Lecture 9

Higgs mechanism and mass generation

We have seen two interesting aspects of our theories in the previous lectures:

- the gauge principle forces gauge bosons to be massless. Massless gauge bosons induce long-range interactions (similar to photons). Except for electromagnetism (and gravity, but this is another story), there are no other (known) long-range interactions in Nature. So, if Nature is to be described by gauge interactions, we need a mechanism to give gauge boson masses in a way that does not break gauge invariance;
- spontaneous symmetry breaking leads to the appearance of massless scalar particles – Goldstone bosons. Again, we do not really need massless scalar particles; we do not have many of them in Nature.

The Higgs mechanism solves these two problems in a spectacular and unexpected fashion: it cleans the spectrum of massless scalars by giving masses to gauge bosons in a way that is consistent with gauge invariance.

To show how this works, we consider a theory of a complex scalar field and an $U(1)$ gauge field. The Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^*(D^\mu\varphi) - V(\varphi), \tag{1}$$

where $D_\mu = \partial_\mu - igA_\mu$ and

$$V(\varphi) = -\frac{\mu^2}{2}|\varphi|^2 + \frac{\lambda}{4}|\varphi|^4. \tag{2}$$

As we have seen in the previous lectures, the minimum of the potential is reached at

$$|\varphi|^2 = \varphi_{\rm vac}^2 = \frac{\mu^2}{\lambda}. \tag{3}$$

We need to understand which configuration of fields minimizes the total energy of the system. The energy functional reads

$$E[A,\varphi] = \int {\rm d}^3\vec{x}\left[\frac{1}{2}(F_{0i})^2 + \frac{1}{4}F_{ij}^2 + (D_0\varphi)^*(D_0\varphi) + (D_i\varphi)^*(D_i\varphi) + V(\varphi)\right]. \tag{4}$$

Since all terms with derivatives in Eq. (4) are positive definite, we need to minimize each of them. Both, the Lagrangian Eq. (1) and the energy functional Eq. (4) are invariant under gauge transformations

$$A_\mu \to A_\mu + \frac{1}{g}\partial_\mu\theta, \quad \varphi \to e^{i\theta(x)}\varphi. \tag{5}$$

We will have to keep this in mind when discussing field configurations that minimize $E[A, \varphi]$.

As the first step, we note that the energy stored in the gauge fields depends on $F_{0i} \sim E_i$ and $F_{ij} \sim \epsilon_{ijk} H_k$, where $\vec{E}$ and $\vec{H}$ are electric and magnetic fields. These contributions are minimized when fields vanish which is equivalent to $F_{\mu\nu} = 0$. This, in general, implies that the vector potential is a "pure gauge" configuration, $A_\mu = \frac{1}{g}\partial_\mu \theta_0(x)$, i.e. they are related to $A_\mu = 0$ via a gauge transformation. We then perform the corresponding gauge transformation on the scalar field, going from φ to $e^{i\theta_0(x)}\varphi$, and find

$$D_\mu[e^{i\theta_0(x)}\varphi] = e^{i\theta_0(x)}\, \partial_\mu \varphi. \tag{6}$$

In the context of the energy functional, this implies that it is minimized for constant values of the field φ. For a constant field φ the minimum is reached for φ_{vac}, as we already know. Hence, in the vacuum, we can always choose

$$\varphi = \varphi_{\text{vac}} = \sqrt{\frac{\mu^2}{\lambda}}, \qquad A_\mu = 0. \tag{7}$$

Our next goal is to consider fluctuations around the vacuum fields. To this end, we write the complex field using "polar coordinates"

$$\varphi(x) = \left(\varphi_{\text{vac}} + \frac{\chi}{\sqrt{2}}\right) e^{i\theta(x)}. \tag{8}$$

Then

$$|\varphi|^2 = \left|\varphi_{\text{vac}} + \frac{\chi}{\sqrt{2}}\right|^2, \tag{9}$$

so that the phase $\theta(x)$ disappears from $V(\varphi)$.

Consider now the term $D_\mu \varphi$. It reads

$$D_\mu \varphi = e^{i\theta(x)} \left[\partial_\mu - ig\left(A_\mu - \frac{\partial_\mu \theta}{g}\right)\right]\left(\varphi_{\text{vac}} + \frac{\chi}{\sqrt{2}}\right) = e^{i\theta(x)}\left[\partial_\mu - igB_\mu\right]\left(\varphi_{\text{vac}} + \frac{\chi}{\sqrt{2}}\right), \tag{10}$$

where we introduced a new gauge field $B_\mu = A_\mu - g^{-1}\partial_\mu\theta$. Since, obviously, $F_{\mu\nu}[A] = F_{\mu\nu}[B]$, the new Lagrangian of the theory reads

$$\begin{aligned} L = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left[(\partial_\mu - igB_\mu)\left(\varphi_{\text{vac}} + \frac{\chi}{\sqrt{2}}\right)\right]^\dagger \left[(\partial^\mu - igB^\mu)\left(\varphi_{\text{vac}} + \frac{\chi}{\sqrt{2}}\right)\right] \\ &- V\left(\left(\varphi_{\text{vac}} + \frac{\chi}{\sqrt{2}}\right)^2\right). \end{aligned} \tag{11}$$

One remarkable thing that can be clearly seen from this Lagrangian is that the phase $\theta(x)$ that used to produce a Goldstone boson has disappeared from the Lagrangian completely. Moreover, the spectrum of particles describes by L is peculiar. First, by expanding the potential $V(\varphi)$ around φ_{vac}, we find that the field χ has the mass μ.

Second, if we focus on the second term on the right hand side in Eq. (11) and expand it, we find

$$\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{2g^2\varphi_{\rm vac}^2}{2}B_\mu B^\mu + g^2 B_\mu B^\mu\left[\sqrt{2}\varphi_{\rm vac}\chi + \frac{\chi^2}{2}\right]. \tag{12}$$

The second term is the mass term for the gauge field B_μ; the mass of this field is

$$m_B^2 = 2g^2\varphi_{\rm vac}^2 = \frac{2g^2\mu^2}{\lambda}. \tag{13}$$

After the symmetry breaking, the theory describes one massive vector boson and one massive real field. As we see, the massless Goldstone boson disappeared from the spectrum and the longitudinal mode of the gauge field appeared. Hence, the total number of degrees of freedom has not changed.

To make this last statement more clear, consider the equation of motion for *free* massive field B^μ. It reads

$$\partial_\mu F^{\mu\nu} + m_B^2 B^\nu = 0. \tag{14}$$

We contract Eq. (14) with ∂_ν and use $F_{\mu\nu} = -F_{\nu\mu}$, to obtain

$$m_B^2\,\partial_\nu B^\nu = 0. \tag{15}$$

Since $m_B^2 \neq 0$, we conclude that

$$\partial_\nu B^\nu = 0. \tag{16}$$

Since we can represent the field B_μ as the sum over Fourier modes, $B_\mu \sim \epsilon^\mu(p)e^{-ip_\mu x^\mu}$, and since $\partial_\nu \to p_\nu$, we find that above equations imply that polarization vectors ϵ are *transverse*

$$p_\mu\epsilon^\mu(p) = 0. \tag{17}$$

For a four-vector $p = (E_p, \vec{p})$ with $E_p^2 - \vec{p}^2 = m_B^2$, there are *three* independent vectors that satisfy Eq. (16). They are

$$\begin{aligned}\epsilon^\mu &= (0,1,0,0)\,,\\ \epsilon^\mu &= (0,0,1,0)\,,\\ \epsilon^\mu &= \frac{1}{m_B}\,(p,0,0,E_p)\,.\end{aligned} \tag{18}$$

where we have assumed that $p^\mu = (E_p, 0, 0, p)$. Note that all polarization vectors are normalized

$$\epsilon_\mu\epsilon^\mu = -1. \tag{19}$$

The above discussion shows that a massive vector boson has *three* polarizations, two transversal and one longitudinal. A *massless* vector boson, e.g. a photon, has only *two* transversal polarizations. The longitudinal polarization that, according to Eq. (18) behaves strangely in $m_B \to 0$ limit, does not appear in case of photons to begin with. This discussion illustrates the statement about the number of degrees of freedom before and after the symmetry breaking that was made right after Eq. (13).

After spontaneous symmetry breaking, the fields in the theory are B_μ and a real scalar field χ. Lets us check how these fields transform under gauge transformations. In these cases,

$$\varphi \to e^{i\tilde{\theta}}\varphi, \qquad A_\mu \to A_\mu + \frac{\partial_\mu \tilde{\theta}}{g}. \tag{20}$$

Since χ is related to the absolute value of φ

$$|\varphi| = \varphi_{\text{vac}} + \frac{\chi}{\sqrt{2}}, \tag{21}$$

the field χ remains unchanged. At the same time since

$$B_\mu = A_\mu - \frac{\partial_\mu \theta}{g}, \tag{22}$$

where θ is the phase of the "original" field φ, if θ and A_μ change together, the field B remains unchanged

$$B_\mu \to A_\mu + \frac{\partial_\mu \tilde{\theta}}{g} - \frac{\partial_\mu(\theta + \tilde{\theta})}{g} = B_\mu. \tag{23}$$

Hence, we conclude that the Lagrangian Eq. (11) is written in terms of physical fields with physical properties.

We will now consider the generalization of this mechanism to the case of the $SU(2)$ non-abelian gauge theory. The Lagrangian reads

$$L = (D_\mu \vec{\varphi})^\dagger (D^\mu \vec{\varphi}) - V(\vec{\varphi}^\dagger \vec{\varphi}) - \frac{1}{2}\text{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right], \tag{24}$$

where $\vec{\varphi}$ is a two-dimensional complex vector that transforms under the fundamental representation of $SU(2)$. Moreover,

$$D_\mu = \partial_\mu - ig\hat{A}_\mu, \quad \hat{A}_\mu = \sum_{a=1}^{3} A^a_\mu \tau^a, \tag{25}$$

where τ^a are the generators of the $SU(2)$ Lie algebra and

$$[\tau^a, \tau^b] = i\epsilon^{abc}\tau^c. \tag{26}$$

The Lagrangian is invariant under $SU(2)$ transformations

$$\vec{\varphi} \to U(x)\vec{\varphi}, \quad \hat{A}_\mu \to U\hat{A}_\mu U^{-1} + \frac{1}{ig}\left(\partial_\mu U\right) U^{-1}. \tag{27}$$

To study the symmetry breaking, we again choose the potential $V(\varphi)$ to be

$$V(\varphi) = -\frac{\mu^2}{2}\vec{\varphi}^\dagger\vec{\varphi} + \frac{\lambda}{4}\left(\vec{\varphi}^\dagger\vec{\varphi}\right)^2. \tag{28}$$

Similar to the abelian case, the energy of the system is minimized for $\hat{A}_\mu = 0$ and the vacuum field $\vec{\varphi} = \vec{\varphi}_{\rm vac}$ where

$$\vec{\varphi}^\dagger_{\rm vac} \vec{\varphi}_{\rm vac} = \frac{\mu^2}{\lambda}. \tag{29}$$

When written in components of the complex vector φ, the above equation reads

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix} \quad \Rightarrow \quad \left[\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2\right]_{\rm vac} = \frac{\mu^2}{\lambda}, \tag{30}$$

so there are infinitely many possibilities to choose the vacuum field. We will choose it as follows

$$\vec{\varphi}_{\rm vac} = \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{31}$$

and we will write the field $\varphi(x)$ in the following way

$$\vec{\varphi}(x) = U(x) \vec{\varphi}_R, \tag{32}$$

where

$$\varphi_R = \begin{pmatrix} 0 \\ v + \frac{\chi}{\sqrt{2}} \end{pmatrix}, \tag{33}$$

and $U(x) \in SU(2)$. It is important to emphasize that the representation of the field φ as in Eq. (32) is *exact*. To see this, write

$$U(x) = \cos \frac{\theta(x)}{2} + i \sin \frac{\theta(x)}{2}\, \vec{n}(x) \cdot \vec{\sigma}, \tag{34}$$

where $\vec{n}(x)$ is an x-dependent unit vector and check that the number of parameters $\theta(x), \vec{n}(x)$ is sufficient to express $\varphi_{1,2}, \varphi_{3,4}$ through them and the field $\chi(x)$. Indeed, if we write

$$\varphi = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}, \tag{35}$$

then

$$\varphi_1 = n_2\, r \sin \frac{\theta}{2}, \quad \varphi_2 = n_1\, r \sin \frac{\theta}{2}, \quad \varphi_3 = r \cos \frac{\theta}{2}, \quad \varphi_4 = -n_3\, r \sin \frac{\theta}{2}, \tag{36}$$

where $r = v + \chi/\sqrt{2}$. It follows that

$$r^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2, \quad \mathrm{ctg} \frac{\theta}{2} = \frac{\varphi_3}{\sqrt{\varphi_1^2 + \varphi_2^2 + \varphi_4^2}} \tag{37}$$

Once we know r and θ from the above equations, we reconstruct $n_{1,2,3}$ from Eq. (36).

Having proved the existence of the representation Eq. (32), it is easy to see that it allows us to express the Lagrangian of the theory in terms of physical degrees of freedom. Indeed, using Eq. (32), we find

$$\begin{aligned} V(\vec{\varphi}) &= -\frac{\mu^2}{2} \left(v + \frac{\chi}{\sqrt{2}} \right)^2 + \frac{\lambda}{4} \left(v + \frac{\chi}{\sqrt{2}} \right)^4, \\ D_\mu \vec{\varphi} &= U \left(\partial_\mu - ig \left[U^{-1} \hat{A}_\mu U - \frac{1}{ig} U^{-1} \partial_\mu U \right] \right) \vec{\varphi}_R. \end{aligned} \tag{38}$$

We now define a new gauge field

$$\hat{B}_\mu = U^{-1}\hat{A}_\mu U - \frac{1}{ig}U^{-1}\partial_\mu U, \tag{39}$$

so that the covariant derivative in Eq. (38) only depends on the field $\hat{B}_\mu$. Since Eq. (39) is a canonical transformation from $\hat{A}_\mu$ to $\hat{B}_\mu$, the field-strength tensor does not change. Hence, the Lagrangian reads

$$\begin{aligned} L = &-\frac{1}{2}\mathrm{Tr}\left[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right] + \left[\left(\partial_\mu - ig\hat{B}_\mu\right)\vec{\varphi}_R\right]^\dagger \left(\partial^\mu - ig\hat{B}^\mu\right)\vec{\varphi}_R \\ &+ \frac{\mu^2}{2}\left(v + \frac{\chi}{\sqrt{2}}\right)^2 - \frac{\lambda}{4}\left(v + \frac{\chi}{\sqrt{2}}\right)^4 . \end{aligned} \tag{40}$$

Similar to the abelian case, the mass spectrum of gauge bosons follows from the kinetic term of the field φ upon replacing φ_R with φ_{vac}. We obtain

$$\begin{aligned} g^2(0 \quad v)\hat{B}_\mu\hat{B}^\mu \begin{pmatrix} 0 \\ v \end{pmatrix} &= g^2v^2 B^a_\mu B^{b,\mu}(0 \quad 1)\tau^a\tau^b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ = \frac{g^2v^2}{4}B^a_\mu B^{b,\mu}(0 \quad 1)\left(\delta_{ab} + i\epsilon_{abc}\sigma_c\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \frac{g^2v^2}{4}B^a_\mu B^{b,\mu}. \end{aligned} \tag{41}$$

Hence, our theory contains three massive gauge bosons with identical masses

$$m^2 = \frac{g^2v^2}{2}. \tag{42}$$

Again, it is instructive to count degrees of freedom. Before the symmetry breaking, our theory was describing three massless gauge fields, each having two polarizations, and a complex doublet which is described by four scalar fields; altogether $3 \times 2 + 4 = 10$ degrees of freedom. After the symmetry breaking we have three massive gauge fields, each with three polarizations and one real (χ) field; altogether $3 \times 3 + 1 = 10$ degrees of freedom.