Lecture 8

Spontaneous symmetry breaking, Goldstone effect

Consider a theory of a single scalar field

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi\right)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4}\varphi^4. \tag{1}$$

As we know, the parameter m is the mass of particle-like excitations of the field φ and λ is the self-coupling. The equation of motion reads

$$\left(\partial_\mu \partial^\mu + m^2\right)\varphi = -\lambda \varphi^3. \tag{2}$$

For small values of λ, we can neglect the right hand side in Eq. (2) and describe excitations of the field φ as plane waves

$$\varphi \sim e^{-i\omega_k t + i\vec{k}\vec{x}}, \quad \text{with} \quad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}. \tag{3}$$

This discussion corresponds to particles that propagate around and, if we put back the r.h.s. in Eq. (2) back into action, interact with each other. It assumes, of course, that $m^2 > 0$. The minimal energy that can be stored in the field in this case can be found from the Hamiltonian

$$E = \int \mathrm{d}^3\vec{x} \left[\frac{1}{2}\left(\partial_t \varphi\right)^2 + \frac{1}{2}\left(\vec{\nabla}\varphi\right)^2 + \frac{m^2\varphi^2}{2} + \frac{\lambda}{4}\varphi^4\right]. \tag{4}$$

The minimum value of E corresponds to $\varphi = 0$.

What happens if we change the sign of m^2, i.e. we take

$$m^2 = -\mu^2, \tag{5}$$

with $\mu^2 > 0$? If we do that, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi\right)^2 - V(\varphi), \tag{6}$$

with

$$V(\varphi) = -\frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4. \tag{7}$$

The minimal energy in this case corresponds to the minimum of

$$E = \int \mathrm{d}^3\vec{x} \left[\frac{1}{2}\left(\partial_t \varphi\right)^2 + \frac{1}{2}\left(\vec{\nabla}\varphi\right)^2 - \frac{\mu^2\varphi^2}{2} + \frac{\lambda}{4}\varphi^4\right]. \tag{8}$$

In this case, the time- and space-independent field φ still minimizes the energy but the value of φ is different from zero. In fact, it corresponds to the minimum of the potential $V(\varphi)$. We find it by computing

$$\frac{\partial V(\varphi)}{\partial \varphi} = 0 \quad \rightarrow \quad \varphi_{\text{min}} = \pm \varphi_{\text{vac}}, \quad \varphi_{\text{vac}} = \sqrt{\frac{\mu^2}{\lambda}}. \tag{9}$$

The energy of the vacuum is then

$$E_{\mathrm{vac}} = \Omega \left[-\frac{\mu^2}{2} \frac{\mu^2}{\lambda} + \frac{\lambda}{4} \frac{\mu^4}{\lambda^2} \right] = -\frac{\Omega \mu^4}{4\lambda}, \tag{10}$$

where $\Omega = \int \mathrm{d}^3\vec{x}$ is the space volume.

The important point is that if we want to describe small excitations of the field φ, we cannot construct such an expansion around $\varphi = 0$. This is because, even for small λ, the equations of motion of the field φ around $\varphi = 0$ is

$$\left(\partial_\mu \partial^\mu - \mu^2 \right) \varphi = 0. \tag{11}$$

The solutions to this equation are $\varphi \sim e^{\pm \mu t}$, so that there is an exponentially growing field that "moves away" from $\varphi = 0$ rather than it oscillates around this value. To have "small oscillations", we need to consider values of the field that are close to $\varphi = \pm \varphi_{\mathrm{vac}}$.

An important question is which of the two minima should be considered. If this were quantum mechanics, the answer to this question is "neither" since the ground state of a quantum-mechanical system with two minima is a symmetric wave function with maxima both at the left and at the right minimum. The reason is the tunneling through a potential barrier; it connects the two minima and forces us to choose a symmetric wave function as the true ground state.

It is very important to understand that in quantum field theory we can choose one of the two ground states and we do not need to care about tunneling. To see why this is so, let us map the quantum field theory problem on a quantum-mechanical problem by considering fields that are $\vec{x}$-independent. Then, the action reads

$$S = \int \mathrm{d}t \, \left[\frac{\Omega}{2} \left(\partial_t \varphi \right)^2 - \Omega V(\varphi) \right], \tag{12}$$

where $\Omega = \int \mathrm{d}^3\vec{x}$ is the space volume on which the field φ has a non-vanishing support. If we identify $\varphi(t)$ with $x(t)$, we can view Eq. (12) as the action of a particle with the mass Ω and and the potential energy $\Omega V(\varphi)$.

We can now compute the tunneling amplitude from one vacuum to the other vacuum using the quantum mechanical formulas

$$\mathcal{A}_{\mathrm{tunnel}} \sim e^{-\int p \mathrm{d}x}, \tag{13}$$

where $p \sim \sqrt{m|U|} \to \Omega \sqrt{|V(\varphi)|}$ and $\mathrm{d}x \to \mathrm{d}\varphi$. Hence, in the quantum field theory, the tunneling amplitude reads

$$\mathcal{A}_{\mathrm{tunnel}} \sim e^{-\Omega \int\limits_{-\varphi_{\mathrm{vac}}}^{\varphi_{\mathrm{vac}}} \sqrt{|V(\varphi)|} \, \mathrm{d}\varphi}. \tag{14}$$

Therefore, if we consider quantum field theory in an infinitely large volume $\Omega \to \infty$ the tunneling amplitude *vanishes*. For this reason, at variance with quantum mechanics, we *must* choose *one* vacuum in quantum field theory; which one it is – the "left" $\varphi = -\varphi_{\mathrm{vac}}$

or the "right" $\varphi = +\varphi_{\rm vac}$, we cannot predict. For this reason, the phenomenon of a system choosing the (left or right) vacuum is called *spontaneous symmetry breaking.*

Let us imagine that the system has chosen the "right" vacuum where

$$\langle 0|\varphi|0\rangle = \varphi_{\rm vac}. \tag{15}$$

We then re-write the Lagrangian Eq. (6) using a new field χ that is defined as $\varphi(x) = \varphi_{\rm vac} + \chi(x)$. Since $\partial_\mu \varphi_{\rm vac} = 0$, we obtain the new Lagrangian

$$L = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - V(\varphi_{\rm vac}) + \frac{1}{2}\left(\mu^2 - 3\lambda\varphi_{\rm vac}^2\right)\chi^2 - \lambda\varphi_{\rm vac}\chi^3 - \frac{\lambda}{4}\chi^4. \tag{16}$$

We use the explicit expression for $\varphi_{\rm vac}$ to simplify Eq. (16) and find

$$L = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - \frac{1}{2}m_\chi^2 \chi^2 - \lambda\varphi_{\rm vac}\chi^3 - \frac{\lambda}{4}\chi^4 - V(\varphi_{\rm vac}), \tag{17}$$

where $m_\chi^2 = 2\mu^2$. Note that Eq. (17) describes a theory of a scalar self-interacting field with the mass m_χ^2. In contrast to the original theory, there is nothing strange about the theory described by Eq. (17) anymore. In particular, the mass of the field χ is *positive.*

As the next step, we extend the original theory by considering a larger number of fields that appear in the Lagrangian in a symmetric way. We consider two real fields φ_1, φ_2 and write them as a two-component vector

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{18}$$

The Lagrangian reads

$$L = \frac{1}{2}\partial_\mu \vec{\varphi} \cdot \partial^\mu \vec{\varphi} - V(\vec{\varphi}\cdot\vec{\varphi}), \tag{19}$$

where

$$V(\vec{\varphi}\cdot\vec{\varphi}) = -\frac{\mu^2}{2}\vec{\varphi}\cdot\vec{\varphi} + \frac{\lambda}{4}\left(\vec{\varphi}\cdot\vec{\varphi}\right)^2. \tag{20}$$

The Lagrangian has an $\mathcal{O}(2)$ symmetry; if we rotate $\vec{\varphi}$ with a 2×2 orthogonal matrix

$$\vec{\varphi} = \hat{R}\vec{\varphi}', \quad \hat{R}^T R = 1, \tag{21}$$

we find

$$L(\vec{\varphi}) = L(\vec{\varphi}'). \tag{22}$$

Since the potential energy $V(\varphi)$ depends on the "length" of the vector $\vec{\varphi}$, we can read off the value of the field that minimizes $V(\vec{\varphi}\cdot\vec{\varphi})$ from the calculation at the beginning of this lecture. We find

$$\vec{\varphi}_{\rm vac}\cdot\vec{\varphi}_{\rm vac} = \varphi_{1,{\rm vac}}^2 + \varphi_{2,{\rm vac}}^2 = \frac{\mu^2}{\lambda}. \tag{23}$$

It follows from Eq. (23) that the "vacuum manifold" is a circle with the radius $r_\varphi = \sqrt{\mu^2/\lambda}$. In contrast to the single-field case, we parameterize the vacuum in that we write

$$\vec{\varphi}_{\rm vac} = \varphi_{\rm vac}\vec{e}_{\rm vac}, \tag{24}$$

where

$$\vec{e}_{\rm vac} = \left(\begin{array}{c} \cos\theta \\ \sin\theta \end{array} \right). \tag{25}$$

Eqs. (24,25) describe a particular choice of the vacuum. To construct an expansion around the vacuum field, we write

$$\vec{\varphi} = \vec{\varphi}_{\rm vac} + \vec{\chi}. \tag{26}$$

Since $\vec{\varphi}_{\rm vac}$ is a constant field, it follows

$$\partial_\mu \vec{\varphi} = \partial_\mu \vec{\chi}. \tag{27}$$

We would like to express the Lagrangian Eq. (19) through the field $\vec{\chi}$. To do that, we note that it is convenient to write $\vec{\varphi}$ as a sum of two vectors

$$\vec{\chi} = h\vec{e}_{\rm vac} + \chi_\perp \vec{e}_\perp, \tag{28}$$

where $\vec{e}_{\rm vac} \cdot \vec{e}_\perp = 0$. Then

$$V(\vec{\varphi}\cdot\vec{\varphi}) = V((\varphi_{\rm vac} + h)^2 + \chi_\perp^2). \tag{29}$$

Using the explicit form of the potential Eq. (20), we find

$$V(\vec{\varphi}\cdot\vec{\varphi}) = -\frac{\mu^2}{2}\left[(\varphi_{\rm vac} + h)^2 + \chi_\perp^2\right] + \frac{\lambda}{4}\left((\varphi_{\rm vac} + h)^2 + \chi_\perp^2\right)^2. \tag{30}$$

It is easy to analyze this potential energy to arrive at the following conclusions:

- there are two fields h and $\chi_\perp$ in the Lagrangian after the symmetry breaking;
- the mass of the field h is $2\mu^2$, similar to the single-field case;
- the mass of the field $\chi_\perp$ is zero;
- there are interactions between h and $\chi_\perp$.
- nothing depends on the chosen vacuum state that is characterized by the vector $\vec{e}_{\rm vac}$. The dependence on that vector disappeared completely.

Particle excitations of the massless field $\chi_\perp$ that appeared in the theory after the spontaneous symmetry breaking are known as *Nambu-Goldstone bosons.*

We will now do the same calculation using a different field parameterization. This is important since choosing a different parameterization offers a different perspective on the Nambu-Goldstone mechanism. We write the field as

$$\vec{\varphi}(x) = \rho(x) \begin{pmatrix} \cos\alpha(x) \\ \sin\alpha(x) \end{pmatrix}, \tag{31}$$

which means that we have chosen "spherical" coordinates in field space. The potential energy is then

$$V(\vec{\varphi}\cdot\vec{\varphi}) = V(\rho^2). \tag{32}$$

However, to compute the kinetic energy stored in the field φ we calculate the derivative

$$\partial_\mu \vec{\varphi} = (\partial_\mu \rho) \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + \rho(\partial_\mu \alpha) \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \tag{33}$$

and find

$$\frac{1}{2}\partial_\mu \vec{\varphi}\cdot\partial^\mu \vec{\varphi} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{\rho^2}{2}\partial_\mu \alpha \partial^\mu \alpha. \tag{34}$$

Again, to account for the spontaneous symmetry breaking, we write

$$\rho = \varphi_{\mathrm{vac}} + r. \tag{35}$$

The Lagrangian becomes

$$L = \frac{1}{2}\partial_\mu r \partial^\mu r + \frac{\varphi_{\mathrm{vac}}^2}{2}\partial_\mu \alpha \partial^\mu \alpha - \frac{(2\mu^2)}{2} r^2 + \dots, \tag{36}$$

where the ellipsis describes the interactions between different fields. We observe from Eq. (36) that the "angular" variable α describes a massless field (whose excitations are Goldstone bosons) and the "radial variable" r describes a massive field with the mass $2\mu^2$. We note that all the terms in the Lagrangian that involve the field α are proportional to $\partial_\mu \alpha$. Since $\partial_\mu \alpha \sim \sum p_\mu \alpha$, this feature implies that interactions of Goldstone bosons become weak at small energies.

We generalize the construction to three fields and a Lagrangian that is symmetric under $SO(3)$ transformations. We again use Eq. (19) but this time the field $\vec{\varphi}$ is a triplet

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}. \tag{37}$$

Similar to the discussion of the two-component vector, we write

$$\vec{\varphi} = (\varphi_{\mathrm{vac}} + h)\vec{e}_{\mathrm{vac}} + \vec{\chi}_\perp, \tag{38}$$

where

$$\vec{e}_{\mathrm{vac}} = \begin{pmatrix} \sin\theta_0 \cos\varphi_0 \\ \sin\theta_0 \sin\varphi_0 \\ \cos\theta_0 \end{pmatrix}, \tag{39}$$

and $\vec{\chi}_\perp \cdot \vec{e}_{\rm vac} = 0$, so that $\vec{\chi}_\perp$ is a two-component field. If we use this representation in the formula for the potential energy, we obtain

$$V(\vec{\varphi}\cdot\vec{\varphi}) = -\frac{\mu^2}{2}\left[(\varphi_{\rm vac}+h)^2+\vec{\chi}_\perp^2\right]+\frac{\lambda}{4}\left((\varphi_{\rm vac}+h)^2+\vec{\chi}_\perp^2\right)^2. \tag{40}$$

As we already remarked after Eq. (30), this form of the potential energy implies that $\vec{\chi}_\perp$ describes two massless fields (Goldstone bosons), whereas h is a massive field with the mass squared being equal to $2\mu^2$. It should be now obvious that if we consider a theory of N fields that is invariant under $SO(N)$ transformations, we will get $N-1$ Goldstone bosons after the symmetry breaking.

To understand how many Goldstone bosons arise in the theory after the symmetry breaking, consider the field $\vec{\varphi}$ that describes N fields. After the symmetry breaking, we write it as

$$\vec{\varphi} = (\varphi_{\rm vac}+h)\vec{e}_{\rm vac}+\vec{\chi}_\perp, \tag{41}$$

where the field $\vec{\chi}_\perp$ describes $N-1$ fields that span the $(N-1)$-dimensional space $D_{\rm vac}$ that is orthogonal to $\vec{e}_{\rm vac}$. Since the potential energy only depends on $\vec{\chi}_\perp^2$, the theory is still invariant under $(N-1)$-rotations in $D_{\rm vac}$. We then say that the symmetry is broken from $SO(N)$ to $SO(N-1)$. We now note that the group $SO(N)$ has $G_N = N(N-1)/2$ "symmetry generators", i.e. "independent rotations". After the symmetry breaking, the symmetry group becomes $SO(N-1)$, so some of the original symmetry transformations are not symmetry transformations anymore. The number of such "broken" symmetry transformations reads

$$G_N - G_{N-1} = N-1. \tag{42}$$

This is exactly the number of massless particles that we have been finding in our examples.

We will now explain why this is not a coincidence and that, indeed, the number of Goldstone bosons equals the number of broken symmetries *in any theory.* To this end, consider a theory with the interaction potential $V(\vec{\varphi})$. The theory is invariant under a symmetry that is described by generators T^a, $a=1,\dots,N_a$. Hence, if we consider an infinitesimal transformation

$$\vec{\varphi}' = \vec{\varphi}+\epsilon_a T^a\vec{\varphi}, \tag{43}$$

the potential energy computed for $\vec{\varphi}'$ and $\vec{\varphi}$ should be the same

$$V(\vec{\varphi}+\epsilon_a T^a \vec{\varphi}) = V(\vec{\varphi}). \tag{44}$$

Expanding the left hand side to first order in ϵ, we find

$$0 = \epsilon_a \frac{\partial V}{\partial \varphi_i} T^a_{ik}\varphi_k. \tag{45}$$

Since different ϵ_a's parameterize independent symmetry transformations, Eq. (45), in fact, splits into N_a independent equations

$$0 = \frac{\partial V}{\partial \varphi_i} T^a_{ik}\varphi_k, \tag{46}$$

one for every symmetry generator.

We now take the derivative of Eq. (46) with respect to φ_m. We obtain

$$0 = \frac{\partial^2 V}{\partial\varphi_i \partial\varphi_m} T^a_{ik} \varphi_k + \frac{\partial V}{\partial\varphi_i} T^a_{im}. \tag{47}$$

Eq. (47) holds for any $\vec{\varphi}$. However, it is instructive to apply it at $\vec{\varphi} = \vec{\varphi}_{\rm vac}$. Since $\vec{\varphi}_{\rm vac}$ minimizes the potential, the last term in Eq. (47) vanishes and we obtain

$$0 = \frac{\partial V}{\partial\varphi_i \partial\varphi_m}|_{\vec{\varphi}=\vec{\varphi}_{\rm vac}} \; T^a_{ik} \; \varphi_{{\rm vac},k}. \tag{48}$$

To understand the meaning of this equation, consider a generic Lagrange function

$$L = \frac{1}{2}\partial_\mu \vec{\varphi} \partial^\mu \vec{\varphi} - V(\vec{\varphi}), \tag{49}$$

and assume that spontaneous symmetry breaking occurs. We then write $\vec{\varphi} = \vec{\varphi}_{\rm vac} + \vec{\chi}$ and expand around $\vec{\chi} = 0$. We find

$$L = \frac{1}{2}\partial_\mu \vec{\chi} \partial^\mu \vec{\chi} - V(\vec{\varphi}_{\rm vac}) - \frac{\partial V}{\partial\varphi_i}|_{\vec{\varphi}=\vec{\varphi}_{\rm vac}} \; \chi_i - \frac{1}{2}\frac{\partial V}{\partial\varphi_i\partial\varphi_j}|_{\vec{\varphi}=\vec{\varphi}_{\rm vac}} \; \chi_i\chi_j + ... \tag{50}$$

Since the potential $V(\vec{\varphi})$ has a minimum at $\vec{\varphi} = \vec{\varphi}_{\rm vac}$, the right hand side of Eq. (50) simplifies. We write

$$L = \frac{1}{2}\partial_\mu \vec{\chi} \partial^\mu \vec{\chi} - V(\vec{\varphi}_{\rm vac}) - \frac{1}{2} m^2_{ij} \; \chi_i\chi_j + ..., \tag{51}$$

where

$$m^2_{ij} = \frac{\partial V}{\partial\varphi_i\partial\varphi_j}|_{\vec{\varphi}=\vec{\varphi}_{\rm vac}} \tag{52}$$

is the *mass matrix*. The name comes from the fact that, upon diagonalising it, we get the information about masses of particles that our theory describes.

We note that this matrix also appears in Eq. (48) that we write in the following way

$$0 = m_{ij}\xi^{(a)}_j, \tag{53}$$

where $\vec{\xi}^{(a)} = T^a \vec{\varphi}_{\rm vac}$. Clearly, $\vec{\xi}^{(a)}$ is what you get if you act on a vacuum field by a generator of a symmetry transformation $T^{(a)}$.

According to Eq. (53) when the mass matrix multiplies *any* $\vec{\xi}^{(a)}$, the result should be zero, however, this can be realized in two ways. If, for a particular a, $\vec{\xi}^{(a)} = 0$, Eq. (53) does not provide any useful information. However, if $\vec{\xi}^{(a)} \neq 0$, Eq. (53) implies that the mass matrix has a non-trivial eigenvector with *zero eigenvalue*, i.e. zero mass squared. The number of such eigenvectors is equivalent to the number of symmetries (number of generators) that *do not* leave the vacuum $\vec{\varphi}_{\rm vac}$ unchanged, since $T^a\vec{\varphi}_{\rm vac} \neq 0$. Hence, *for each broken symmetry, there is a massless mode that is a Nambu-Goldstone boson.*