Introduction to Theoretical Particle Physics

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Exercise Sheet 3

Issue: 25.10. – Submission: 31.10. @ 18:00 Uhr – Discussion: 05.11/06.11

Exercise 6: Two-point correlation function — 3 points

In the lecture you have seen how the matrix element $\langle x_f, t_f | q(t_1) | x_i, t_i \rangle$, where $q(t_1)$ is the position operator in the Heisenberg representation, can be written as a path integral with an additional factor $x(t_1)$ in the integrand:

$$\langle x_f, t_f | q(t_1) | x_i, t_i \rangle = \int [\mathcal{D}x(t)] x(t_1) e^{\frac{\hbar}{i} S} \Big|_{x(t_f)=x_f, x(t_i)=x_i} . \tag{6.1}$$

Show that a similar identity holds for the matrix element of two position operators at times t_1 and t_2,

$$\langle x_f, t_f | T q(t_1) q(t_2) | x_i, t_i \rangle = \int [\mathcal{D}x(t)] x(t_1) x(t_2) e^{\frac{i}{\hbar} S} \Big|_{x(t_f)=x_f, x(t_i)=x_i} , \tag{6.2}$$

Where T is the time-ordering operator,

$$T q(t_1) q(t_2) = \theta(t_1 - t_2) q(t_1) q(t_2) + \theta(t_2 - t_1) q(t_2) q(t_1) . \tag{6.3}$$

Explain why the time-ordering operator has to be introduced.

Exercise 7: Harmonic oscillator with external force — 9 points

Consider a free harmonic oscillator with the Lagrangian

$$L_{\text{H.O.}}(\dot{q}, q) = \frac{1}{2} m \dot{q}(t)^2 - \frac{1}{2} m \omega^2 q(t)^2 , \tag{7.1}$$

where m is the mass parameter and ω is the frequency. The action of an external force $mf(t)$ on the oscillator is modelled by an additional term in the Lagrangian

$$\delta L_{\text{ext}} = m f(t) q(t) . \tag{7.2}$$

In the lectures it was shown that taking $H \to (1 - i\epsilon) H$, where ϵ is infinitesimal and positive, the transition amplitude from the ground state in the remote past ($t = -\infty$) to the ground state in the remote future ($t = +\infty$) is given by

$$\langle 0 | 0 \rangle = \int [\mathcal{D}q] \exp \{ iS \} , \tag{7.3}$$

where $[\mathcal{D}q]$ denotes the usual path integral. In this exercise we study the path integral in the presence of an external force where the full Lagrangian is given by $L_{\text{full}} = (L_{\text{H.O.}} + \delta L_{\text{ext}})$.

(a) What is the Hamiltonian H of the free harmonic oscillator? Show that the limit, $H \to (1 - i\epsilon)H$, is equivalent to the replacements

$$m \to m(1 + i\epsilon)\,, \qquad m\omega^2 \to m\omega^2(1 - i\epsilon)\,. \tag{7.4}$$

Since ϵ is infinitesimally small, keep only linear terms and neglect all terms of $\mathcal{O}(\epsilon^2)$.

(b) Perform the shift of Eq. (7.4) on the H.O. Lagrangian (7.1) and write it in terms of the Fourier transformed variables

$$\tilde{q}(E) = \int_{-\infty}^{+\infty} dt\, e^{iEt} q(t)\,, \qquad q(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, e^{-iEt}\tilde{q}(E)\,. \tag{7.5}$$

Show that the action, $S = \int_{-\infty}^{+\infty} dt\, L_{\text{full}}$, can be written in terms of an integral over one Fourier parameter, E,

$$S = \frac{m}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi}\left[\left(E^2 - \omega^2 + i\epsilon\right)\tilde{q}(E)\tilde{q}(-E) + \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E)\right]. \tag{7.6}$$

Hints:

⋆ Use a separate Fourier parameter for each power of $q(t)$ and perform the integral over time

$$\int_{-\infty}^{+\infty} dt\, e^{i(E+E')t} = (2\pi)\delta(E + E')\,. \tag{7.7}$$

⋆ Since the $i\epsilon$ prescription is considered in the $\epsilon \to 0$ limit, you can rescale $i(E^2 + \omega^2)\epsilon \to i\epsilon$.

(c) Perform a shift of the variable $\tilde{q}$,

$$\tilde{q}(E) \longrightarrow \tilde{x}(E) = \tilde{q}(E) + \frac{\tilde{f}(E)}{E^2 - \omega^2 + i\epsilon}\,. \tag{7.8}$$

Argue that the path integral measure does not change under this transformation, i.e. $[\mathcal{D}q] = [\mathcal{D}x]$. Verify that the action splits into two pieces, one that is quadratic in $\tilde{x}$ and another which is independent of $\tilde{x}$. Show that

$$\langle 0|0\rangle_f = \exp\left\{-\frac{i}{2}m\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{E^2 - \omega^2 + i\epsilon}\right\}\cdot \langle 0|0\rangle_{f=0}\,. \tag{7.9}$$

(d) The second factor on the right-hand-side of Eq. (7.9) is the transition amplitude in the absence of an external force and hence $\langle 0|0\rangle_{f=0} = 1$. Perform an inverse Fourier transform and show that the transition amplitude can be written as

$$\langle 0|0\rangle_f = \exp\left\{\frac{i}{2}m\int_{-\infty}^{+\infty} dt\, dt'\, f(t)G(t - t')f(t')\right\}\,, \tag{7.10}$$

where

$$G(t - t') = -\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{e^{iE(t-t')}}{E^2 - \omega^2 + i\epsilon}. \tag{7.11}$$

(e) Verify that the function $G(t-t')$ of Eq. (7.11) is a Green's function for the harmonic oscillator, i.e. check that after taking $\epsilon \to 0$ limit the following equation holds

$$\left(\partial_t^2 + \omega^2\right) G(t-t') = \delta(t-t') \,. \tag{7.12}$$

(f) Using the residue theorem, show that the Green's function of Eq. (7.11) can be written explicitly as

$$G(t-t') = \frac{i}{2\omega} e^{-i\omega|t-t'|} \,. \tag{7.13}$$

Hint: See the discussion in Sec. 2.4 of Peskin & Schroeder.