Lecture 6

Gauge invariance: the abelian case and the electromagnetic field

Let us go back to the discussion of the complex field φ in the previous lecture. We have seen that the following Lagrangian

$$\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi - V(\varphi^\dagger \varphi), \tag{1}$$

is invariant under the transformation

$$\varphi \to e^{i\alpha} \varphi, \tag{2}$$

where α is a constant parameter. The consequence of this invariance is the conservation of the following current

$$J^\mu = i(\varphi^\dagger \partial^\mu \varphi - (\partial^\mu \varphi^\dagger)\varphi), \tag{3}$$

and the time-independence of the quantity

$$Q = \int \mathrm{d}^3\vec{x} J^0(t, \vec{x}) \tag{4}$$

that we identified with the electric charge.

The interpretation of the invariance Eq. (2) is that we are free to choose the phase of the field φ in any way we want but we should do this "globally", i.e. we should do this in the same way everywhere in space-time. We may not like this fact, viewing it in contradiction with locality. We may then want to ask the following question – if we are allowed to choose different phases of the field φ at different points x^μ without changing the theory, how does the Lagrangian of the theory need to be modified?

To answer this question, we will try to perform a transformation $\varphi \to e^{i\alpha(x)}\varphi$. Clearly, $\varphi^\dagger\varphi$ is invariant under this transformation but $\partial_\mu \varphi$ is not. Indeed, we find

$$\partial_\mu \varphi \to \partial_\mu e^{i\alpha(x)} \varphi = e^{i\alpha(x)} \left[\partial_\mu \varphi + i \left(\partial_\mu \alpha \right) \varphi \right]. \tag{5}$$

It is, however, possible to force the theory to allow local phase transformations; the price to pay is the introduction of an additional field. Indeed, imagine that we replace the derivative of the field φ in Eq. (1) with the *covariant derivative*

$$\partial^\mu \varphi \to D^\mu \varphi, \tag{6}$$

where

$$D^\mu = \partial^\mu - igA^\mu(x). \tag{7}$$

We then obtain a new Lagrange function that reads

$$\mathcal{L} = \left(D_\mu \varphi \right)^\dagger \left(D^\mu \varphi \right) - m^2 \varphi^\dagger \varphi - V(\varphi^\dagger \varphi). \tag{8}$$

Upon making the transformation $\varphi \to e^{i\alpha(x)}\varphi$, we obtain

$$D^\mu \varphi \to e^{i\alpha(x)} \left[\partial^\mu - ig(A^\mu - g^{-1}\partial^\mu \alpha)\right] \varphi(x). \tag{9}$$

If, in addition, we change the field A as

$$A^\mu \to A^\mu + g^{-1}\partial^\mu \alpha, \tag{10}$$

we observe that the covariant derivative transforms in the same way as the field φ itself, i.e

$$D^\mu \varphi \to e^{i\alpha(x)} D^\mu \varphi, \tag{11}$$

and this ensures the invariance of the Lagrange function Eq. (8) under local phase transformations.

The field A^μ couples to the field φ and this coupling follows uniquely from the covariant derivative. We write

$$\begin{aligned} &(D_\mu\varphi)^\dagger (D^\mu\varphi) = \left(\left(\partial^\mu \varphi^\dagger\right) + igA^\mu\varphi^\dagger\right) (\partial_\mu\varphi - igA_\mu\varphi) \\ &= \partial^\mu\varphi^\dagger\partial_\mu\varphi + igA_\mu(\varphi^\dagger\partial^\mu\varphi - (\partial^\mu\varphi)^\dagger\varphi) + \mathcal{O}(A^2) = \partial^\mu\varphi^\dagger\partial_\mu\varphi + gA_\mu J^\mu + \mathcal{O}(A^2). \end{aligned} \tag{12}$$

It follows that the massless field A^μ interacts with the *conserved current* that, as we saw, arises as the result of the "global phase symmetry" we have studied earlier. This conserved current allowed us to define an electric charge; hence, we can identify the current with the electric current and the field A^μ with the electromagnetic field.

To make A^μ a dynamical field, we need a kinetic term for it. The simplest option is to write

$$\mathcal{L}_{\rm kin} = -\frac{1}{2}\left[\partial_\nu A^\mu \partial^\nu A_\mu\right], \tag{13}$$

which arises if we treat each component of a four-vector $A^\mu = (A^0, A^1, A^2, A^3)$ as a scalar field. The problem with Eq. (13) is that it is not invariant under the (gauge) transformation in Eq. (10) that, as we saw earlier, is needed to ensure the invariance of the Lagrangian in Eq. (8).

A way out is to use the so-called field-strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ that is invariant under these transformations. Indeed,

$$F_{\mu\nu} \to \partial_\mu(A_\nu + g^{-1}\partial_\nu\alpha) - \partial_\nu(A_\mu + g^{-1}\partial_\mu\alpha) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}. \tag{14}$$

We can construct the kinetic term by writing

$$\mathcal{L}_{\rm kin} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{15}$$

Then, the full Lagrangian for a theory with a complex scalar field φ and the massless gauge field A^μ reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger (D^\mu\varphi) - m^2\varphi^\dagger\varphi - V(\varphi^\dagger\varphi). \tag{16}$$

Note that we cannot add a mass term for the field A^μ to the Lagrangian Eq. (16) since a term $\sim \lambda^2 A_\mu A^\mu$ is not invariant under gauge transformations in Eq. (10). Hence, if we take invariance under gauge transformations as a principle for building theories of Nature, we have a natural explanation for the zero mass of photons.

Note also that the field-strength tensor $F_{\mu\nu}$ can be obtained from a commutator of covariant derivatives.

$$[D_\mu, D_\nu] = -igF_{\mu\nu}. \tag{17}$$

We will use this formula in the next lecture when we talk about non-abelian gauge transformations.

We can now write the action $S = \int \mathrm{d}^4 x L$ and derive equations of motion for the field A_μ. We find

$$\delta S = \int \mathrm{d}^4 x \, \left[-\frac{1}{2} F_{\mu\nu} \delta F^{\mu\nu} + g J_\mu \delta A^\mu + \mathcal{O}(g^2) \right]. \tag{18}$$

We will neglect the $\mathcal{O}(g^2)$ term in Eq. (18) since the coupling is small. We then integrate by parts in the first term in the integrand of Eq. (18) and obtain

$$\partial_\mu F^{\mu\nu} = -gJ^\nu. \tag{19}$$

This equation combines two of Maxwell's equations (the one's with currents and charge densities on the r.h.s.). Two other Maxwell's equations follow from the Bianci identity for the field-strength tensor

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0. \tag{20}$$