Lecture 3

Canonical quantization, particles

One of the things that we did in the previous lecture was the computation of the energy stored in the scalar field

$$E = \int \mathrm{d}^3\vec{x} \left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + \frac{1}{2}\left(\vec{\nabla}\varphi\right)^2 + \frac{m^2\varphi^2}{2} + V(\varphi) \right]. \quad (1)$$

We will identify this quantity with the Hamiltonian of the system.

The Hamiltonian should be a function of canonical momenta and canonical coordinates. The canonical momentum is computed in a standard way

$$\frac{\delta L}{\delta \partial_t \varphi(t,\vec{x})} = \partial_t \varphi(t,\vec{x}) = \pi(t,\vec{x}). \quad (2)$$

We then obtain

$$E = H = \int \mathrm{d}^3\vec{x} \left[\frac{1}{2}\pi^2 + \frac{1}{2}\left(\vec{\nabla}\varphi\right)^2 + \frac{m^2\varphi^2}{2} + V(\varphi) \right] \quad (3)$$

The quantization procedure amounts to the choice of the *equal-time* commutator of *operators* φ and π

$$[\pi(t,\vec{x}), \varphi(t,\vec{y})] = -i\delta^{(3)}(\vec{x}-\vec{y}). \quad (4)$$

Moreover,

$$[\pi(t,\vec{x}), \pi(t,\vec{y})] = 0, \quad [\varphi(t,\vec{x}), \varphi(t,\vec{y})] = 0. \quad (5)$$

We will now consider a free theory ($V(\varphi) = 0$) and derive equations that φ and π satisfy. We will use the fact that the commutator of any operator with H gives a time derivative of this operator. For example

$$i\partial_t \pi(t,\vec{x}) = [\pi(t,\vec{x}), H]. \quad (6)$$

It is important that the Hamiltonian is time-independent (energy is conserved). This observation simplifies the computation of the commutator since we can take operators π and φ in the integrand of H at *any* time. It is this possibility that allows us to use Eq. (3) to compute the commutator of H with π and φ.

Indeed, we write

$$[\pi(t,\vec{x}), H] = \int \mathrm{d}^3\vec{y} \left[\frac{1}{2}[\pi(t,\vec{x}), \pi(t,\vec{y})^2] + \frac{1}{2}[\pi(t,\vec{x}), \left(\vec{\nabla}\varphi(t,\vec{y})\right)^2] + \frac{m^2}{2}[\pi(t,\vec{x}), \varphi(t,\vec{y})^2] \right]. \quad (7)$$

The first commutator vanishes since $\pi(t,\vec{x})$ commutes with $\pi(t,\vec{y})$. To compute the second and the third terms, we use the following equations

$$\begin{aligned}&\frac{1}{2}[\pi(t,\vec{x}),\left(\vec{\nabla}\varphi(t,\vec{y})\right)^2] = \vec{\nabla}_y\varphi(t,\vec{y})\cdot[\pi(t,\vec{x}),\vec{\nabla}_y\varphi(t,\vec{y})] \\ &= \vec{\nabla}_y\varphi(t,\vec{y})\cdot\vec{\nabla}_y[\pi(t,\vec{x}),\varphi(t,\vec{y})] = -i\vec{\nabla}_y\varphi(t,\vec{y})\cdot\vec{\nabla}_y\delta^{(3)}(\vec{x}-\vec{y}).\end{aligned} \tag{8}$$

and

$$\frac{1}{2}[\pi(t,\vec{x}),\varphi(t,\vec{y})^2] = \varphi(t,\vec{y})[\pi(t,\vec{x}),\varphi(t,\vec{y})] = -i\varphi(t,\vec{y})\delta^{(3)}(\vec{x}-\vec{y}). \tag{9}$$

We substitute Eqs. (8, 9) into Eq. (7), integrate by parts once and obtain

$$\partial_t\pi(t,\vec{x}) = \left[\vec{\nabla}^2 - m^2\right]\varphi(t,\vec{x}). \tag{10}$$

Since $\pi = \partial_t\varphi$, we find

$$\left[\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) + m^2\right]\varphi = \left[\partial_\mu\partial^\mu + m^2\right]\varphi = 0. \tag{11}$$

This is the Klein-Gordon equation for the field operator φ.

Since the operator φ satisfies the Klein-Gordon equation, we can write

$$\varphi(t,\vec{x}) = \int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}}\left[a_{\vec{k}}e^{-i\omega_k t+i\vec{k}\vec{x}} + a^\dagger_{\vec{k}}e^{i\omega_k t-i\vec{k}\vec{x}}\right], \tag{12}$$

where $\omega_k = \sqrt{\vec{k}^2+m^2}$, and the $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ operators are referred to as "creation" and "annihilation" operators, respectively.[1] In writing Eq. (12) we have used the fact that the field φ is real; the consequence of this for the quantum operator φ is that it is hermitian

$$\varphi^\dagger(t,\vec{x}) = \varphi(t,\vec{x}). \tag{13}$$

The momentum operator $\pi(t,\vec{x})$ is obtained by computing $\partial_t\varphi$. We find

$$\pi(t,\vec{x}) = -i\int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}}\,\omega_k\,\left[a_{\vec{k}}e^{-i\omega_k t+i\vec{k}\vec{x}} - a^\dagger_{\vec{k}}e^{i\omega_k t-i\vec{k}\vec{x}}\right], \tag{14}$$

The momentum operator π and the field operator φ must satisfy canonical equal-time commutation relations, Eq. (4). To check that they do, we write

$$\left[a_{\vec{k}},a^\dagger_{\vec{q}}\right] = f_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{q}), \tag{15}$$

where the function $f_{\vec{k}}$ at this point is arbitrary. We also assume that $[a_{\vec{k}},a_{\vec{q}}] = 0$ and $[a^\dagger_{\vec{k}},a^\dagger_{\vec{q}}] = 0$.

[1] We will see what is being "created" or "annihilatated" by these operators shortly.

To make expressions more compact, we introduce four-vectors $x^\mu = (t, \vec{x}), y^\mu = (t, \vec{y})$ and $k^\mu = (\omega_k, \vec{k})$ and write

$$\omega_k t - \vec{k}\vec{x} = kx, \quad \omega_k t - \vec{k}\vec{y} = ky. \tag{16}$$

We then find

$$[\pi(t,\vec{x}), \varphi(t,\vec{y})] = -i \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3\sqrt{2\omega_q}}\, \omega_k \left\{ [a_{\vec{k}}, a^\dagger_{\vec{q}}] e^{-ikx+iqy} + [a_{\vec{q}}, a^\dagger_{\vec{k}}] e^{ikx-iqy} \right\}$$
$$= -i \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2}\, f_{\vec{k}} \left\{ e^{i\vec{k}(\vec{x}-\vec{y})} + e^{-i\vec{k}(\vec{x}-\vec{y})} \right\} = -i\delta^{3)}(\vec{x}-\vec{y}), \tag{17}$$

where the last step requires $f_{\vec{k}} = 1$. Hence,

$$\left[a_{\vec{k}}, a^\dagger_{\vec{q}}\right] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{q}). \tag{18}$$

To understand the meaning of creation and annihilation operators, we need to express the Hamiltonian H through them. We start with the term $m^2\varphi^2$. Then

$$\int \mathrm{d}^3\vec{x}\; \varphi^2(t,\vec{x})$$
$$= \int \mathrm{d}^3\vec{x} \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3\sqrt{2\omega_1}} \frac{\mathrm{d}^3\vec{k}_2}{(2\pi)^3\sqrt{2\omega_2}} \left\{ a_{\vec{k}_1} e^{-ik_1x} + a^\dagger_{\vec{k}_1} e^{ik_1x} \right\} \left\{ a_{\vec{k}_2} e^{-ik_2x} + a^\dagger_{\vec{k}_2} e^{ik_2x} \right\}, \tag{19}$$

where $k_i x = \omega_i t - \vec{k}_i \vec{x}$. We integrate over $\vec{x}$ and obtain

$$\int \mathrm{d}^3\vec{x}\; \varphi^2(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3\sqrt{2\omega_1}} \frac{\mathrm{d}^3\vec{k}_2}{(2\pi)^3\sqrt{2\omega_2}} \left[\, a_{\vec{k}_1} a_{\vec{k}_2} e^{-i(\omega_1+\omega_2)t} (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) \right.$$
$$\left. + a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} e^{i(\omega_1+\omega_2)t} (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) + a^\dagger_{\vec{k}_1} a_{\vec{k}_2} \delta^{(3)}(\vec{k}_1 - \vec{k}_2) + a_{\vec{k}_1} a^\dagger_{\vec{k}_2} (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \right]. \tag{20}$$

As you see, this operator is *explicitly* time-dependent. As we have argued earlier, the Hamiltonian should be independent of time; therefore, we should expect important cancellation of the time-dependence of $\int \mathrm{d}^3x\; \varphi^2$ and the time-dependences of other contributions to H.

The next contribution is

$$\int \mathrm{d}^3\vec{x}\; (\vec{\nabla}\varphi)^2(t,\vec{x}) = -\int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3\sqrt{2\omega_1}} \frac{\mathrm{d}^3\vec{k}_2}{(2\pi)^3\sqrt{2\omega_2}} \vec{k}_1 \cdot \vec{k}_2 \left[\, a_{\vec{k}_1} a_{\vec{k}_2} e^{-i(\omega_1+\omega_2)t} (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) \right.$$
$$\left. + a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} e^{i(\omega_1+\omega_2)t} (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) - a^\dagger_{\vec{k}_1} a_{\vec{k}_2} \delta^{(3)}(\vec{k}_1 - \vec{k}_2) - a_{\vec{k}_1} a^\dagger_{\vec{k}_2} (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \right]. \tag{21}$$

A similar computation for $\int \mathrm{d}^3\vec{x}\, \pi^2$ gives

$$\int \mathrm{d}^3 x\, \pi^2(t,\vec{x}) = -\int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3\sqrt{2\omega_1}} \frac{\mathrm{d}^3\vec{k}_2}{(2\pi)^3\sqrt{2\omega_2}} \omega_1\omega_2 \Big[\, a_{\vec{k}_1} a_{\vec{k}_2} e^{-i(\omega_1+\omega_2)t} (2\pi)^3 \delta^{(3)}(\vec{k}_1+\vec{k}_2)$$
$$+ a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} e^{i(\omega_1+\omega_2)t} (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2) - a^\dagger_{\vec{k}_1} a_{\vec{k}_2} \delta^{(3)}(\vec{k}_1-\vec{k}_2) - a_{\vec{k}_1} a^\dagger_{\vec{k}_2} (2\pi)^3 \delta^{(3)}(\vec{k}_1-\vec{k}_2)\Big] . \tag{22}$$

To obtain the Hamiltonian H, we add the relevant contributions together, integrate over $\vec{k}_2$ and, using that $\omega_k^2 - m^2 - \vec{k}^2 = 0$, find that all time-dependent contributions cancel. The result becomes

$$H = \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3} \frac{\omega_1}{2} \left[a^\dagger_{\vec{k}_1} a_{\vec{k}_1} + a_{\vec{k}_1} a^\dagger_{\vec{k}_1}\right] = \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3} \left[\omega_1 a^\dagger_{\vec{k}_1} a_{\vec{k}_1} + \frac{\omega_1}{2}\delta^{(3)}(\vec{0})\right]. \tag{23}$$

In the last step we used the commutation relation Eq. (18) to re-order creation and annihilation operators; $\delta^{(3)}(\vec{0})$ is $\delta^{(3)}(\vec{q}-\vec{k})$, for $\vec{q}=\vec{k}$. To interpret $\delta^{(3)}(\vec{0})$, we write

$$(2\pi)^3\delta^{(3)}(\vec{0}) = (2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{k}_2)|_{\vec{k}_1=\vec{k}_2} = \int \mathrm{d}^3\vec{x} e^{i(\vec{k}_1-\vec{k}_2)\vec{x}}|_{\vec{k}_1=\vec{k}_2} = \int \mathrm{d}^3\vec{x} = V, \tag{24}$$

where V is the volume of the region where the field φ is defined.

Hence, we write

$$H = VE_0 + \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3} \omega_1 a^\dagger_{\vec{k}_1} a_{\vec{k}_1}, \tag{25}$$

where

$$E_0 = \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3}\frac{\omega_1}{2}. \tag{26}$$

We would like to find eigenstates of the Hamiltonian H. To this end, we define a quantum state with the minimal energy (the vacuum state) $|0\rangle$ as a state which is annihilated by *all* annihilation operators[2]

$$a_{\vec{k}}|0\rangle = 0\ , \quad \forall \vec{k}\ . \tag{27}$$

The energy of the vacuum state easily follows

$$H|0\rangle = VE_0|0\rangle. \tag{28}$$

It is interesting to know how large is the energy of the vacuum state. An unexpected answer to this question is that E_0 is *infinite* since the integral over $|\vec{k}|$ in Eq. (26) does not converge. We will discuss this issue in the next lecture. For now just note that the absolute value of the vacuum energy is not important for us; rather, we are interested in

[2]If this point is not clear to you, go back to your Quantum Mechanics lectures and check out the discussion of a quantum oscillator based on creation and annihilation operators. Then generalize.

how much energy is needed to *excite* the vacuum. Hence, we redefine the Hamiltonian by subtracting the vacuum energy and define

$$H - E_0 V \to H = \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3} \omega_1 a^\dagger_{\vec{k}_1} a_{\vec{k}_1}. \tag{29}$$

The excited states of the new Hamiltonian are constructed by acting with creation operators $a^\dagger_{\vec{k}}$ on the vacuum state

$$|\vec{k}_1, ...\vec{k}_N\rangle = a^\dagger_{\vec{k}_1} ... a^\dagger_{\vec{k}_N} |0\rangle. \tag{30}$$

It is easy to see that these quantum states are eigenstates of the Hamiltonian

$$H|\vec{k}_1, ...\vec{k}_N\rangle = \left(\sum_{i=1}^{N} \omega_i \right) |\vec{k}_1, ...\vec{k}_N\rangle. \tag{31}$$

For example, a state $a^\dagger_{\vec{k}_1}|0\rangle$ is an eigenstate of the Hamiltonian with the energy $\omega_1 = \sqrt{\vec{k}_1^2 + m^2}$. If we interpret $\vec{k}_1$ as the three-momentum of the state $|\vec{k}_1\rangle$, the above formula shows that the relation between energy and momentum of this state is identical to the relation between energy and momentum of a relativistic particle with the mass m. We will show below that $\vec{k}_1$ is indeed the three-momentum of the state $|\vec{k}_1\rangle$.

Before we discuss this, it is useful to say a few things about the normalization. If we compute $\langle\vec{k}_1|\vec{k}_2\rangle$ and use $\langle 0|0\rangle$=1, we easily find

$$\langle\vec{k}_1|\vec{k}_2\rangle = \langle 0|a_{\vec{k}_1} a^\dagger_{\vec{k}_2}|0\rangle = \langle 0|[a_{\vec{k}_1}, a^\dagger_{\vec{k}_2}]|0\rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2). \tag{32}$$

Unfortunately, this normalization is not Lorentz invariant. We therefore redefine the states

$$|\vec{k}\rangle = \sqrt{2\omega_k} a^\dagger_{\vec{k}} |0\rangle. \tag{33}$$

The relativistic normalization now reads

$$\langle\vec{k}_1|\vec{k}_2\rangle = 2\omega_{k_1} (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2). \tag{34}$$

From now on, we will always use states that are normalized in a Lorentz-invariant way.

Note that with this normalization the following result is valid

$$\langle 0|\varphi(t, \vec{x})|\vec{k}\rangle = e^{-i\omega_k t + i\vec{k}\vec{x}}, \tag{35}$$

which looks like a wave function of a relativistic particle with momentum $\vec{k}$ and energy ω_k.

To show that $\vec{k}$ is indeed the three-momentum of a particle described by the state $|\vec{k}\rangle$, we require the operator of the three-momentum. To find it, we will discuss certain properties of the action of a free field. The action reads

$$S = \int \mathrm{d}^4 x \, \mathcal{L}(\partial_\mu \varphi, \varphi). \tag{36}$$

The important point is that the Lagrangian density does not explicitly depend on x^μ. We can use this property to find quantities that do not change during the time evolution of the system – integrals of motion.

To this end, consider a general coordinate transformation

$$x^\mu = \tilde{x}^\mu + a^\mu(\tilde{x}). \tag{37}$$

We assume that a^μ is small. Then,

$$\varphi(x) = \varphi(\tilde{x} + a(\tilde{x})) \approx \varphi(\tilde{x}) + a^\mu(\tilde{x})\tilde{\partial}_\mu\varphi(\tilde{x}). \tag{38}$$

Now, writing

$$\varphi(\tilde{x} + a(\tilde{x})) = \tilde{\varphi}(\tilde{x}) = \varphi(\tilde{x}) + \delta\varphi(\tilde{x}), \tag{39}$$

we find

$$\delta\varphi(\tilde{x}) = a^\mu(\tilde{x})\tilde{\partial}_\mu\varphi(\tilde{x}). \tag{40}$$

To find the change in $\partial_\mu\varphi$, we need to be a bit more careful. We write

$$\begin{aligned} \frac{\partial\varphi(x)}{\partial x^\mu} &= \frac{\partial}{\partial x^\mu}\varphi(\tilde{x} + a^\mu(\tilde{x})) = \frac{\partial\tilde{x}^\nu}{\partial x^\mu}\frac{\partial}{\partial\tilde{x}^\nu}\varphi(\tilde{x} + a^\mu(\tilde{x})) \\ &= \left(g^\nu_\mu - \tilde{\partial}_\mu a^\nu\right)\tilde{\partial}_\nu\left(\varphi + \delta\varphi\right) \approx \tilde{\partial}_\mu\varphi - (\tilde{\partial}_\mu a^\nu)\tilde{\partial}_\nu\varphi + \tilde{\partial}_\mu\delta\varphi, \end{aligned} \tag{41}$$

where in the last step we neglected $\mathcal{O}(a^2)$ contributions.

Finally, we need to calculate[3]

$$\mathrm{d}^4x = \det\left[\frac{\partial x}{\partial\tilde{x}}\right]\,\mathrm{d}^4\tilde{x} \approx \mathrm{d}^4\tilde{x}\left(1 + \tilde{\partial}_\mu a^\mu\right). \tag{42}$$

Putting everything together, we find[4]

$$\begin{aligned} S &= \int \mathrm{d}^4\tilde{x}\left(1 + \tilde{\partial}_\mu a^\mu\right)\mathcal{L}\left[\varphi(\tilde{x}) + \delta\varphi(\tilde{x}), \tilde{\partial}_\mu\varphi - (\tilde{\partial}_\mu a^\nu)\tilde{\partial}_\nu\varphi + \tilde{\partial}_\mu\delta\varphi\right] \\ &= \int \mathrm{d}^4x\left\{\mathcal{L}(\varphi, \partial\varphi) + \partial_\mu a^\mu\mathcal{L} - \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right]\partial_\mu a^\nu\partial_\nu\varphi + \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right]\partial^\mu\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi\right\} \\ &= S + \int \mathrm{d}^4x\left\{\partial_\mu a^\mu\mathcal{L} - \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right]\partial_\mu a^\nu\partial_\nu\varphi - \partial^\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right]\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi\right\}, \end{aligned} \tag{43}$$

where in the last step we used integration-by-parts and neglected the integral of the total derivative. The last two terms cancel for arbitrary $\delta\varphi$ thanks to the equations of motion.

To simplify the first two terms, we use

$$\begin{aligned} &\partial_\mu a^\mu\mathcal{L} = \partial_\mu\left(a^\mu\mathcal{L}\right) - a^\mu\partial_\mu\mathcal{L}, \\ &\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right]\partial_\mu a^\nu\partial_\nu\varphi = \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}a^\nu\partial_\nu\varphi\right] - a^\nu\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\partial_\nu\varphi\right]. \end{aligned} \tag{44}$$

[3] We note that $\det(1 + \epsilon A) \approx 1 + \epsilon\mathrm{Tr}[A]$, for small ϵ.

[4] We replace $\tilde{x}$ with x in the second step, for simplicity.

Hence, neglecting total derivatives, we arrive at

$$0 = \int d^4 x a^\mu \left\{ \partial_\mu \mathcal{L} - \partial_\nu \left[\frac{\delta \mathcal{L}}{\delta \partial_\nu \varphi} \partial_\mu \varphi \right] \right\} = \int d^4 x a^\mu \partial_\nu \left\{ g^\nu_\mu \mathcal{L} - \left[\frac{\delta \mathcal{L}}{\delta \partial_\nu \varphi} \partial_\mu \varphi \right] \right\}. \tag{45}$$

Since a^μ is arbitrary, it follows that

$$\partial_\mu T^{\mu\nu} = 0, \tag{46}$$

where

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \, \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}, \tag{47}$$

is the *energy-momentum tensor*.

Conservation of the energy-momentum tensor Eq. (46) implies energy conservation. Indeed, Eq. (46) reads

$$\partial_t T^{00} = -\partial_i T^{i0}. \tag{48}$$

Now, integrating over the entire space and setting the integral of the total derivative to zero (we imagine that there is no flux through an infinitely-remote surface), we find

$$\partial_t \int \mathrm{d}^3 \vec{x} \; T^{00} = 0. \tag{49}$$

Hence, $\int \mathrm{d}^3 \vec{x} T^{00}$ is indeed time-independent; comparing it with Eq. (1), we find $H = \int \mathrm{d}^3 \vec{x} T^{00}$. This is the total energy stored in the field.

We can follow the same lines of reasoning to show that

$$P^i = \int \mathrm{d}^3 \vec{x} \; T^{0i} = -\int \mathrm{d}^3 \vec{x} \; \pi(t, \vec{x}) \vec{\nabla} \varphi(t, \vec{x}). \tag{50}$$

is also time-independent. To this end, we start with $\partial_\mu T^{\mu i} = 0$ and integrated over $\mathrm{d}^3 x$. Since the integral of the total derivative vanishes, we find

$$\partial_t \int \mathrm{d}^3 x T^{0i} = \partial_t P^i = 0. \tag{51}$$

The quantity P^i is the three-momentum operator,

To write P^i through creation and annihilation operators, we follow what we have done for the Hamiltonian. We find

$$\vec{P} = \int \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^3} \; \vec{k} \; a^\dagger_{\vec{k}} a_{\vec{k}}. \tag{52}$$

Hence, for $|\vec{k}\rangle = \sqrt{2\omega_k} a^\dagger_{\vec{k}} |0\rangle$, we obtain

$$\vec{P} |\vec{k}\rangle = \vec{k} |\vec{k}\rangle. \tag{53}$$

It follows, that the state $|\vec{k}\rangle$ is an eigenstate of the momentum operator $\vec{P}$ and the Hamiltonian H; the relation between eigenvalues of $\vec{P}$ and H corresponds to the relation between energy and momentum of a free relativistic particle with the mass m. We conclude that a quantum theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \tag{54}$$

describes free relativistic particles with the mass m.