Introduction to Theoretical Particle Physics

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Exercise Sheet 1

Issue: 14.10 – Submission: 18.10 @ 12:00 Uhr – Discussion: 22.10 and 23.10

Exercise 1: Gaussian integrals

5 points

An important integral which appears in path integrals is the gaussian integral,

$$G = \int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{1}{2}x^2}. \tag{1.1}$$

In this exercise we will evaluate this integral as well as some of its generalisations.

(a) The gaussian integral 1.1 can be evaluated easily using a trick. Write the square of the integral

$$G^2 = \left(\int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{1}{2}x^2}\right)\left(\int_{-\infty}^{\infty} \mathrm{d}y e^{-\frac{1}{2}y^2}\right), \tag{1.2}$$

and perform the change from cartesian to polar coordinates. You can now evaluate G^2 and show that $G = \sqrt{2\pi}$.

(b) Calculate the integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{1}{2}ax^2}, \tag{1.3}$$

where $a > 0$ is a constant, and

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{1}{2}ax^2 + Jx}, \tag{1.4}$$

with J also a constant. For the latter, it is useful to complete the square in the integrand.

(c) Show that

$$G_N = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_N e^{-\frac{1}{2}\vec{x}^{\mathrm{T}}A\vec{x} + \vec{J}\cdot\vec{x}} = \left(\frac{(2\pi)^N}{\det(A)}\right)^{\frac{1}{2}} e^{\frac{1}{2}\vec{J}^{\mathrm{T}}A^{-1}\vec{J}}, \tag{1.5}$$

where A is a real symmetric $N \times N$ matrix and $\vec{x} = \{x_1, \dots, x_N\}$ and $\vec{J}$ are vectors in N-dimensional space. In order to show this relation, diagonalise A by an orthogonal transformation

$$A = O^{-1}DO, \tag{1.6}$$

where D is a diagonal matrix.

Exercise 2: Lorentz transformations $\boxed{\textbf{7 points}}$

The coordinate change $x \to x'$ can be written as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \tag{2.1}$$

where $\Lambda^{\mu}{}_{\nu}$ is a matrix describing a Lorentz transformation. We use the Lorentz-invariant metric tensor $g_{\mu\nu}$ to raise, $x^{\mu} = g^{\mu\nu} x_{\nu}$, and lower, $x_{\mu} = g_{\mu\nu} x^{\nu}$, indices.

(a) Requiring the invariance of the space-time interval

$$x^2 = x^{\mu} x_{\mu} = g_{\mu\nu} x^{\mu} x^{\nu}, \tag{2.2}$$

show that the Lorentz transformation (2.1) satisfies

$$g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}. \tag{2.3}$$

(b) Using (2.3) show that $|\det(\Lambda^{\mu}{}_{\nu})| = 1$ and further that the volume element $\mathrm{d}^4 x$ is invariant under Lorentz transformations, i.e. show that $\mathrm{d}^4 x' = \mathrm{d}^4 x$.

(c) A Lorentz transformation for momentum vectors can also be written as $k'^{\mu} = \Lambda^{\mu}{}_{\nu} k^{\nu}$. Argue that the integration measure

$$\mathrm{d}^4 k \, \delta\left(k^2 - m^2\right) \Theta(k^0), \tag{2.4}$$

is also Lorentz invariant (under orthochronous transformations). m is the mass of the particle carrying momentum k.

(d) Perform the integration over k^0 in (2.4).

(e) Given the matrix for a rotation in the xy-plane

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.5}$$

write down the matrices for rotations in the xz- and yz-planes.

(f) A boost in the z-direction has the transformation matrix

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}, \tag{2.6}$$

where $\eta \in \mathbb{R}$ is the rapidity. Find a Lorentz transformation that takes a particle with mass m and momentum $p^{\mu} = (E, p_x, 0, p_z)$ to its rest frame. Check your result by calculating the energy component of the four-momentum in the rest frame.

Exercise 3: Harmonic oscillator **4 points**

The quantum mechanical harmonic oscillator is an important tool in the description of quantum fields. In this exercise its representation in terms of ladder operators is being reminded.
Consider the following Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2, \tag{3.1}$$

where m is the mass of the oscillator and $\hat{p}$ and $\hat{q}$ are the momentum and position operators satisfying the canonical commutation relation $[\hat{q}, \hat{p}] = i\hbar$.

(a) Introduce the two operators

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{q} + \frac{i}{\sqrt{m\hbar\omega}}\hat{p}\right) \quad , \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{q} - \frac{i}{\sqrt{m\hbar\omega}}\hat{p}\right). \tag{3.2}$$

Show that the new operators satisfy the commutation relation $[a, a^\dagger] = 1$. Show also that the Hamiltonian can be written in terms of the new operators as

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag{3.3}$$

(b) Define now the operator $N = a^\dagger a$, such that the Hamiltonian can be written as $\hat{H} = \hbar\omega\left(N + \frac{1}{2}\right)$. Show the commutation relations

$$[N, a] = -a \quad , \quad [N, a^\dagger] = a^\dagger. \tag{3.4}$$

Show also that the eigenvalues of N are nonnegative.

(c) Let $|\nu\rangle$ be an eigenvector of N with eigenvalue ν, that is, $N|\nu\rangle = \nu|\nu\rangle$. Show that $a|\nu\rangle$ is zero if $\nu = 0$ and an eigenvector of N with eigenvalue $(\nu - 1)$ if $\nu > 0$. Similarly, show that $a^\dagger|\nu\rangle$ is an eigenvector of N with eigenvalue $(\nu + 1)$.

(d) Show that the the statements from the previous subquestions lead to a contradiction if the eigenvalue ν of N is allowed to take noninteger values. Conclude from this that the eigenvalues of N are nonnegative integers. What does this imply for the eigenvalues of the Hamiltonian?