Introduction to Theoretical Particle Physics

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Exercise Sheet 2

Issue: 18.10. – Submission: 25.10. @ 12:00 Uhr – Discussion: 29.10 / 30.10

Exercise 3: Path integral in quantum mechanics — 4 points

We saw in the lecture that the transition amplitude $U(x_f, x_i; t_f, t_i) = \langle x_f | \mathrm{e}^{-\frac{i}{\hbar}H(t_f - t_i)} | x_i \rangle$ can be expressed using the path integral as

$$U(x_f, x_i; t_f, t_i) = \int [\mathcal{D}x(t)] \exp\left(\frac{i}{\hbar} \int_{t_i}^{t_f} \mathrm{d}\tau L[x, \dot{x}] \right), \tag{3.1}$$

where $L[x, \dot{x}]$ is the Lagrange function, which is a functional of the trajectory $x(t)$ and its derivative, and $\int [\mathcal{D}x(t)]$ is the path integral which integrates over all possible trajectories. It can be explicitly constructed by discretising the integration over time into $n+1$ steps of length $\delta t = \frac{t_f - t_i}{n+1}$, integrating over the values x_k of the trajectory at each time step and then taking the continuum limit $n \to \infty$,

$$U(x_f, x_i; t_f, t_i) = \lim_{n\to\infty} \left(\frac{m}{2\pi i \hbar \delta t} \right)^{\frac{n+1}{2}} \left(\prod_{k=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}x_k \right) \exp\left(\frac{i}{\hbar} \sum_{j=1}^{n+1} L(x_j, \tfrac{x_j - x_{j-1}}{\delta t}) \delta t \right). \tag{3.2}$$

Given the Lagrange function of a free, non-relativistic particle

$$L = \frac{m}{2} \dot{x}^2, \tag{3.3}$$

calculate the transition amplitude $U(x_f, x_i; t_f, t_i)$ explicitly via the path integral, i.e. by starting with Eq. (3.2), performing the integrals over all generalised coordinates and finally taking the continuum limit.
Hint: Rewrite each integral over $\mathrm{d}x_j$ as a Gaussian integral and use the result recursively.

Exercise 4: Coupled harmonic oscillators — 8 points

Quantum Field Theory is essentially Quantum Mechanics with infinitely many degrees of freedom. In this exercise we investigate a quantum mechanical system with N degrees of freedom and at the end take the $N \to \infty$ limit.
Consider a chain of N coupled quantum mechanical harmonic oscillators with mass m and frequency Ω_0. The distance between the equilibrium position of one oscillator

to the next one is a. The deviation of the n-th oscillator from its equilibrium position is denoted as q_n, such that its position with respect to the equilibrium position of the zeroth oscillator is given by $x_n = a_n + q_n$, with $a_n = a \cdot n$. The coupling between two neighbouring oscillators is given by a harmonic potential as well with frequency Ω, such that the Hamiltonian of the system is given by

$$H = \sum_{n=1}^{N} \frac{p_n^2}{2m} + \frac{m\Omega^2}{2}(q_n - q_{n-1})^2 + \frac{m\Omega_0^2}{2} q_n^2 \ , \tag{4.1}$$

where we used natural units, $\hbar = 1$. The chain has periodic boundary conditions such that $q_0 = q_N$.

(a) The canonical commutation relations are given by $[x_n, p_m] = i\delta_{nm}$. What are the commutation relations

$$[q_n, p_m] \quad , \quad [q_n, q_m] \quad , \quad [p_n, p_m] \quad ? \tag{4.2}$$

(b) Determine from the Hamiltonian the equations of motion in the Heisenberg picture. Show that they can be combined into a second order differential equation for $q_n(t)$,

$$\ddot{q}_n(t) = \Omega^2 \left(q_{n+1}(t) + q_{n-1}(t) - 2q_n(t)\right) - \Omega_0^2 q_n(t) \ . \tag{4.3}$$

(c) In order to diagonalise the Hamiltonian it is convenient to decompose the motion into individual Fourier modes:

$$\begin{aligned} q_n &= \frac{1}{\sqrt{mN}} \sum_j e^{ik_j a_n} Q_j \quad \Leftrightarrow \quad Q_j = \sqrt{\frac{m}{N}} \sum_n e^{-ik_j a_n} q_n \\ p_n &= \sqrt{\frac{m}{N}} \sum_j e^{-ik_j a_n} P_j \quad \Leftrightarrow \quad P_j = \frac{1}{\sqrt{mN}} \sum_n e^{ik_j a_n} p_n . \end{aligned} \tag{4.4}$$

Here $k_j = \frac{2\pi j}{Na}$ and j takes integer values $-\frac{N}{2} < j \leq \frac{N}{2}$ for even N respectively $-\frac{N-1}{2} \leq j \leq \frac{N-1}{2}$ for odd N due to the periodic boundary conditions. The Fourier coefficients satisfy orthogonality and completeness relations:

$$\frac{1}{N} \sum_n e^{ik_j a_n} e^{-ik_l a_n} = \delta_{jl} \quad , \quad \frac{1}{N} \sum_j e^{ik_j a_n} e^{-ik_j a_m} = \delta_{nm} \tag{4.5}$$

Show that in terms of the new coordinates Q_n and P_n the Hamiltonian becomes

$$H = \frac{1}{2} \sum_j \left(P_j P_j^\dagger + \omega_j^2 Q_j Q_j^\dagger \right) , \tag{4.6}$$

where

$$\omega_j^2 = \Omega^2 \left(2 \sin\left(\frac{k_j a}{2}\right)\right)^2 + \Omega_0^2 . \tag{4.7}$$

Use the fact that due to the hermeticity of q_n and p_n, one has $Q_j^\dagger = Q_{-j}$ and $P_j^\dagger = P_{-j}$.

(d) In the Hamiltonian of the previous subquestion, modes with positive and negative j are still coupled. In order to deal with this one introduces the operators

$$\begin{aligned} a_j &= \tfrac{1}{\sqrt{2\omega_j}}\left(\omega_j Q_j + iP_j^\dagger\right) \\ a_j^\dagger &= \tfrac{1}{\sqrt{2\omega_j}}\left(\omega_j Q_j^\dagger - iP_j\right) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} Q_j &= \tfrac{1}{\sqrt{2\omega_j}}(a_j + a_{-j}^\dagger) \\ P_j &= -i\sqrt{\tfrac{\omega_j}{2}}(a_{-j} - a_j^\dagger). \end{aligned} \tag{4.8}$$

Calculate the commutators

$$[a_j, a_l] \quad , \quad [a_j^\dagger, a_l^\dagger] \quad , \quad [a_j, a_l^\dagger] \tag{4.9}$$

and find the Hamiltonian in terms of those new operators.

(e) Consider now the limit $a \to 0$, $N \to \infty$, while the length $L = aN$, density $\rho = \frac{m}{a}$ and tension $v^2 = (\Omega a)^2$ stay constant. This limit describes for instance an oscillating string. Let

$$q(x) = q_n\sqrt{\frac{m}{a}} \quad , \quad p(x) = p_n\sqrt{\frac{1}{ma}} \, , \tag{4.10}$$

where $x = a_n$. Rewrite the equation of motion from subquestion b) in this limit. Replace further

$$v \to c \quad , \quad \frac{\Omega_0^2}{c^2} \to m^2 \tag{4.11}$$

in the equation. What equation have you recovered?