Introduction to Theoretical Particle Physics

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Exercise Sheet 4

Issue: 01.11. – Submission: 08.11. @ 12:00 Uhr – Discussion: 12.11. and 13.11

Exercise 8: Scalar Lagrangians

6 points

The canonical Lagrangian for a free scalar field reads

$$\mathcal{L}_{\phi,\text{canonical}} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}m^2\phi^2 \,. \tag{8.1}$$

The behaviour of the field is given by minimising the action

$$S_\phi = \int d^4x \mathcal{L}_{\phi,\text{canonical}} \,. \tag{8.2}$$

When constructing the canonical Lagrangian of Eq. (8.1), we have required that it contains terms which are at most quadratic in the field ϕ and that this whole functional is Lorentz invariant.

(a) In principle, there are more terms which satisfy these two conditions and could be added to the scalar field Lagrangian. Consider

$$\mathcal{L}_{\phi,\text{extra}} = \tfrac{1}{2}(\partial_\mu \phi)^2 + B\partial_\mu\partial^\mu\phi + C\phi\partial_\mu\partial^\mu\phi + D\phi - \tfrac{1}{2}m^2\phi^2 \,, \tag{8.3}$$

where B, C, D are real constants. Show that the theory defined by the Lagrangian of Eq. (8.3) can be described by the Lagrangian in canonical form, Eq. (8.1).
Hints:

- In order to do so, show that the Lagrangian of Eq. (8.3) can be brought into the canonical form of Eq. (8.1) up to a total derivative/constant terms – as these will not change the minimum of the action.
- You can use integration-by-parts inside the action.
- You can rescale and shift the scalar field, as well as rescale the mass,

$$\begin{aligned} \phi &\longrightarrow a(\phi + b) \,, \\ m^2 &\longrightarrow cm^2 \,, \end{aligned} \tag{8.4}$$

where a, b and c are real constants.

Now consider a slightly richer theory with two real scalar fields ϕ_1 and ϕ_2, governed by the Lagrangian

$$\mathcal{L}_{\phi_1,\phi_2} = \tfrac{1}{2}(\partial_\mu\phi_1)^2 + \tfrac{1}{2}(\partial_\mu\phi_2)^2 - \tfrac{1}{2}M_{11}^2\phi_1^2 - M_{12}^2\phi_1\phi_2 - \tfrac{1}{2}M_{22}^2\phi_2^2. \tag{8.5}$$

(b) Rewrite the Lagrangian of Eq. (8.5) in terms of a doublet $\vec{\phi} = (\phi_1, \phi_2)$. Write the mass terms as a 2×2 matrix $\mathbf{M}$, known as mass matrix.

(c) Now set

$$M_{11} = M_{22} = m\,, \qquad\qquad M_{12} = \xi \cdot m\,, \tag{8.6}$$

where ξ is a real parameter, $\xi < 1$. Consider a rotation in the field space $(\phi_1, \phi_2) \to (\tilde{\phi}_1, \tilde{\phi}_2)$ and rewrite the Lagrangian of Eq. (8.5) in a basis where the mass matrix is diagonal. What are the masses of the new scalar fields?

(d) Now consider our theory of Eq. (8.5) with

$$M_{11} = M_{22} = m\,, \qquad\qquad M_{12} = 0\,. \tag{8.7}$$

Introduce another scalar field η into our theory which interacts with the two scalars, ϕ_1 and ϕ_2, through the term

$$\mathcal{L}_{\text{interaction}} = -\frac{g}{2!}\eta\phi_1\phi_2\,, \tag{8.8}$$

where g is a real coupling constant and $0 < g < m$. Assume that in the ground state the field η developes a non-vanishing expectation value $\langle\eta\rangle = \eta_0$. You can rewrite it using small perturbations around this minimum, i.e.

$$\eta = \eta_0 + \delta\eta\,. \tag{8.9}$$

Find the modification of the masses of the scalar fields ϕ_1 and ϕ_2 in terms of the coupling g and the expectation value η_0.

Exercise 9: Angular momentum of a scalar field — 7 points

In this exercise we are going to investigate the conserved currents related to Lorentz transformations and identify conserved charges associated with angular momentum of the field. We consider the Lagrangian for a scalar field.

(a) During the lecture we have considered a translation of coordinates which leads to a conserved current which was associated with the energy-momentum tensor, $T^{\mu\nu}$. Show that

$$M^{\mu\nu\sigma} = x^\nu T^{\mu\sigma} - x^\sigma T^{\mu\nu}\,, \tag{9.1}$$

is also a conserved current, i.e. $\partial_\mu M^{\mu\nu\sigma} = 0$.
Hint: Use the fact that the energy-momentum tensor is symmetric, $T^{\mu\nu} = T^{\nu\mu}$. You can verify that in case of a free scalar field theory; in general it is not guaranteed by construction but $T^{\mu\nu}$ can always be brought into a symmetric form.

(b) Consider the conserved quantities corresponding to the Lorentz currents of Eq. (9.1), i.e.

$$I^{\mu\nu} = \int d^3\vec{x}\, M^{0\mu\nu}\,, \tag{9.2}$$

and write them for spatial indices $\mu = j$ and $\nu = k$ using the scalar field $\varphi(t, \vec{x})$ and its conjugate momentum $\pi(t, \vec{x})$.

(c) It is useful to define the quantity $J_i = \frac{1}{2}\epsilon_{ijk} I^{jk}$ which may be interpreted as the total angular momentum of the scalar field. Show that

$$\vec{J} = \int d^3\vec{x}\ (\vec{x} \times \vec{p})\ , \tag{9.3}$$

where $\vec{p}$ is the 3-momentum density of the field.

(d) Express the components of the angular momentum operator in terms of creation and annihilation operators.

(e) Consider a single particle state with zero-momentum, i.e.

$$|\vec{0}\rangle = a^\dagger_{\vec{0}}|0\rangle\,. \tag{9.4}$$

Show that the expectation value of the total angular momentum operator of this state vanishes, i.e.

$$\langle\vec{0}|\vec{J}|\vec{0}\rangle = 0\,. \tag{9.5}$$

(f) From your quantum mechanics course you know that one of the terms contributing to the total angular momentum is the spin. What can you say about the spin of a scalar field knowing the result of previous question?

Optional: Do you see how this constraint can be avoided so that a particle at rest has a non-zero spin?[1]

[1]Consult for example Sec. 3.5 of M. Peskin & D. Schroeder or Sec. 1.3 and Sec. 6.1 of "An Introduction to Quantum Field Theory" by G. Sterman.