Lecture 2

With the path integral to field theory

In the previous lecture, we talked about the path integral formulation of Quantum Mechanics. We will now show how a small modification of that discussion will give a description of quantum field theory. We consider a chain of identical masses connected to each other by identical springs. The Hamiltonian reads

$$H = \sum_{a=1}^{N} \left[\frac{p_a^2}{2m} + \frac{k}{2}(q_a - q_{a-1})^2 \right]. \tag{1}$$

We assume that the equilibrium distance between two neighbors is l and x_a describes a displacement of a particle a from its equilibrium position. We will identify q_0 with zero.

The computation of the time evolution operator has to be modified but only slightly. Indeed, the position operator is now a vector $\vec{q} = (q_1, ..., q_N)$. We are interested in a transition from an eigenstate of the operator $\vec{q}$, that we denote as $\vec{x}_i$, at $t = t_i$ to an eigenstate of the operator $\vec{q}$ that we denote as $\vec{x}_f$, at $t = t_f$. Specifically,

$$\vec{q}|\vec{x}_{i,f}\rangle = \vec{x}_{i,f}|\vec{x}_{i,f}\rangle. \tag{2}$$

We then write the matrix element of the evolution operator as

$$U(\vec{x}_f, \vec{x}_i; t_f, t_i) = \langle \vec{x}_f | e^{-iH(t_f - t_i)/\hbar} | \vec{x}_i \rangle. \tag{3}$$

To re-write the quantity $U(\vec{x}_f, \vec{x}_i; t_f, t_i)$ we proceed in exactly the same way as in Lecture 1, i.e. we split the time interval into segments, insert identity operators into strategic places and replace them with integrals over coordinates using completeness relations

$$1 = \int \mathrm{d}\vec{x} \; |\vec{x}\rangle\langle\vec{x}|. \tag{4}$$

It is then obvious that when all is said and done, we obtain

$$U(\vec{x}_f, \vec{x}_i; t_f, t_i) = \langle \vec{x}_f | e^{-iH(t_f - t_i)/\hbar} | \vec{x}_i \rangle = \int [\mathcal{D}\vec{x}(t)] e^{iS/\hbar}|_{\vec{x}(t_i)=\vec{x}_i, \vec{x}(t_f)=\vec{x}_f}. \tag{5}$$

The action reads

$$S = \int_{t_i}^{t_f} \mathrm{d}\tau L(\dot{\vec{x}}(\tau), \vec{x}(\tau)) = \int_{t_i}^{t_f} \mathrm{d}\tau \sum_{a=1}^{N} \left[\frac{m\dot{x}_a^2}{2} - \frac{k(x_a - x_{a-1})^2}{2} \right]. \tag{6}$$

We would like to take the limit $N \to \infty$, $l \to 0$, keeping $lN = L$ fixed. We then write

$$x_a(t) = \varphi(t, \xi), \tag{7}$$

where $\xi = al$ is the equilibrium position of a particle "a" along the chain. We replace the sum with the integral

$$\sum_{a=1}^{N} \to \int_{l}^{L} \frac{\mathrm{d}\xi}{l}, \tag{8}$$

and write

$$\lim_{l\to 0} \sum_{a=1}^{N} \left[\frac{m\dot{x}_a^2}{2} - \frac{k(x_a - x_{a-1})^2}{2} \right] = \lim_{l\to 0} \int_{l}^{L} \frac{\mathrm{d}\xi}{l} \left[\frac{m}{2} \left(\frac{\partial \varphi(t,\xi)}{\partial t} \right)^2 - \frac{k}{2} \left(\varphi(t,\xi) - \varphi(t,\xi - l) \right)^2 \right]. \tag{9}$$

The last term on the right hand side reads

$$\lim_{l\to 0} \left(\varphi(t,\xi) - \varphi(t,\xi - l) \right) \to l \, \frac{\partial \varphi(t,\xi)}{\partial \xi}. \tag{10}$$

We obtain

$$S = \int_{t_i}^{t_f} \mathrm{d}t \int_{0}^{L} \mathrm{d}\xi \left[\frac{m}{2l} \left(\frac{\partial \varphi(t,\xi)}{\partial t} \right)^2 - \frac{kl}{2} \left(\frac{\partial \varphi(t,\xi)}{\partial \xi} \right)^2 \right]. \tag{11}$$

We remove the prefactor in the term with time derivatives by redefining φ

$$\varphi \to \sqrt{\frac{l}{m}} \varphi \tag{12}$$

and find

$$S = \int_{t_i}^{t_f} \mathrm{d}t \int_{0}^{L} \mathrm{d}\xi \left[\frac{1}{2} \left(\frac{\partial \varphi(t,\xi)}{\partial t} \right)^2 - \frac{kl^2}{2m} \left(\frac{\partial \varphi(t,\xi)}{\partial \xi} \right)^2 \right]. \tag{13}$$

The combination of parameters kl^2/m has the dimension of velocity squared; we denote it as $kl^2/m = c^2$. Taking in addition the $L \to \infty$ limit, we find

$$S = \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}\xi \left[\frac{1}{2} \left(\frac{\partial \varphi(t,\xi)}{\partial t} \right)^2 - \frac{c^2}{2} \left(\frac{\partial \varphi(t,\xi)}{\partial \xi} \right)^2 \right]. \tag{14}$$

We see that in the continuous limit, our system of oscillators is described by a "field" $\varphi(t,\xi)$; this field parameterizes the displacement of a particle at the point ξ and at the time t from its equilibrium position. The quantum transition amplitude from a quantum state with the definite value of the field φ_i at $t = t_i$ to a quantum field with the definite value of the field $\varphi = \varphi_f$ at $t = t_f$ is given by a path integral

$$\langle \varphi_f(\xi), t_f | \varphi_i(\xi), t_i \rangle = U(\varphi_f, \varphi_i; t_f, t_i) = \int [\mathcal{D}\varphi] e^{\frac{iS}{\hbar}}, \tag{15}$$

where the integration goes over all fields with the following boundary condition $\varphi(t_{f,i}, x) = \varphi_{f,i}(x)$. Note that Eq.(15) implies that in our quantum theory the field $\hat{\varphi}(\xi)$ is a quantum operator, just like the position operator $\vec{q} = (q_1, ..., q_N)$ in N-body quantum mechanics used to be. Since we went from the latter to the former by taking the $N \to \infty$ limit, quantum field theory is quantum mechanics with infinitely many degrees of freedom.

We will see first implications of the above statement in the next lecture. In the remaining part of this lecture, we will discuss classical aspects of the theory. Classical "trajectories" of our system correspond to the case when the action S is minimal and its variation vanishes, $\delta S = 0$. Computing the variation, we find

$$\delta S = \int \mathrm{d}t\mathrm{d}\xi \left[\frac{\partial \varphi}{\partial t}\frac{\partial \delta\varphi}{\partial t} - c^2 \frac{\partial \varphi}{\partial \xi}\frac{\partial \delta\varphi}{\partial \xi} \right] = -\int \mathrm{d}t\mathrm{d}\xi \left[\frac{\partial^2 \varphi}{\partial t^2} - c^2 \frac{\partial^2 \varphi}{\partial \xi^2} \right] \delta\varphi. \tag{16}$$

Since δS must vanish for any $\delta\varphi$, the classical field φ must satisfy the following equation of motion

$$\frac{\partial^2 \varphi}{\partial t^2} - c^2 \frac{\partial^2 \varphi}{\partial \xi^2} = 0. \tag{17}$$

The general solution to this equation is

$$\varphi(t, \xi) = f_1(\xi - ct) + f_2(\xi + ct). \tag{18}$$

Hence, the path integral Eq.(15) describes a theory that is a quantum generalization of a classical theory that describes propagation of waves with the velocity c. Identifying c with the speed of light, we obtain a relativistic theory.

There are multiple generalizations of this construction that are useful to do. First, we can extend the theory of coupled oscillators to three dimensions. The action gets modified as follows

$$S = \int \mathrm{d}t\mathrm{d}^3\vec{x} \left[\frac{1}{2}\left(\frac{\partial \varphi}{\partial t}\right)^2 - \frac{c^2}{2}\sum_{i=1}^{3}\left(\frac{\partial \varphi}{\partial x_i}\right)^2 \right] = \int \mathrm{d}^4 x \; \frac{c}{2}\; \partial_\mu \varphi \partial^\mu \varphi, \tag{19}$$

where we introduced a four-vector $x^\mu = (ct, \vec{x})$ and used the notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\partial t}, \vec{\nabla} \right), \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{c\partial t}, -\vec{\nabla} \right). \tag{20}$$

Deriving classical equations of motion for this theory, we obtain

$$\partial_\mu \partial^\mu \varphi = 0. \tag{21}$$

This is the wave equation in four dimensions; waves propagate with the velocity c quite similar to electromagnetic waves.

It is easy to see that the theory in Eq.(19) is invariant under a Lorentz transformation $x^\mu = \Lambda^{\mu\nu} x'_\nu$ provided that the field φ transforms as a Lorentz scalar (i.e. does not change)

$$\varphi(x) = \varphi(\Lambda x') = \varphi'(x'). \tag{22}$$

The invariance under Lorentz transformations follows from the properties of the matrix Λ

$$\det\Lambda = 1, \Lambda^{\mu\nu}\Lambda_{\mu\alpha} = g^{\nu}_{\alpha}. \tag{23}$$

As the consequence,

$$\mathrm{d}^4x = \mathrm{d}^4x', \quad \partial_\mu\varphi\partial^\mu\varphi = \partial_{\mu'}\varphi'\partial^{\mu'}\varphi', \tag{24}$$

and so

$$S = \int \mathrm{d}^4x \, \frac{c}{2} \, \partial_\mu\varphi\partial^\mu\varphi = \int \mathrm{d}^4x' \, \frac{c}{2} \, \partial_{\mu'}\varphi'\partial^{\mu'}\varphi'. \tag{25}$$

We can now extend the action S by adding additional terms to the action that leave the action S Lorentz-invariant. One of the simplest terms that we can add is $-m^2c^3/(2\hbar^2)\varphi^2$.[1] The action becomes

$$S = \int \mathrm{d}^4x \, \left[\frac{c}{2} \, \partial_\mu\varphi\partial^\mu\varphi - \frac{m^2c^3\varphi^2}{2\hbar^2}\right]. \tag{26}$$

Computing δS and setting it to zero, we obtain "classical" equations of motion for the field φ

$$\left[\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right]\varphi = 0. \tag{27}$$

This is the Klein-Gordon equation that you discussed in Quantum Mechanics as a relativistic alternative to the Schrödinger equation. In our theory, the Klein-Gordon equation follows from classical equations of motion of the action S.

To understand the physical meaning of Eq.(27), we write

$$\varphi(t,x) = e^{-iEt/\hbar + i\vec{p}\cdot\vec{x}/\hbar} \tag{28}$$

and find that Eq.(27) is satisfied if the following condition

$$E^2 - c^2\vec{p}^2 - m^2c^4 = 0 \quad \Rightarrow \quad E = \pm\sqrt{c^2\vec{p}^2 + m^2c^4}, \tag{29}$$

is fulfilled. Leaving the sign ambiguity in front of the square root aside, we recognize in Eq.(29) the dispersion relation[2] of a relativistic particle with the mass m.

We can proceed further by modifying the action S even more. For example, we can add a quartic $\sim \varphi^4$ term to the action without violating Lorentz invariance. We find

$$S = \int \mathrm{d}^4x \, \left[\frac{c}{2} \, \partial_\mu\varphi \, \partial^\mu\varphi - \frac{m^2c^3\varphi^2}{2\hbar^2} - \frac{c\lambda\varphi^4}{4}\right]. \tag{30}$$

This action will lead to the classical equation of motion of the the following form

$$\left[\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right]\varphi = -\lambda\varphi^3. \tag{31}$$

[1] If you go back to the original Hamiltonian Eq.(1) you recognize that a term like that corresponds to an additional term $\kappa q_a^2/2$ for each site, i.e. an external linear force applied independently to each particle.

[2] Dispersion relation is the relation between energy and momentum.

This equation in non-linear (i.e. the superposition principle gets violated) and its solution is not known. We say in this case that the field φ is *self-interacting*.

The most general form of the action of a scalar field, consistent with Lorentz invariance and with the canonical form of the kinetic energy, is

$$S = \int \mathrm{d}^4 x \, \left[\frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2 \varphi^2}{2} - V(\varphi) \right], \tag{32}$$

where $V(\varphi)$ is an arbitrary function of a scalar field φ. Moreover, in Eq.(32) we set $c = 1$ and $\hbar = 1$ which is what we often do in quantum field theory computations to simplify the notation. So, from now on, we will not show $\hbar$ and c anywhere. The quantity in square brackets in Eq.(32)

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2 \varphi^2}{2} - V(\varphi), \tag{33}$$

is referred to as the "Lagrange function density". The Lagrange function itself is then

$$L = \int \mathrm{d}^3 \vec{x} \mathcal{L}. \tag{34}$$

Since we understand the connection between field theory and a system of many particles we can compute the energy that is stored in our system. Indeed, for a mechanical system the energy is computed from the following equation

$$E = \sum_a \frac{\partial L}{\partial \dot{q}_a} \, \dot{q}_a - L. \tag{35}$$

Since $q_a \to \varphi(t, \xi)$ and $\dot{q}_a \to \partial \varphi(t, \xi)/\partial t$, we find

$$E = \int \mathrm{d}^3 \vec{x} \left(\frac{1}{2} \left(\frac{\partial \varphi}{\partial t} \right)^2 + \frac{1}{2} \left(\vec{\nabla} \varphi \right)^2 + \frac{m^2 \varphi^2}{2} + V(\varphi) \right). \tag{36}$$

Note that a static, constant field φ, i.e. $\partial_t \varphi = 0, \vec{\nabla} \varphi = 0$, has energy

$$E = \left(\frac{m^2 \varphi^2}{2} + V(\varphi) \right) L^3, \tag{37}$$

where L^3 is the space volume. If $V(\varphi) \sim \varphi^n$, $n > 0$, it follows that the configuration with minimal energy is an empty space with vanishing field $\varphi = 0$.