Lecture 4

The Casimir force

We have seen that the energy of a scalar field contains infinite contributions of "zero modes". According to our construction, the vacuum energy density reads

$$\rho_{\rm vac} = \frac{E_{\rm vac}}{V} = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{\omega_k}{2}, \tag{1}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$.

Let us calculate it. We integrate over directions of the vector $\vec{k}$ and find

$$\rho_{\rm vac} = \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}k\, k^2\sqrt{k^2+m^2}. \tag{2}$$

The resulting integral over k *diverges at large* k. To give it some meaning nevertheless, we cut it off at $k_{\max} = \Lambda \gg m$. We obtain

$$\rho_{\rm vac} = \frac{1}{4\pi^2}\int_0^\Lambda \mathrm{d}k\, k^2\sqrt{k^2+m^2} \approx \frac{\Lambda^4}{16\pi^2}\left(1 + \mathcal{O}(m^2/\Lambda^2)\right). \tag{3}$$

What value of Λ can be expected on physical grounds? We can argue in the following way. Our theory clearly ignores gravity. At which values of k do gravity effects become important? The wavelength of a particle with momentum k is $\lambda \sim k^{-1}$. The self-interaction gravitational energy that a particle of this "size" will have is $E_{\rm grav} \sim G\omega_k^2/\lambda$, where G is Newton's constant. Gravity effects can't be neglected anymore if the gravitational energy is comparable to the "total" energy of a particle computed ignoring gravity altogether. Hence only if

$$\frac{G\omega_k^2}{\lambda} \ll \omega_k \tag{4}$$

are we allowed to ignore gravity. By taking $\lambda \sim k^{-1} \sim \omega_k^{-1}$, the above equation simplifies to

$$Gk^2 \ll 1 \quad \to k \ll \frac{1}{\sqrt{G}}. \tag{5}$$

Hence, the maximal value of the cut-off Λ should be $G^{-1/2}$. We therefore estimate

$$\rho_{\rm vac} \approx \frac{G^{-2}}{16\pi^2}. \tag{6}$$

A constant vacuum energy density plays the role of a "cosmological constant" which contributes to the right hand side of Einstein's equations forcing the Universe to expand with an acceleration. However, the value of the cosmological constant that we observe is, roughly, *one hundred orders of magnitude smaller than what our theory predicts*

$$\Lambda_{CC} \sim 10^{-122}\rho_{\rm vac}. \tag{7}$$

This *cosmological constant problem* is a remarkable, dramatic and not-at-all-understood failure of the current theory of Nature!

However, if we are not completely devastated by this failure, we can still ask if there are other ways to probe the vacuum energy and, if there are, how does quantum field theory stack up against observations in those cases? It turns out that there is at least one case where we can probe the vacuum energy and get satisfactory results.

Imagine that we put two plates into an empty space. The plates are infinitely large; they are placed in such a way that they are parallel to the $(x-y)$-plane and the z-axis is orthogonal to them. The distance between the plates is a. We assume that the plates force particular boundary conditions on the field, i.e. that the scalar field satisfies

$$\varphi(x,y,z=0)=\varphi(x,y,z=a)=0, \tag{8}$$

where we assumed that one plate intersects the z-axis at $z=0$ and the other at $z=a$.[1] We now need to repeat the quantization procedure in such a way that the boundary conditions are satisfied.

Recall that without plates the field is written as

$$\varphi=\int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}}\left[a_{\vec{k}}e^{-i\omega_k t+i\vec{k}\vec{x}}+a^{\dagger}_{\vec{k}}e^{i\omega_k t-i\vec{k}\vec{x}}\right]. \tag{9}$$

Once the plates are introduced, the integral over k_z becomes a sum over all discrete modes that satisfy boundary conditions. We can easily arrange for that if we do the following replacements in Eq. (9)

$$e^{ikz}\to\Psi_n(z),\quad \Psi_n(z)=\sqrt{\frac{2}{a}}\sin\frac{\pi n z}{a},\quad \int\frac{\mathrm{d}k_z}{2\pi}\to\sum_{n=0}^{\infty}. \tag{10}$$

Moreover, the following formulas are useful

$$\sum_{n=0}^{\infty}\Psi_n(z)\Psi_n(z')=\delta(z-z'),\qquad \int\limits_0^a\mathrm{d}z\Psi_n(z)\Psi_{n'}(z)=\delta_{nn'}. \tag{11}$$

We write expressions for the field operator that is valid in case there are plates

$$\varphi=\sum_{n=0}^{\infty}\int\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^2\sqrt{2\omega_{k,n}}}\left[a_{\vec{k}_\perp,n}e^{-i\omega_k t+i\vec{k}_\perp\vec{x}_\perp}\Psi_n(z)+a^{\dagger}_{\vec{k}_\perp,n}e^{i\omega_k t-i\vec{k}_\perp\vec{x}_\perp}\Psi_n(z)\right], \tag{12}$$

where $\vec{k}_\perp=k_x\vec{e}_x+k_y\vec{e}_y$ and $\omega_{k,n}=\sqrt{k_\perp^2+(\pi n/a)^2+m^2}$.

It is instructive to check that φ and the canonical momentum $\pi=\partial_t\varphi$ satisfy canonical commutation relations. We find

$$\pi=\sum_{n=0}^{\infty}\int\frac{\mathrm{d}^2\vec{l}_\perp}{(2\pi)\sqrt{2\omega_{l,n}}}(-i\omega_{l,n})\left\{a_{\vec{l}_\perp,n}e^{-i\omega_{l,n}t+i\vec{l}_\perp\vec{x}_\perp}-a^{\dagger}_{\vec{l}_\perp,n}e^{i\omega_{l,n}t-i\vec{l}_\perp\vec{x}_\perp}\right\}\Psi_n(z) \tag{13}$$

[1] In a realistic case of electromagnetism, we would impose boundary conditions on the (vacuum) electric and magnetic fields, as usual.

We then compute

$$
\begin{aligned}
[\pi(t,x_1),\varphi(t,x_2)] = \sum_{n_1,n_2=0}^{\infty}\int \frac{\mathrm{d}^2\vec{k}_{1,\perp}}{(2\pi)^2\sqrt{2\omega_1}}\frac{\mathrm{d}^2\vec{k}_{2,\perp}}{(2\pi)^2\sqrt{2\omega_2}}\,(-i\omega_1)\,\Psi_{n_1}(z_1)\Psi_{n_2}(z_2) \\
\times\left[[a_{\vec{k}_{1,\perp},n_1},a^{\dagger}_{\vec{k}_{2,\perp},n_2}]e^{-i(\omega_1-\omega_2)t+i\vec{k}_{1,\perp}\vec{x}_{\perp,1}-i\vec{k}_{2,\perp}\vec{x}_{\perp,2}}-h.c.\right].
\end{aligned}
\tag{14}
$$

We choose

$$
[a_{\vec{k}_{1,\perp},n_1},a^{\dagger}_{\vec{k}_{2,\perp},n_2}] = (2\pi)^2\delta^{(2)}(\vec{k}_{1,\perp}-\vec{k}_{2,\perp})\delta_{n_1n_2} \tag{15}
$$

and obtain

$$
\begin{aligned}
[\pi(t,\vec{r}_1),\varphi(t,\vec{r}_2)] = -i\sum_{n=0}^{\infty}\int\int\frac{\mathrm{d}^2\vec{k}_{1,\perp}}{(2\pi)^2}\frac{1}{2}\left[e^{i\vec{k}_{1,\perp}(\vec{x}_{1,\perp}-\vec{x}_{2,\perp})}+h.c.\right]\Psi_n(z_1)\Psi_n(z_2) \\
= -i\delta^{(2)}\left(\vec{x}_{1,\perp}-\vec{x}_{2,\perp}\right)\delta(z_1-z_2) = -i\delta^{(3)}(\vec{r}_1-\vec{r}_2).
\end{aligned}
\tag{16}
$$

It is now easy to see that if we re-write the Hamiltonian in terms of creation and annihilation operators, we will obtain the following result for the vacuum energy

$$
E_{\mathrm{vac}} = L^2\sum_{n=0}^{\infty}\int\frac{\mathrm{d}^2\vec{k}_{\perp}}{(2\pi)^2}\,\frac{\omega_n}{2}, \tag{17}
$$

where ω_n reads

$$
\omega_n = \sqrt{k_{\perp}^2+\left(\frac{\pi n}{a}\right)^2+m^2}. \tag{18}
$$

We will start analyzing this formula by considering a toy one-dimensional model (A. Zee). That is, we set $L\to 1$ and remove integration over $k_{\perp}$ in Eq. (17) and set $k_{\perp}\to 0$ and $m\to 0$ in Eq. (18). We obtain the vacuum energy

$$
E^{\mathrm{toy}}_{\mathrm{vac}} = \frac{\pi}{2a}\sum_{n=0}^{\infty} n. \tag{19}
$$

The result is peculiar and is in line with our earlier discussion of the fact that the vacuum energy that we obtain in quantum field theory is infinite and that this infinity comes from vacuum fluctuations with very large energies.[2] However, for fluctuations with extremely high energies the plates should be invisible so that the high-frequency modes should leak through the plates. Their contribution to the vacuum energy should not depend on the plates and should rather match the continuum result.

To account for this, we introduce a cut-off in Eq. (19) and write

$$
E^{\mathrm{toy}}_{\mathrm{vac}} = \frac{\pi}{2a}\sum_{n=0}^{\infty} n e^{-\omega_n/\omega_{\mathrm{cut}}}, \tag{20}
$$

[2]The sum in Eq. (19) diverges because we extend the sum up to infinity.

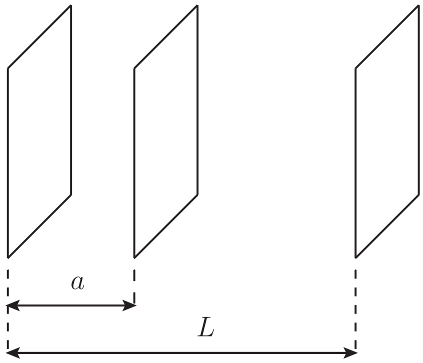

Figure 1: Three plates

where $\omega_n = \pi n/a$. To compute this sum, we write

$$E_{\rm vac}^{\rm toy} = -\frac{\pi}{2a}\frac{a}{\pi}\frac{\partial}{\partial\left[\frac{1}{\omega_{\rm cut}}\right]}\sum_{n=0}^{\infty} e^{-\frac{n\pi}{a\omega_{\rm cut}}}. \tag{21}$$

The sum in Eq. (21) is a geometric progression. We obtain

$$E_{\rm vac}^{\rm toy} = -\frac{\pi}{2a}\frac{a}{\pi}\frac{\partial}{\partial\left[\frac{1}{\omega_{\rm cut}}\right]}\frac{1}{1-e^{-\frac{\pi}{a\omega_{\rm cut}}}} = \frac{\pi}{2a}\frac{e^{-\frac{\pi}{a\omega_{\rm cut}}}}{\left(1-e^{-\frac{\pi}{a\omega_{\rm cut}}}\right)^2}. \tag{22}$$

Since $\omega_{\rm cut} \gg \pi/a$, we can expand the exponents and find

$$E_{\rm vac}^{\rm toy} = \frac{a\omega_{\rm cut}^2}{2\pi} - \frac{\pi}{24a} + ... \tag{23}$$

The two terms in Eq. (23) have different properties: the first one is proportional to the distance between the plates a. This contribution matches the vacuum energy without plates. To see that it plays no role, consider a system of three plates, separated by distances a and $L-a$ with $L \gg a$, see Fig. 1. Computing the total vacuum energy of the system, we obtain

$$E_{\rm vac}^{\rm toy}(L,a) = \frac{(L-a)\omega_{\rm cut}^2}{2\pi} - \frac{\pi}{24a} + \frac{a\omega_{\rm cut}^2}{2\pi} - \frac{\pi}{24(L-a)} ... = \frac{L\omega_{\rm cut}^2}{2\pi} - \frac{\pi}{24a} + ... \tag{24}$$

Note how terms proportional to a and $L-a$ combined to produce a term proportional to the total volume of the system $\sim L$. For this contribution the presence of the plate in the middle is immaterial as should be the case for the vacuum energy without plates. The residual dependence of the total energy on a is present nevertheless; this means that there is a force that acts on the intermediate plate. It is given by

$$f = -\frac{\partial E_{\rm vac}^{\rm toy}(L,a)}{\partial a} = -\frac{\pi}{24a^2}. \tag{25}$$

The force is attractive: plates placed into an absolutely empty space get attracted to each other. Note that this conclusion was verified experimentally with decent precision so there seems to be no doubt that certain properties of the vacuum energy do follow from a quantum field theory.

We will now go back to Eq. (18) and show how to compute the force in the three-dimensional case. This will look like a miracle and that's why I want to show it to you. It is definitely quite technical so if you don't feel like doing miracles, and especially the technical ones, just skip this part.

The problem with Eq. (17) is that it is poorly defined. At large $k_\perp$ the integral diverges as $k_\perp^3$ and the sum over n diverges as n^2. We need to "regularize" the integral, i.e. do something to make it "finite" and calculable. Once this happens, we can compute it and check what happens if we try to remove the regulator.

There is no unique way to regularize the integral Eq. (17). We will do it by changing the number of "transverse" dimensions,

$$\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^2} \to \frac{\mathrm{d}^d\vec{k}_\perp}{(2\pi)^d}, \tag{26}$$

where $d = 2 - 2\epsilon$ and ϵ is a continuous parameter. We will call this "dimensional regularization".

We will now compute the integral over $k_\perp$ in Eq. (18) choosing (mentally) ϵ as if the integrals converge. Then (setting $m \to 0$, for simplicity), we find

$$I_n = \int \frac{\mathrm{d}^d\vec{k}_\perp}{(2\pi)^d} \sqrt{\vec{k}_\perp^2 + \left(\frac{\pi n}{a}\right)^2}. \tag{27}$$

The integral depends on the absolute value of $k_\perp$ only. We introduce spherical coordinates in d-dimensional space and write

$$I_n = \frac{\Omega_d}{2(2\pi)^d} \int\limits_0^\infty \left(k_\perp^2\right)^{\frac{d-2}{2}} \mathrm{d}k_\perp^2 \sqrt{k_\perp^2 + \left(\frac{\pi n}{a}\right)^2}. \tag{28}$$

We now change variables

$$k_\perp^2 = \left(\frac{\pi n}{a}\right)^2 \frac{x}{1-x}, \tag{29}$$

and obtain

$$I_n = \frac{\Omega_d}{2(2\pi)^d} \left(\frac{\pi n}{a}\right)^{3-2\epsilon} \int\limits_0^1 \mathrm{d}x\, x^{-\epsilon}(1-x)^{-5/2+\epsilon}. \tag{30}$$

The last integral is evaluated using the following formula

$$\int\limits_0^1 \mathrm{d}x\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{31}$$

where on the right hand side we have the so-called Γ-functions. These functions are generalizations of factorials; for integer n, $\Gamma(n+1) = n!$. You need to know that $\Gamma(z)$ is an analytic function in the complex plane with simple poles at $z = 0, -1, -2, ...$ We use Eq. (31) to find

$$I_n = \frac{\Omega_d}{2(2\pi)^d} \left(\frac{\pi n}{a}\right)^{3-2\epsilon} \frac{\Gamma(-3/2+\epsilon)\Gamma(1-\epsilon)}{\Gamma(-1/2)}. \tag{32}$$

It remains to compute the sum over n

$$\begin{aligned} E_{\rm vac} &= L^2 \sum_{n=0}^{\infty} I_n = L^2 \frac{\Omega_d}{2(2\pi)^d} \frac{\Gamma(-3/2+\epsilon)\Gamma(1-\epsilon)}{\Gamma(-1/2)} \sum_{n=0}^{\infty} \left(\frac{\pi n}{a}\right)^{3-2\epsilon} \\ &= L^2 \frac{\Omega_d}{2(2\pi)^d} \frac{\Gamma(-3/2+\epsilon)\Gamma(1-\epsilon)}{\Gamma(-1/2)} \left(\frac{\pi}{a}\right)^{3-2\epsilon} \zeta(-3+2\epsilon), \end{aligned} \tag{33}$$

where we have used another special function (Riemann zeta function) to re-write the sum over n

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{34}$$

Finally, I comment on how to compute the solid angle in d-dimensions Ω_d. This is done by considering a product of d Gaussian integrals

$$\begin{aligned} \pi^{d/2} &= \prod_{i=1}^{d} \int_{-\infty}^{\infty} \mathrm{d}x_i e^{-x_i^2} = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \mathrm{d}x_1 ... \mathrm{d}x_d \, e^{-\sum_{i=1}^{d} x_i^2} = \int \mathrm{d}^d\vec{x} e^{-\vec{x}^2} = \Omega_d \int_0^{\infty} r^{d-1} \, \mathrm{d}r e^{-r^2} \\ &= \frac{\Omega_d}{2} \int_0^{\infty} \left[r^2\right]^{d/2-1} \, \mathrm{d}r^2 \, e^{-r^2} = \frac{\Omega_d}{2} \Gamma\left(\frac{d}{2}\right), \end{aligned} \tag{35}$$

where we used an integral representation of the Γ-function

$$\Gamma(z) = \int_0^{\infty} \mathrm{d}\, x \, x^{z-1} \, e^{-x}. \tag{36}$$

We finally find

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{37}$$

Using this result in Eq. (33) and expanding in ϵ,[3] we find

$$E_{\rm vac} = \frac{L^2 \pi^{d/2}}{\Gamma(1-\epsilon)(2\pi)^d} \frac{\Gamma(-3/2+\epsilon)\Gamma(1-\epsilon)}{\Gamma(-1/2)} \left(\frac{\pi}{a}\right)^{3-2\epsilon} \zeta(-3+2\epsilon) = -\frac{L^2\pi^2}{720a^3}. \tag{38}$$

[3]This expansion is not trivial and requires knowing properties of Γ and Zeta-function. This can, however, be done using Mathematica.

The *attractive* force between the plates is then

$$F = -\frac{\partial E_{\text{vac}}}{\partial a} = -\frac{\pi^2 L^2}{240 a^4}. \tag{39}$$

The strange aspect of this computation is that the result Eq. (38) comes out *finite* in $d = 2$ although the starting point Eq. (18) is a divergent integral. The reason the divergence has disappeared is the use of "dimensional regularization" that separates "scale-less" infinities from "scale-dependent" quantities in a very efficient way. To give an example, let us compute the vacuum energy in a volume $L^2 a$ in an empty space without plates using dimensional regularization. We find

$$E_{\text{vac}} = L^2 a \int \frac{\mathrm{d}^{d+1}\vec{k}_\perp}{(2\pi)^{d+1}} \frac{\omega_k}{2}, \tag{40}$$

where $\omega_k = |k|$ and $d = 2 - 2\epsilon$. Then

$$E_{\text{vac}} = L^2 a \int \frac{\mathrm{d}^{d+1}\vec{k}_\perp}{(2\pi)^{d+1}} \frac{\omega_k}{2} = \frac{L^2 a}{2} \frac{\Omega_{d+1}}{(2\pi)^{d+1}} \int\limits_0^\infty k^{d+1} \mathrm{d}k \tag{41}$$

To compute the integral over k, we write

$$\int\limits_0^\infty k^{d+1} \mathrm{d}k = \int\limits_0^1 k^{d+1} \mathrm{d}k + \int\limits_1^\infty k^{d+1} \mathrm{d}k \tag{42}$$

and define the two integrals as analytic continuations from values of d where each of them is well-defined. We then find

$$\int\limits_0^\infty k^{d+1} \mathrm{d}k = \frac{1}{d+2} - \frac{1}{d+2} = 0. \tag{43}$$

Hence, we observe that, if we use dimensional regularization, the vacuum energy of an empty space in the massless theory vanishes identically. This explain why Eq. (38) only contains finite $\mathcal{O}(L^2/a^3)$ terms and no $\mathcal{O}(L^2 a)$ term that will correspond to the vacuum energy of an empty space without plates.