Lecture 7

Gauge invariance: the non-abelian case

In the previous lecture we discussed how to ensure that local phase transformations of a scalar complex field are allowed. If we go back from a single complex field to two real scalar fields,

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}, \quad \varphi^\dagger = \frac{\varphi_1 - i\varphi_2}{\sqrt{2}}, \tag{1}$$

the phase transformations become rotations of a vector $\vec{\varphi} = (\varphi_1, \varphi_2)$. Indeed, a transformation $\varphi \to e^{-i\alpha(x)}\varphi$ is equivalent to

$$\vec{\varphi} \to \mathcal{O}(x)\vec{\varphi}, \tag{2}$$

where

$$\mathcal{O}(x) = \begin{pmatrix} \cos\alpha(x) & \sin\alpha(x) \\ -\sin\alpha(x) & \cos\alpha(x) \end{pmatrix}. \tag{3}$$

As you know, requiring the invariance of the Lagrangian under such a transformation leads to the appearance of the gauge field that we associated with the electromagnetic field.

We imagine now that we have a theory that contains *two complex fields* that we combine into a complex *doublet*

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \frac{\varphi_{1R}+i\varphi_{1I}}{\sqrt{2}} \\ \frac{\varphi_{2R}+i\varphi_{2I}}{\sqrt{2}} \end{pmatrix}. \tag{4}$$

We write the Lagrangian that contains this doublet as

$$\mathcal{L} = \partial^\mu \vec{\varphi}^\dagger \partial_\mu \vec{\varphi} - m^2 \vec{\varphi}^\dagger \vec{\varphi} + V(\vec{\varphi}^\dagger \vec{\varphi}). \tag{5}$$

The Lagrangian Eq. (5) is invariant under transformations of the field $\vec{\varphi}$ with 2×2 special unitary matrices. These matrices can be written as

$$U(\theta) = e^{i\theta\vec{n}\cdot\vec{\sigma}/2} = \cos\frac{\theta}{2}\hat{1} + i\sin\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}, \tag{6}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}$$

are the Pauli matrices. It is clear that $\sigma_i^\dagger = \sigma_i$,

$$U^\dagger(\theta) = e^{-i\theta\vec{n}\cdot\vec{\sigma}/2}, \tag{8}$$

and $U^\dagger(\theta)U(\theta) = 1$. It is then obvious that, if θ is independent of x, the Lagrangian Eq. (5) is invariant under the following field transformations

$$\vec{\varphi} \to U(\theta)\vec{\varphi}. \tag{9}$$

Suppose we require that the theory is invariant under local x-dependent transformations

$$\vec{\varphi} \to U(\theta(x))\vec{\varphi}. \tag{10}$$

Similar to the QED case, we replace ordinary derivatives with the covariant ones

$$\partial_\mu \vec{\varphi} \to D_\mu \vec{\varphi}, \quad D_\mu = \partial_\mu - ig\hat{A}_\mu. \tag{11}$$

The field $\hat{A}_\mu$ is a 2×2 matrix, so that an expression $\hat{A}_\mu \vec{\varphi}$ implies that a two-component complex field is multiplied with this matrix. To understand how the covariant derivative transforms, we write

$$\begin{aligned} D_\mu \vec{\varphi} &\to \left[\partial_\mu - ig\hat{A}_\mu\right] U\vec{\varphi} = \left[(\partial_\mu U)\vec{\varphi} + U\partial_\mu \vec{\varphi} - ig\hat{A}_\mu U\vec{\varphi}\right] \\ &= U\left[\partial_\mu - igU^{-1}\hat{A}_\mu U + U^{-1}\partial_\mu U\right]\vec{\varphi}. \end{aligned} \tag{12}$$

Although the formula for the covariant derivative is modified, similar to the Abelian case, we can absorb the modification by redefining the field

$$\hat{A}_\mu \to U\hat{A}_\mu U^{-1} + \frac{1}{ig}\left(\partial_\mu U\right)U^{-1}. \tag{13}$$

Then we find

$$D_\mu \vec{\varphi} \to UD_\mu \vec{\varphi}, \tag{14}$$

so that the covariant derivative transforms in the same way as the field. Note that you have to be very careful about keeping the correct order of the various matrices because arbitrary 2×2 matrices do not commute with each other.

We therefore arrive at the following result. The Lagrangian, Eq. (5) is invariant under simultaneous transformations of $\vec{\varphi}$ and $\hat{A}$ shown in Eqs. (10, 13).

We would like to argue that the gauge field $\hat{A}_\mu$ belongs to the *algebra* of the respective gauge group. Concretely, for the $SU(2)$ group, the gauge field can always be written as

$$\hat{A}_\mu = \sum_{a=1}^{3} A_\mu^a \tau^a, \quad \tau^a = \frac{\sigma^a}{2}, \tag{15}$$

where A_μ^a are three real fields.

To show that this is sufficient, we first consider infinitesimal gauge transformations

$$U \approx \hat{1} + i\epsilon^a \tau^a. \tag{16}$$

The field A_μ transforms as

$$\hat{A}_\mu \to \hat{A}_\mu + i\epsilon^a\left[\tau^a, \hat{A}_\mu\right] + \frac{1}{g}(\partial_\mu \epsilon^a)\tau^a. \tag{17}$$

Hence, if the original field $\hat{A}_\mu$ belongs to the Lie algebra, the field obtained as the result of infinitesimal transformations also belongs to the Lie algebra.

We can extend this computation to cover a generic $SU(2)$ transformation. The key property of Eq. (15) is that $\hat{A}_\mu$ is traceless, i.e.

$$\mathrm{Tr}[\hat{A}_\mu] = 0. \tag{18}$$

Since a general 2×2 matrix can be represented as a linear combination of the identity matrix and the Pauli matrices, proving the tracelessness in this case is equivalent to proving that $\hat{A}_\mu$ is part of the $SU(2)$ algebra. Hence, we need to show that if we start with a traceless field $\hat{A}_\mu$, the transformation rule Eq. (13) still produces a traceless field. This is quite obvious for the first term in Eq. (15)

$$\mathrm{Tr}\left[U\hat{A}_\mu U^{-1}\right] = 0. \tag{19}$$

For the second term the situation is a bit more complex and an explicit computation is required. A general $SU(2)$ matrix is written as

$$U = \cos\frac{\theta}{2}\hat{1} + i\sin\frac{\theta}{2}\,\vec{n}\cdot\vec{\sigma}, \tag{20}$$

where $\vec{n}^2 = 1$. Hence,

$$\partial_\mu U = \frac{\partial_\mu\theta}{2}\left(-\sin\frac{\theta}{2}\hat{1} + i\cos\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}\right) + i\sin\frac{\theta}{2}\,(\partial_\mu\vec{n})\cdot\vec{\sigma}. \tag{21}$$

Then,

$$\begin{aligned}
&\mathrm{Tr}\left[U^{-1}\partial_\mu U\right] = \mathrm{Tr}\left[U^\dagger\partial_\mu U\right] \\
&= \mathrm{Tr}\left[\left(\cos\frac{\theta}{2}\hat{1} - i\sin\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}\right)\left(\frac{\partial_\mu\theta}{2}\left(-\sin\frac{\theta}{2}\hat{1} + i\cos\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}\right) + i\sin\frac{\theta}{2}\,(\partial_\mu\vec{n})\cdot\vec{\sigma}\right)\right]
\end{aligned} \tag{22}$$

We need the following formulas

$$\mathrm{Tr}[\sigma_i] = 0, \quad \mathrm{Tr}[\sigma_i\sigma_j] = 2\delta_{ij}, \quad (\vec{n}\cdot\vec{\sigma})^2 = 1. \tag{23}$$

We use these formulas to simplify Eq. (22) and find

$$\begin{aligned}
&\mathrm{Tr}\left[\left(\cos\frac{\theta}{2}\hat{1} - i\sin\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}\right)\left(\frac{\partial_\mu\theta}{2}\left(-\sin\frac{\theta}{2}\hat{1} + i\cos\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}\right)\right)\right] \\
&= \frac{\partial_\mu\theta}{2}\,\mathrm{Tr}\left[-\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\,(\vec{n}\cdot\vec{\sigma})^2\right] = 0. \\
&\mathrm{Tr}\left[\left(\cos\frac{\theta}{2}\hat{1} - i\sin\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}\right)\left(i\sin\frac{\theta}{2}\,(\partial_\mu\vec{n})\cdot\vec{\sigma}\right)\right] \propto 2\vec{n}\partial_\mu\vec{n} = \partial_\mu(\vec{n}\cdot\vec{n}) = 0,
\end{aligned} \tag{24}$$

since $\vec{n}^2 = 1$.

Hence, we conclude that

$$\mathrm{Tr}[U^{-1}\partial_\mu U] = 0, \tag{25}$$

and this implies that if the field $\hat{A}_\mu$ belongs to the Lie algebra of the $SU(2)$ group before the transformation, then the transformed field also belongs to the Lie algebra. You will generalize this argument to the case of a more complex gauge group in the homework.

We also need the kinetic term for the field $\hat{A}_\mu$. We have seen that in the abelian case the kinetic term can be obtained from a commutator of covariant derivatives

$$[D_\mu, D_\nu] = -ig\hat{F}_{\mu\nu}. \tag{26}$$

It is easy to check that covariant derivatives transform in the following way under gauge transformations

$$D_\mu \to U D_\mu U^{-1}. \tag{27}$$

It follows that

$$\hat{F}_{\mu\nu} \to U\hat{F}_{\mu\nu}U^{-1}. \tag{28}$$

As the result, the kinetic energy is written as

$$L_{\mathrm{kin}} = -\frac{1}{2}\mathrm{Tr}\left[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right]. \tag{29}$$

We can find an explicit expression for the field-strength tensor by substituting covariant derivatives with their explicit expressions. We obtain

$$\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - ig[\hat{A}_\mu, \hat{A}_\nu]. \tag{30}$$

Since $\hat{A}_\mu = \sum A^a_\mu \tau^a$ and $[\tau^a, \tau^b] = if^{abc}\tau^c$, it follows that $\hat{F}_{\mu\nu}$ is in the Lie algebra

$$\hat{F}_{\mu\nu} = \sum F^{(a)}_{\mu\nu}\tau^a. \tag{31}$$

The complete Lagrangian that describes the scalar field and the gauge field reads

$$L_{\mathrm{kin}} = -\frac{1}{2}\mathrm{Tr}\left[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right] - [D_\mu\vec{\varphi}]^\dagger[D^\mu\vec{\varphi}] - m^2\vec{\varphi}^\dagger\vec{\varphi} - V(\vec{\varphi}^\dagger\vec{\varphi}). \tag{32}$$

Note that the non-abelian gauge fields interact with each other (in variance with photons, c.f. last term on the r.h.s. of Eq. (30)); also note that there is just *one* coupling constant g that determines the strength of self-interaction of gauge-fields and the strength with which non-abelian particles interact with complex scalars.