Exercise 10: Matrix identity

Let $M$ be an arbitrary $n \times n$ matrix of complex numbers with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

(a) Show that the determinant of the exponential of $M$ can be expressed in terms of the exponential of its trace, i.e. $\det(\exp(M)) = \exp(\text{Tr}(M))$.

(b) Using the above identity, prove the identity you have used in the lecture to derive the Noether theorem,

$$\det(1 + \epsilon M) = 1 + \epsilon \text{Tr}(M) + O(\epsilon^2),$$

where $\epsilon$ is a small parameter.

Exercise 11: Relativistic normalisation

Under a boost of velocity $v$ in the z-direction we have

$$p'_z = \gamma(p_z + \beta E)$$
$$E' = \gamma(E + \beta p_z),$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$.

(a) Using the delta function identity

$$\delta(f(x) - f(x_0)) = \frac{1}{|df/dx(x_0)|} \delta(x - x_0),$$

show that

$$E \delta^{(3)}(\vec{p} - \vec{q}) = E' \delta^{(3)}(\vec{p}' - \vec{q}').$$

*Hint:* Outside the delta function you can set $E_{\vec{p}} = E_{\vec{q}} = E$.

(b) Use the result above to argue that by defining momentum states as

$$|\vec{p}\rangle = \sqrt{2E_p a_{\vec{p}}^\dagger|0\rangle},$$

we obtain a relativistically invariant normalisation of $\langle \vec{p} | \vec{q} \rangle$.  

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Exercise 12: Multiparticle states

You have seen in the lecture how excited states of the Hamiltonian for a scalar field can be obtained by repeated action of the creation operator on the vacuum:

\[ |\vec{k}_1, \ldots, \vec{k}_N\rangle = \left( \prod_i \sqrt{2\omega_{k_i}} \right) a_{\vec{k}_1}^\dagger \cdots a_{\vec{k}_N}^\dagger |0\rangle, \quad (12.1) \]

and that the energy of those states is given by

\[ H |\vec{k}_1, \ldots, \vec{k}_N\rangle = \left( \sum_i \omega_i \right) |\vec{k}_1, \ldots, \vec{k}_N\rangle. \quad (12.2) \]

In this exercise we will investigate the action of two more operators on such states.

(a) It has been shown in the lecture that the three-momentum operator for a scalar field \( \varphi \) can be determined as

\[ P^i = \int d^3\vec{x} \ T^{0i} = \int d^3\vec{x} \pi(t, \vec{x}) \partial^i \varphi(t, \vec{x}), \quad (12.3) \]

where \( T^{\mu \nu} \) is the energy-momentum tensor for a scalar field. Use the representation of \( \varphi \) and \( \pi \) in terms of creation and annihilation operators and show that the momentum operator can be written as

\[ \vec{P} = \int \frac{d^3\vec{k}}{(2\pi)^3} \vec{k} a_{\vec{k}}^\dagger a_{\vec{k}}. \quad (12.4) \]

(b) Show that by acting with the momentum operator on the state \( |\vec{k}_1, \ldots, \vec{k}_N\rangle \) as defined above you obtain the sum over the momenta \( \vec{k}_i \),

\[ \vec{P} |\vec{k}_1, \ldots, \vec{k}_N\rangle = \left( \sum_i \vec{k}_i \right) |\vec{k}_1, \ldots, \vec{k}_N\rangle. \quad (12.5) \]

(c) In analogy to the harmonic oscillator one can also define a particle number operator,

\[ \hat{N} = \int \frac{d^3\vec{k}}{(2\pi)^3} a_{\vec{k}}^\dagger a_{\vec{k}}. \quad (12.6) \]

Use the calculation from the previous subquestion to argue that

\[ \hat{N} |\vec{k}_1, \ldots, \vec{k}_N\rangle = N |\vec{k}_1, \ldots, \vec{k}_N\rangle. \quad (12.7) \]

(you don’t have to repeat the full calculation).

With this you have shown that the state \( |\vec{k}_1, \ldots, \vec{k}_N\rangle \) is an eigenstate of the energy, momentum and number operators with eigenvalues

\[ \sum_i \omega_i, \quad \sum_i \vec{k}_i, \quad N. \quad (12.8) \]

One can thus identify this state as a multiparticle state containing \( N \) particles with relativistic energy \( \omega_i = \sqrt{\vec{k}_i^2 + m^2} \) and momentum \( \vec{k}_i \), respectively.

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Exercise 13: Vacuum energy density

In the lecture, we have encountered the energy density of the vacuum

\[ \rho_{\text{vac}} = \frac{E_{\text{vac}}}{V} = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2}, \quad (13.1) \]

where \( \omega_k = \sqrt{k^2 + m^2} \). We saw that the integral is divergent in general. We introduced a cut-off, which was motivated by the energy scale at which gravity effects become important and therefore, where we would expect our theory to break down at the latest. This yielded a value for \( \rho_{\text{vac}} \) which was 122 orders of magnitude larger than the value expected from measurements of the cosmological constant \( \Lambda_{\text{CC}} \). Here, we would like to get an idea of how large the contribution to \( \rho_{\text{vac}} \) from, for example, the electron is.

(a) As in the lecture, switch to spherical coordinates and regulate the integral by replacing the upper integration limit of the radial integral by a cutoff \( \Lambda \). Solve the integral for \( 0 < m \leq \Lambda < \infty \).

(b) We expect the theory to hold at least up to scales of the electron mass \( m_e \). Calculate numerical values for \( \rho_{\text{vac}} \) for \( m = m_e \) and \( \Lambda = 2m_e \). Compare those numbers to the vacuum energy density expected from the cosmological constant, \( \frac{\Lambda_{\text{CC}}}{8\pi G} \). Use \( \Lambda_{\text{CC}} = 4.30 \cdot 10^{-66} \text{ eV}^2 \) for the cosmological constant and \( G = 6.71 \cdot 10^{-39} \text{ GeV}^{-2} \) for Newton’s gravitational constant. Think about what you can learn from this comparison.