Lecture 5

Symmetries

We have discussed a single scalar field ϕ and its quantization in Lecture 3. We have seen that excitations of the field can be interpreted as scalar, spin-less particles with a particular mass. One of the things that we can do to extend our theory is to add more fields. For example, we can write

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{m_1^2}{2}\phi_1^2 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{m_2^2}{2}\phi_2^2 - V(\phi_1, \phi_2) \tag{1}$$

If $m_1 \neq m_2$ and V is an arbitrary function of the two fields, then there is not much to say beyond the fact that our theory describes the physics of two (self-) interacting particles.

However, let us consider a special case when $m_1 = m_2$ and $V(\phi_1, \phi_2) = V(\phi_1^2 + \phi_2^2)$. Let us also imagine that we do not want to use ϕ_1 and ϕ_2 as our "fields" and that we would prefer to employ ϕ_1', ϕ_2' instead. The relations between $\phi_{1,2}$ and $\phi_{1,2}'$ read

$$\begin{aligned} \phi_1 &= \cos\theta\ {\phi'}_1 + \sin\theta\ {\phi'}_2, \\ \phi_2 &= -\sin\theta\ {\phi'}_1 + \cos\theta\ {\phi'}_2. \end{aligned} \tag{2}$$

The parameter θ is arbitrary. Since

$$\phi_1^2 + \phi_2^2 = {\phi'}_1^2 + {\phi'}_2^2, \tag{3}$$

we find that

$$\mathcal{L} = \frac{1}{2}\partial_\mu {\phi'}_1 \partial^\mu {\phi'}_1 - \frac{m_1^2}{2}{\phi'}_1^2 + \frac{1}{2}\partial_\mu {\phi'}_2 \partial^\mu {\phi'}_2 - \frac{m_2^2}{2}{\phi'}_2^2 - V({\phi'}_1, {\phi'}_2) \tag{4}$$

Comparing Eqs. (1) and Eq. (4), we find that the two Lagrangians are the same. We then say that the theory is symmetric under rotations in the field space, Eq. (2).

To make this more transparent, we introduce a new notation. We will combine ϕ_1 and ϕ_2 into a field vector

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{5}$$

and write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{m^2}{2}\vec{\phi} \cdot \vec{\phi} - V(\vec{\phi} \cdot \vec{\phi}). \tag{6}$$

The symmetry transformation in Eq. (2) is represented as a matrix

$$\vec{\phi} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \vec{\phi}' = \hat{\mathcal{O}}(\theta)\vec{\phi}'. \tag{7}$$

The matrix $\hat{\mathcal{O}}(\theta)$ is an orthogonal matrix,

$$\mathcal{O}^T(\theta)\mathcal{O}(\theta) = 1, \tag{8}$$

which implies

$$\vec{\phi} \cdot \vec{\phi} = \vec{\phi}' \cdot \vec{\phi}', \tag{9}$$

and, obviously, $\mathcal{L}(\phi') = \mathcal{L}(\phi)$.

There is yet another way to write the same theory. Indeed, we introduce

$$\phi = \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right), \quad \phi^\dagger = \frac{1}{\sqrt{2}}\left(\phi_1 - i\phi_2\right). \tag{10}$$

Then since

$$\phi\phi^\dagger = \frac{1}{2}\left(\phi_1^2 + \phi_2^2\right), \tag{11}$$

and

$$\partial_\mu\phi\,\partial^\mu\phi^\dagger = \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right), \tag{12}$$

we obtain

$$\mathcal{L} = \partial_\mu\phi\,\partial^\mu\phi^\dagger - m^2\phi\phi^\dagger - V(2\phi\phi^\dagger). \tag{13}$$

The rotation in Eq. (7) is now represented by a phase transformation

$$\phi = e^{i\theta}\phi'. \tag{14}$$

Clearly, this transformation does not change the Lagrangian Eq. (13).

As it is known from classical mechanics, symmetries imply the existence of time-independent quantities (integrals of motion). The corresponding proof is provided by the Noether theorem. For our purposes, the proof goes as follows. Consider a field transformation $\phi \to \phi + \Delta\phi$ that leaves the Lagrangian invariant

$$\mathcal{L}(\phi) = \mathcal{L}(\phi + \Delta\phi). \tag{15}$$

We expand the right hand side of Eq. (15) to $\mathcal{O}(\Delta\phi)$, neglect further terms in the expansion and we find

$$\mathcal{L}(\phi) = \mathcal{L}(\phi) + \frac{\delta\mathcal{L}}{\delta\phi}\Delta\phi + \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\phi}\partial_\mu\Delta\phi. \tag{16}$$

We use the equations of motion

$$\frac{\delta\mathcal{L}}{\delta\phi} = \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\phi}\right], \tag{17}$$

to show that the two last terms on the r.h.s. of Eq. (16) read

$$\mathcal{L}(\phi) = \mathcal{L}(\phi) + \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\phi}\Delta\phi\right]. \tag{18}$$

It follows from Eq. (18) that

$$\partial_\mu J^\mu = 0, \tag{19}$$

where

$$J^\mu = \frac{\delta \mathcal{L}}{\delta\, \partial_\mu \phi}\, \Delta\phi. \tag{20}$$

is the Noether current. Note that the $\Delta\phi$ that appears in the above equation is a particular change in the field induced by a symmetry transformation, rather than an arbitrary variation of the field.

A direct consequence of the current conservation Eq. (19) is the time-independence of the following quantity

$$Q(t) = \int \mathrm{d}^3\vec{x}\; J^0(t,\vec{x}). \tag{21}$$

To prove this, we compute the time derivative and find

$$\frac{\partial}{\partial t}\, Q(t) = \int \mathrm{d}^3\vec{x}\; \partial_0 J^0(t,\vec{x}) = -\int \mathrm{d}^3\vec{x}\; \vec{\nabla}\cdot\vec{J}(t,\vec{x}) = -\int\limits_{|\vec{x}|=\infty} \mathrm{d}^2\vec{S}\cdot\vec{J} = 0. \tag{22}$$

The last steps follow from Gauss' theorem and the flux absence at spatial infinity.

For our example with two fields, it is straightforward to find the conserved current J^μ. For small θ, we find from Eq. (2)

$$\delta\phi_1 = \theta\;\phi_2, \quad \delta\phi_2 = -\theta\;\phi_1, \tag{23}$$

so that (upon setting $\theta \to -1$)

$$J^\mu = -\sum_{a=1}^{2} \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^a}\;\delta\phi^a = -\left(\partial^\mu\phi_1\right)\;\phi_2 + \left(\partial^\mu\phi_2\right)\;\phi_1 \tag{24}$$

In case we use the complex field representation, we have

$$\delta\phi = i\theta\phi, \quad \delta\phi^\dagger = -i\theta\phi^\dagger. \tag{25}$$

We then compute (we set again $\theta \to -1$)

$$J^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\;\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^\dagger}\;\delta\phi^\dagger = -i\left[\left(\partial^\mu\phi^\dagger\right)\;\phi - \phi^\dagger\;\partial^\mu\phi\right]. \tag{26}$$

To understand better what the time-independence of Q implies, we quantize the theory. Since we have two fields, we quantize them using the following field decomposition

$$\phi_{i=1,2} = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \left(a_{i,\vec{k}}\; e^{-ik_\mu x^\mu} + a^\dagger_{i,\vec{k}}\; e^{ik_\mu x^\mu}\right), \tag{27}$$

where

$$[a_{i,\vec{k}}, a^\dagger_{i,\vec{q}}] = \delta_{ij}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{q}). \tag{28}$$

We now use the formula Eq. (10) to write the complex field ϕ as

$$\phi = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \left(\frac{a_{1,\vec{k}} + i a_{2,\vec{k}}}{\sqrt{2}}\, e^{-ik_\mu x^\mu} + \frac{a^\dagger_{1,\vec{k}} + i a^\dagger_{2,\vec{k}}}{\sqrt{2}}\, e^{ik_\mu x^\mu} \right). \tag{29}$$

We will call the two combinations of creation and annihilation operators in the above formula as

$$a_{\vec{k}} = \frac{a_{1,\vec{k}} + i a_{2,\vec{k}}}{\sqrt{2}}, \quad \text{and} \quad b^\dagger_{\vec{k}} = \frac{a^\dagger_{1,\vec{k}} + i a^\dagger_{2,\vec{k}}}{\sqrt{2}}. \tag{30}$$

Note that $a^\dagger_{\vec{k}} \neq b^\dagger_{\vec{k}}$; this implies that $a_{\vec{k}}$ and $b_{\vec{k}}$ should be considered as independent. It is straightforward to compute the commutation relations between $a, a^\dagger, b, b^\dagger$ using the known commutation relations for $a_{1,2}, a^\dagger_{1,2}$. The result reads

$$[a_{\vec{k}}, a^\dagger_{\vec{q}}] = [b_{\vec{k}}, b^\dagger_{\vec{q}}] = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{q}), \quad [a_{\vec{k}}, b_{\vec{q}}] = [a^\dagger_{\vec{k}}, b^\dagger_{\vec{q}}] = 0. \tag{31}$$

The formula for the quantized complex fields reads then

$$\begin{aligned} \phi &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \left(a_{\vec{k}}\, e^{-ik_\mu x^\mu} + b^\dagger_{\vec{k}}\, e^{ik_\mu x^\mu} \right), \\ \phi^\dagger &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \left(a^\dagger_{\vec{k}}\, e^{ik_\mu x^\mu} + b_{\vec{k}}\, e^{-ik_\mu x^\mu} \right), \end{aligned} \tag{32}$$

The Hamiltonian for the complex field is then[1]

$$H = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \omega_k \left[a^\dagger_{\vec{k}} a_{\vec{k}} + b^\dagger_{\vec{k}} b_{\vec{k}} \right]. \tag{33}$$

The Hilbert space is again constructed by acting with the creation operators on the vacuum state

$$|\vec{k}\rangle_a = \sqrt{2\omega_k} a^\dagger_{\vec{k}} |0\rangle, \quad |\vec{k}\rangle_b = \sqrt{2\omega_k} b^\dagger_{\vec{k}} |0\rangle, \tag{34}$$

etc. We see that there are two types of particles.

To understand the difference between them, we will express the time-independent quantity Q through creation and annihilation operators. We use Eq. (26) and write

$$Q = -i \int \mathrm{d}^3\vec{x} \left[\phi(t,\vec{x})\partial_0\phi^\dagger(t,\vec{x}) - \phi^\dagger(t,\vec{x})\partial_0\phi(t,\vec{x}) \right]. \tag{35}$$

We substitute expressions for fields ϕ and $\phi^\dagger$ in terms of creation and annihilation operators, integrate over $\vec{x}$ and find

$$Q = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \left[a^\dagger_{\vec{k}} a_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}} \right]. \tag{36}$$

[1] We discarded the vacuum energy density contribution.

Eq. (36) implies

$$Q|\vec{k}\rangle_a = |\vec{k}\rangle_a, \qquad Q|\vec{k}\rangle_b = -|\vec{k}\rangle_b. \tag{37}$$

Hence, the two types of particles are eigenstates of the operator Q with different quantum numbers. We refer to these quantum numbers as *charges*. For example, we can associate the *electric charge* with the quantity Q, so that a-particles have positive and b-particle negative electric charges. Since Q is time-independent, it commutes with the Hamiltonian H

$$[H, Q] = 0. \tag{38}$$

This implies that one can choose quantum states in such a way that they diagonalize both H and Q simultaneously.

The idea of symmetry transformations is not restricted to the two field case. In fact, we can consider a theory with N fields $\phi_1, .., \phi_N$, and treat the N fields as a vector $\vec{\phi} = (\phi_1, ..., \phi_N)$. We take the Lagrangian to be

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{m^2}{2}\vec{\phi}\cdot\vec{\phi} - V(\vec{\phi}\cdot\vec{\phi}). \tag{39}$$

The symmetry then consists of rotating the field vector $\vec{\phi}$ in all possible ways; the Lagrangian density in Eq. (39) is invariant under these transformations. Such rotations are described by an $SO(N)$ group; the total number of "independent rotations" is $N(N-1)/2$ and they can be represented by $N \times N$ special orthogonal matrices. The transformation reads

$$\vec{\phi} = \mathcal{O}_N\vec{\phi}'. \tag{40}$$

It is now easy to find independent charges. To this end, we have to consider infinitesimal transformations as given by Eq. (40). In full generality, they read

$$\Delta\vec{\phi} = \sum_{a=1}^{N(N-1)/2} \epsilon_a T^a \vec{\phi}. \tag{41}$$

The quantities T^a are generators of the Lie Algebra of the $SO(N)$ group and ϵ_a are independent parameters of the symmetry transformation. It follows that there are $N(N-1)/2$ conserved currents

$$J^a_\mu = \partial_\mu\vec{\phi}T^a\vec{\phi}, \tag{42}$$

and the same number of conserved charges

$$Q^a(t) = \int \mathrm{d}^3x J^a_0(t, \vec{x}). \tag{43}$$

It is easy to check using canonical commutation relations between fields ϕ_i and the respective canonical momenta that charges satisfy the following commutation relations

$$[Q_a, Q_b] = i f_{abc} Q_c. \tag{44}$$

The constants f_{abc} are the structure constants of the $SO(N)$ Lie algebra. Hence, conserved charges provide a representation of a Lie algebra on the Hilbert space of the theory.

We discussed the extended symmetry transformations in the context of $SO(N)$ symmetry group. However, it should be clear from the context that this discussion is generic and, in principle, applies to arbitrary symmetry groups.