\vec{p} - \vec{q}) = E'\,\delta^{(3)}(\vec{p}\,' - \vec{q}\,'). \tag{11.2}$$

Hint: Outside the delta function you can set $E_{\vec{p}} = E_{\vec{q}} = E$.

(b) Use the result above to argue that by defining momentum states as

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\,a^\dagger_{\vec{p}}|0\rangle, \tag{11.3}$$

we obtain a relativistically invariant normalisation of $\langle\vec{p}|\vec{q}\rangle$.

Exercise 12: Multiparticle states **5 points**

You have seen in the lecture how excited states of the Hamiltonian for a scalar field can be obtained by repeated action of the creation operator on the vacuum:

$$|\vec{k}_1, \dots, \vec{k}_N\rangle = \left(\prod_i \sqrt{2\omega_{k_i}}\right) a^\dagger_{\vec{k}_1} \dots a^\dagger_{\vec{k}_N} |0\rangle, \tag{12.1}$$

and that the energy of those states is given by

$$H|\vec{k}_1, \dots, \vec{k}_N\rangle = \left(\sum_i \omega_i\right) |\vec{k}_1, \dots, \vec{k}_N\rangle \ . \tag{12.2}$$

In this exercise we will investigate the action of two more operators on such states.

(a) It has been shown in the lecture that the three-momentum operator for a scalar field φ can be determined as

$$P^i = \int \mathrm{d}^3\vec{x}\ T^{0i} = -\int \mathrm{d}^3\vec{x}\pi(t,\vec{x})\partial^i\varphi(t,\vec{x}) \ , \tag{12.3}$$

where $T^{\mu\nu}$ is the energy-momentum tensor for a scalar field. Use the representation of φ and π in terms of creation and annihilation operators and show that the momentum operator can be written as

$$\vec{P} = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \vec{k} a^\dagger_{\vec{k}} a_{\vec{k}} \ . \tag{12.4}$$

(b) Show that by acting with the momentum operator on the state $|\vec{k}_1, \dots, \vec{k}_N\rangle$ as defined above you obtain the sum over the momenta $\vec{k}_i$,

$$\vec{P}|\vec{k}_1, \dots, \vec{k}_N\rangle = \left(\sum_i \vec{k}_i\right) |\vec{k}_1, \dots, \vec{k}_N\rangle \ . \tag{12.5}$$

(c) In analogy to the harmonic oscillator one can also define a particle number operator,

$$\hat{N} = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} a^\dagger_{\vec{k}} a_{\vec{k}} \ . \tag{12.6}$$

Use the calculation from the previous subquestion to argue that

$$\hat{N}|\vec{k}_1, \dots, \vec{k}_N\rangle = N|\vec{k}_1, \dots, \vec{k}_N\rangle \ . \tag{12.7}$$

(you don't have to repeat the full calculation).

With this you have shown that the state $|\vec{k}_1, \dots, \vec{k}_N\rangle$ is an eigenstate of the energy, momentum and number operators with eigenvalues

$$\sum_i \omega_i \ , \quad \sum_i \vec{k}_i \ , \quad N \ . \tag{12.8}$$

One can thus identify this state as a multiparticle state containing N particles with relativistic energy $\omega_i = \sqrt{\vec{k}_i^2 + m^2}$ and momentum $\vec{k}_i$, respectively.

Exercise 13: Vacuum energy density **3 points**

In the lecture, we have encountered the energy density of the vacuum

$$\rho_{\text{vac}} = \frac{E_{\text{vac}}}{V} = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{\omega_k}{2}\,, \tag{13.1}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$. We saw that the integral is divergent in general. We introduced a cut-off, which was motivated by the energy scale at which gravity effects become important and therefore, where we would expect our theory to break down at the latest. This yielded a value for ρ_{vac} which was 122 orders of magnitude larger than the value expected from measurements of the cosmological constant Λ_{CC}. Here, we would like to get an idea of how large the contribution to ρ_{vac} from, for example, the electron is.

(a) As in the lecture, switch to spherical coordinates and regulate the integral by replacing the upper integration limit of the radial integral by a cutoff Λ. Solve the integral for $0 < m \leq \Lambda < \infty$.

(b) We expect the theory to hold at least up to scales of the electron mass m_e. Calculate numerical values for ρ_{vac} for $m = m_e$ and $\Lambda = 2m_e$. Compare those numbers to the vacuum energy density expected from the cosmological constant, $\frac{\Lambda_{\text{CC}}}{8\pi G}$. Use $\Lambda_{\text{CC}} = 4.30 \cdot 10^{-66}\,\text{eV}^2$ for the cosmological constant and $G = 6.71 \cdot 10^{-39}\,\text{GeV}^{-2}$ for Newton's gravitational constant. Think about what you can learn from this comparison.