Lecture 16

Green's functions and Feynman diagrams

We have seen in the last lecture that we can understand interesting physics by analyzing *free* field theory coupled to external sources. However, ultimately, we would like to understand how to deal with *interacting* field theories. In this lecture we briefly discuss how this can be done.

Recall that Green's functions of a quantum field theory defined by the action

$$S[\varphi] = \int \mathrm{d}^4x \, \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - V(\varphi) \right) \tag{1}$$

can be computed using the generating functional

$$Z[J] = \frac{\int \mathcal{D}\varphi \; e^{iS[\varphi,J]}}{\int \mathcal{D}\varphi \; e^{iS[\varphi,0]}}. \tag{2}$$

In Eq. (2),

$$S[\varphi, J] = S[\varphi] + \int \mathrm{d}^4x \; J(x)\varphi(x). \tag{3}$$

The Green's functions are computed by taking functional derivatives of $Z[J]$

$$\langle 0|T\varphi(x_1)\varphi(x_2)...\varphi(x_N)|0\rangle = \frac{1}{Z[0]} \frac{\delta^N Z[J]}{i\delta J(x_1)\; i\delta J(x_2)\; ...i\delta J(x_N)}\bigg|_{J=0}. \tag{4}$$

To get an idea of how Green's functions look like, we start with a free field theory where $V(\varphi) = 0$. Then,

$$Z[J] = e^{-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y J(x)D_F(x-y)J(y)}. \tag{5}$$

We compute Green's functions by taking derivatives of $Z[J]$ in Eq. (5). Upon taking two derivatives, we find

$$\frac{\delta^2 Z[J]}{i\delta J(x_3)\; i\delta J(x_4)} = D_F(x_3-x_4)Z[J] - \int \mathrm{d}^4y_3 D_F(x_3-y_3)J(y_3) \int \mathrm{d}^4y_4 D_F(x_4-y_4)J(y_4)\; Z[J]. \tag{6}$$

If we set $J = 0$ in Eq. (6), we obtain the two-point function.

$$\frac{\delta^2 Z[J]}{i\delta J(x_3)\; i\delta J(x_4)}\bigg|_{J=0} = \langle 0|T\varphi(x_3)\varphi(x_4)|0\rangle = D_F(x_3 - x_4). \tag{7}$$

We can represent a two-point function by a straight line that connects the two points x_3 and x_4, see Fig. (1); this is an example of a Feynman diagram in position space

$$D_F(x_3 - x_4) =$$

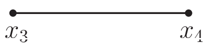


Figure 1: Feynman diagram that describes a two-point function in a free theory.

If we differentiate Eq. (6) two more times and set $J = 0$ afterwards, we find the four-point function. It reads

$$\begin{aligned}
&\left.\frac{\delta^4 Z[J]}{i\delta J(x_1)\, i\delta J(x_2)\, i\delta J(x_3)\, i\delta J(x_4)}\right|_{J=0} = \\
&= D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_3 - x_1)D_F(x_4 - x_2) + D_F(x_3 - x_2)D_F(x_4 - x_1) \\
&= \sum_{\text{pairs}} D_F(x_{i_1} - x_{i_2})D_F(x_{i_3} - x_{i_4}).
\end{aligned} \tag{8}$$

This formula suggests that a four-point function in a free theory can be obtained by drawing four points x_1, x_2, x_3, x_4 and then connecting them, pairwise, in all possible ways and taking a sum over all permutations. The picture obtained in this way (c.f. Fig. (2)) is the Feynman diagram for a four-point function in a free theory.

$$\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle =$$

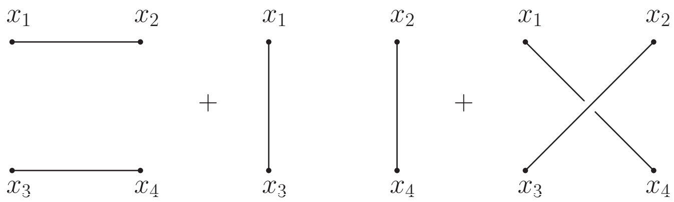


Figure 2: A Feynman diagram for a four-point function in a free theory.

An obvious generalization of Eq. (8) provides an N-point Green's function

$$\left.\frac{\delta^N Z[J]}{i\delta J(x_1)\, i\delta J(x_2)\, ...\, i\delta J(x_N)}\right|_{J=0} = \sum_{\text{pairs}} D_F(x_{i_1} - x_{i_2})D_F(x_{i_3} - x_{i_4})...D_F(x_{i_{N-1}} - x_{i_N}). \tag{9}$$

Suppose, we would like to calculate Green's functions in an interacting theory. The problem is that in this case the generating functional $Z[J]$ is not known exactly. However, it is possible to compute it in perturbation theory. To do that, we introduce an auxiliary functional $\tilde{Z}[j]$ and write

$$\begin{aligned}
\tilde{Z}[J] &= \int \mathcal{D}\varphi\, e^{i\int \mathrm{d}^4x\, \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2\varphi^2}{2} - V(\varphi) + J\varphi\right]} \\
&= \sum_{n=0}^{\infty} \int \mathcal{D}\varphi\, \frac{(-i)^n}{n!} \prod_{i=1}^{n} \int \mathrm{d}^4y_i V(\varphi(y_i))\ \ e^{i\int \mathrm{d}^4x\, \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2\varphi^2}{2} + J\varphi\right]} \\
&= \sum_{n=0}^{\infty} \int \mathcal{D}\varphi\, \frac{(-i)^n}{n!} \prod_{i=1}^{n} \int \mathrm{d}^4y_i V\left(\frac{\delta}{i\delta J(y_i)}\right)\ \ e^{i\int \mathrm{d}^4x\, \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2\varphi^2}{2} + J\varphi\right]} \\
&= e^{-i\int \mathrm{d}^4y V\left(\frac{\delta}{i\delta J(y)}\right)} \int \mathcal{D}\varphi\, e^{i\int \mathrm{d}^4x\, \left[\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2\varphi^2}{2} + J\varphi\right]} \\
&= \tilde{N}_0\, e^{-i\int \mathrm{d}^4y V\left(\frac{\delta}{i\delta J(y)}\right)}\ \tilde{Z}_0[J],
\end{aligned} \tag{10}$$

where $\tilde{N}_0$ is a normalization constant in a free theory that we can set to one without further ado. Also,

$$\tilde{Z}_0[J] = e^{-\frac{1}{2}\int \mathrm{d}^4x \mathrm{d}^4y J(x) D_F(x-y) J(y)}. \tag{11}$$

The formula that we just derived is extremely general. To understand how it works, we consider a theory with potential energy

$$V(\varphi) = \frac{\lambda}{4!}\varphi^4. \tag{12}$$

We assume that λ is small and expand $\tilde{Z}[J]$ in powers of λ. Working to first order, we find

$$\tilde{Z}[J] = \tilde{Z}_0[J] - \frac{i\lambda}{4!}\int \mathrm{d}^4x\ \left[\frac{\delta}{i\delta J(x)}\right]^4 \tilde{Z}_0[J] + \mathcal{O}(\lambda^2). \tag{13}$$

To proceed further, we need to compute four derivatives of $Z_0[J]$ w.r.t. $J(x)$. We find

$$\begin{aligned}
\frac{\delta \tilde{Z}_0[J]}{\delta J(x)} &= -\int \mathrm{d}^4y\ D_F(x-y)J(y)\ \tilde{Z}_0[J], \\
\frac{\delta^2 \tilde{Z}_0[J]}{\delta J(x)^2} &= -D_F(0)\tilde{Z}_0[J] + \left[\int \mathrm{d}^4y\ D_F(x-y)J(y)\right]^2\ \tilde{Z}_0[J], \\
\frac{\delta^3 \tilde{Z}_0[J]}{\delta J(x)^3} &= 3D_F(0)\left[\int \mathrm{d}^4y\ D_F(x-y)J(y)\right]\ \tilde{Z}_0[J] - \left[\int \mathrm{d}^4y\ D_F(x-y)J(y)\right]^3\ \tilde{Z}_0[J], \\
\frac{\delta^4 \tilde{Z}_0[J]}{\delta J(x)^4} &= 3D_F^2(0)\tilde{Z}_0[J] - 6D_F(0)\left[\int \mathrm{d}^4y\ D_F(x-y)J(y)\right]^2\ \tilde{Z}_0[J] \\
&+ \left[\int \mathrm{d}^4y\ D_F(x-y)J(y)\right]^4\ \tilde{Z}_0[J],
\end{aligned} \tag{14}$$

Suppose that we are interested in the computation of a two-point function through first order in the coupling constant λ. We need to compute

$$\left.\frac{\delta^2 Z[J]}{i\delta J(x_1) i\delta J(x_2)}\right|_{J=0}, \tag{15}$$

and we do that using Eq. (13) and Eq. (14). We find

$$\begin{aligned}
&\left.\frac{\delta^2 \tilde{Z}[J]}{i\delta J(x_1) i\delta J(x_2)}\right|_{J=0} = D_F(x_1 - x_2) \\
&- \frac{i\lambda}{4!}\int \mathrm{d}^4 x \left[3D_F(0)D_F(0)\ D_F(x_1-x_2) + 12 D_F(0) D_F(x_1-x)D_F(x-x_2)\right] \\
&= D_F(x_1-x_2)\left(1 - \frac{i\lambda}{8}\int \mathrm{d}^4 x\ D_F^2(0)\right) - \frac{i\lambda}{2} D_F(0)\int \mathrm{d}^4 x D_F(x_1-x)D_F(x-x_2).
\end{aligned} \tag{16}$$

To find $Z[J]$, we also need $\tilde{Z}[0]$, to compute the normalization constant. We obtain it from Eqs. (13,14). It reads

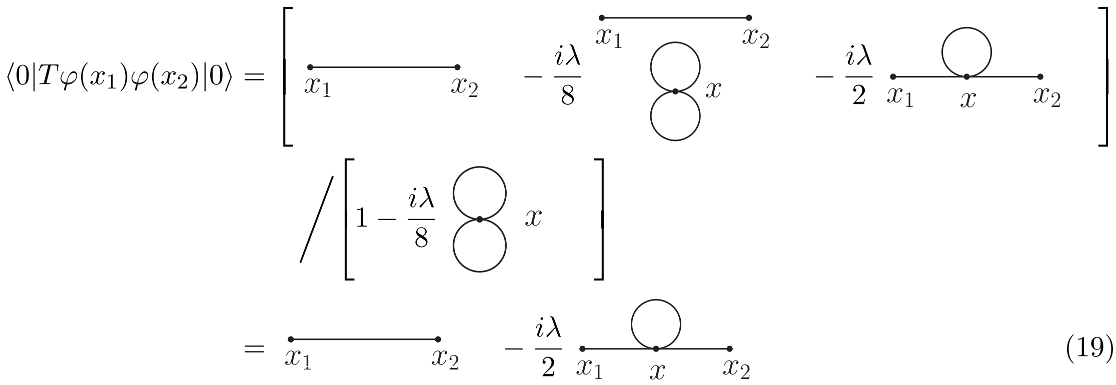
$$\tilde{Z}[0] = 1 - \frac{3i\lambda}{4!}\int \mathrm{d}^4 x\ D_F(0)D_F(0). \tag{17}$$

Since $Z[J] = \tilde{Z}[J]/\tilde{Z}[0]$, we combine Eqs. (16,17), expand to first order in λ and find our final result for the two-point function in an interacting theory

$$\begin{aligned}
\left.\frac{\delta^2 Z[J]}{i\delta J(x_1) i\delta J(x_2)}\right|_{J=0} &= \langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle \\
&= D_F(x_1-x_2) - \frac{i\lambda}{2}D_F(0)\int \mathrm{d}^4 x\ D_F(x_1-x)D_F(x-x_2).
\end{aligned} \tag{18}$$

We can develop a pictorial representation of Eq. (18) using what we already did earlier in a free field theory. In a free field theory a two-point function is represented by a line that connects the two points x_1 and x_2. This is the first term in the right hand side of Eq. (18). The second term in the r.h.s. of Eq. (18) tells us that, in an interacting field theory, a particle may interact with vacuum fluctuations on its way from x_1 to x_2. The interaction can happen at *any* point x (hence, integration over x in Eq. (18)) and, at the point x, the interaction introduces a fluctuation that is described by a particle propagating from the point x to the same point x (hence, $D_F(0)$). The interaction in our theory is $V \sim \varphi^4$ and, hence, necessarily involves four particles (or fields). For this reasons four fields have to meet at the "interaction point" x.

The Feynman diagram that is obtained by following these rules to describe Eq. (18) is shown below, c.f. Eq. (19). Note that there we show separately the expansion of the numerator and denominator of $Z[J]$ defined in Eq. (2).



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