Lecture 15

Green's functions and the path integral. Forces.

We have seen in Lecture 2 that a probability amplitude that describes a transition from a vacuum state to a vacuum state over very large time T is given by the following formula

$$\langle 0|e^{-iHT}|0\rangle = \int \mathcal{D}\varphi \; e^{iS}, \tag{1}$$

where

$$S = \int \mathrm{d}^4x \; \left[\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - V(\varphi) \right], \tag{2}$$

and the integration goes over all possible field configurations. In general, the matrix element on the left hand side of Eq. (1) is not very interesting because the vacuum state is an eigenstate of the Hamiltonian H. This implies that $e^{-iHT}|0\rangle = |0\rangle$ (since we define the energy of the vacuum to be zero) so that $\langle 0|e^{-iHT}|0\rangle = 1$.

We can make the path integral more interesting if we modify the action by introducing an external source $J(t,\vec{x})$ that couples to the quantum field φ

$$S \to S[\varphi, J] = \int \mathrm{d}^4x \; \left[\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - V(\varphi) + J\varphi \right]. \tag{3}$$

We define a functional $Z[J]$ as follows

$$Z[J] = \int \mathcal{D}\varphi \; e^{iS[\varphi,J]}. \tag{4}$$

We know from the discussion above that $Z[0] = 1$; we can make this explicit by writing

$$Z[J] = \frac{\int \mathcal{D}\varphi \; e^{iS[\varphi,J]}}{\int \mathcal{D}\varphi \; e^{iS[\varphi,0]}}. \tag{5}$$

We have discussed a similar quantity in Lecture 1 in the context of quantum mechanics. Here we will discuss it once again in connection with quantum field theory. We will start with a free theory, $V(\varphi) = 0$, where the path integral can be explicitly computed. We start by re-writing the kinetic term in the action by integrating by parts

$$\frac{1}{2}\int \mathrm{d}^4x \; (\partial_\mu \varphi)^2 = -\frac{1}{2}\int \mathrm{d}^4x \; \varphi \; \partial_\mu \partial^\mu \varphi. \tag{6}$$

The functional $Z[J]$ becomes

$$Z[J] = \frac{\int \mathcal{D}\varphi \; e^{i\int \mathrm{d}^4x \; \left[-\frac{1}{2}\varphi(\partial^2+m^2)\varphi + J\varphi\right]}}{\int \mathcal{D}\varphi \; e^{i\int \mathrm{d}^4x \; \left[-\frac{1}{2}\varphi(\partial^2+m^2)\varphi\right]}}. \tag{7}$$

To compute the integral, we change the integration variable $\varphi \to \bar{\varphi}$ in the numerator as follows $\varphi = \bar{\varphi} + \chi$. Then, $\mathcal{D}\varphi = \mathcal{D}\bar{\varphi}$ and

$$\begin{aligned}-\frac{1}{2}\varphi(\partial^2+m^2)\varphi + J\varphi = &-\frac{1}{2}\bar{\varphi}(\partial^2+m^2)\bar{\varphi} \\ &+ \bar{\varphi}J - \frac{1}{2}\bar{\varphi}(\partial^2+m^2)\chi - \frac{1}{2}\chi(\partial^2+m^2)\bar{\varphi} \\ &+ J\chi - \frac{1}{2}\chi(\partial^2+m^2)\chi.\end{aligned} \tag{8}$$

We can remove the entire second line in the above equation by choosing χ in an appropriate way. To this end, we integrate the last term in the second line of Eq.(8) by parts twice and find

$$\bar{\varphi}J - \frac{1}{2}\bar{\varphi}(\partial^2+m^2)\chi - \frac{1}{2}\chi(\partial^2+m^2)\bar{\varphi} \to \bar{\varphi}\left(J - (\partial^2+m^2)\chi\right). \tag{9}$$

By choosing

$$\chi = \left[\partial^2+m^2\right]^{-1} J, \tag{10}$$

we ensure that $\bar{\varphi}J - \frac{1}{2}\bar{\varphi}(\partial^2+m^2)\chi - \frac{1}{2}\chi(\partial^2+m^2)\bar{\varphi}$ vanishes. Note that the right hand side of Eq.(10) is actually an integral

$$\chi(x) = \int \mathrm{d}^4y\, D(x,y)J(y), \tag{11}$$

where $D(x,y)$ is symmetric and defined as

$$\left(\partial_x^2+m^2\right) D(x,y) = \delta^{(4)}(x-y). \tag{12}$$

We use Eq.(10) to simplify $Z[J]$

$$Z[J] = \frac{\int \mathcal{D}\bar{\varphi}\; e^{i\int \mathrm{d}^4x\, \left[-\frac{1}{2}\bar{\varphi}(\partial^2+m^2)\bar{\varphi}\right]}}{\int \mathcal{D}\varphi\; e^{i\int \mathrm{d}^4x\, \left[-\frac{1}{2}\varphi(\partial^2+m^2)\varphi\right]}}\; e^{iW[J]} = e^{iW[J]}, \tag{13}$$

where

$$W[J] = \int \mathrm{d}^4x \left(J\chi - \frac{1}{2}\chi(\partial^2+m^2)\chi\right) = \frac{1}{2}\int \mathrm{d}^4x \mathrm{d}^4y\; J(x)D(x,y)\; J(y). \tag{14}$$

We can now compute the functional derivatives of $Z[J]$ with respect to J. We find

$$\frac{\delta Z[J]}{i\delta J(x)} = \left[\int \mathrm{d}^4x_1 D(x,x_1)J(x_1)\right]\; Z[J], \tag{15}$$

and

$$\frac{\delta^2 Z[J]}{i\delta J(x)\; i\delta J(y)} = -iD(x,y)\; Z[J] + \left[\int \mathrm{d}^4x_1 D(x,x_1)J(x_1)\right]\left[\int \mathrm{d}^4y_1 D(y,y_1)J(y_1)\right]\; Z[J]. \tag{16}$$

Taking $J \to 0$ in both equations and using $Z[0] = 1$, we find

$$\frac{\delta Z[J]}{i\delta J(x)}|_{J=0} = 0, \qquad \frac{\delta^2 Z[J]}{i\delta J(x)\; i\delta J(y)}|_{J=0} = -iD(x,y). \tag{17}$$

On the other hand, taking derivatives with respect to J in Eq.(5) and setting $J = 0$, we find e.g.

$$\frac{\delta^2 Z[J]}{i\delta J(x)\; i\delta J(y)}|_{J=0} = \frac{\int \mathcal{D}\varphi\; \varphi(x)\varphi(y) e^{iS[\varphi,0]}}{\int \mathcal{D}\varphi e^{iS[\varphi,0]}} = \langle 0|T\varphi(x)\varphi(y)|0\rangle, \tag{18}$$

where the last step follows from the discussion in Lecture 1. We have seen in the previous lecture that for a free theory that we discuss now

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle = D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i0} e^{i(x-y)p}. \tag{19}$$

We therefore identify

$$D(x,y) = iD_F(x-y). \tag{20}$$

In fact, the following equation is valid in an interacting theory

$$\frac{\delta^n Z[J]}{i\delta J(x_1)\; i\delta J(x_2) ...\; i\delta J(x_N)} = \langle 0|T\varphi(x_1)\varphi(x_2)....\varphi(x_N)|0\rangle. \tag{21}$$

Note that the exact form of $Z[J]$ in a free theory Eq.(13) implies that *all* Green's functions in a free theory are given by products of Feynman propagators $D_F(x_i - x_j)$.

One interesting thing that we can discuss using path integral formalism is the connection between "particles" and "forces". We sometimes call particles "force carriers" and we will see now how this comes about. To this end, we use Eq.(20) to write $Z[J]$ in a free theory as

$$Z[J] = e^{-1/2 \int \mathrm{d}^4 x\; \mathrm{d}^4 y\; J(x) D_F(x-y) J(y)}. \tag{22}$$

Let us take $J(x) = J_1(x) + J_2(x)$, where $J_{1,2}(x) = \delta^{(3)}(\vec{x} - \vec{x}_{1,2})$. This means that we took two time-independent, similar sources, located at two different spatial points. If we substitute $J = J_1 + J_2$ into JD_FJ in Eq.(22), we will find four terms: two of the form $J_1 D_F J_1$ and $J_2 D_F J_2$ and two of the form $J_1 D_F J_2$ and $J_2 D_F J_1$. We are interested in the latter two since they describe the influence of one point-like source on another one. We need

$$\begin{aligned}
&-\frac{1}{2}\int \mathrm{d}^4 y\; \mathrm{d}^4 z\; J_1(y) D_F(y-z) J_2(z) = -\frac{1}{2}\int \mathrm{d}y_0 \mathrm{d}z_0 D_F(y_0 - z_0, \vec{x}_1 - \vec{x}_2) \\
&= -\frac{i}{2}\int \mathrm{d}y_0 \mathrm{d}z_0 \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik_0(y_0 - z_0) - i\vec{k}(\vec{x}_1 - \vec{x}_2)}}{k^2 - m^2 + i0} \\
&= -\frac{i}{2}\int \mathrm{d}y_0 \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{2\pi\delta(k_0)\; e^{ik_0 y_0}\; e^{-i\vec{k}(\vec{x}_1 - \vec{x}_2)}}{k^2 - m^2 + i0} = \frac{i}{2}\, T \int \frac{\mathrm{d}^3 \vec{k}}{(2\pi)^3} \frac{e^{-i\vec{k}(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + m^2},
\end{aligned} \tag{23}$$

where in the last step we replaced $\int \mathrm{d}y_0 = T$. Hence, we find

$$Z[J] = e^{iT\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\, \frac{e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}.}{\vec{k}^2+m^2}}. \tag{24}$$

Since, on the other hand,

$$Z[J] = \langle 0|e^{-iHT}|0\rangle_J = e^{-iE_{\mathrm{vac}}T}, \tag{25}$$

we identify

$$E_{\mathrm{vac}} = -\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\, \frac{e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}.}{\vec{k}^2+m^2}, \tag{26}$$

with the change in the vacuum energy due to the interaction between the two sources. This energy can be easily evaluated; it reads (Yukawa potential).

$$E_{\mathrm{vac}}(\vec{x}_1-\vec{x}_2) = -\frac{1}{4\pi|\vec{x}_1-\vec{x}_2|}e^{-m|\vec{x}_1-\vec{x}_2|}. \tag{27}$$

The vacuum energy decreases if the distance between the two sources decreases; hence, the interaction is *attractive*. We have seen in the previous lecture that the Feynman propagator, that played an essential role in the derivation of this result (c.f. Eq. (23)), can be associated with creation and annihilation of particles through field quantization. Hence, we say that exchanges of virtual scalar particles between like objects lead to an *attractive force* between sources.

As the next step, we discuss what happens in a gauge theory. We write

$$S[A,J] = \int \mathrm{d}^4x\, \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu + J_\mu A^\mu\right], \tag{28}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We have added the mass to the vector field; although such a term breaks gauge-invariance, we will imagine that this term comes e.g. from spontaneous symmetry breaking. We will also require that the current J_μ is conserved, $\partial_\mu J^\mu = 0$. We begin by rewriting the kinetic term by integrating by parts. We find

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\partial^\mu A^\nu)\, F_{\mu\nu} \to \frac{1}{2}A^\nu\left(\partial^2 g_{\nu\mu} - \partial_\nu\partial_\mu\right)A^\mu. \tag{29}$$

Hence, $S[A,J]$ becomes

$$S[A,J] = \int \mathrm{d}^4x\, \left[\frac{1}{2}A^\nu\left(\partial^2 g_{\nu\mu} - \partial_\nu\partial_\mu\right)A^\mu + \frac{m^2}{2}A_\mu A^\mu + J_\mu A^\mu\right]. \tag{30}$$

Similar to the scalar field case, we need a functional $Z[J]$ that reads

$$Z[J] \sim \int \prod_{\mu=0}^{3} \mathcal{D}A_\mu\, e^{iS[A,J]}. \tag{31}$$

To compute $Z[J]$ for the vector field, we shift $A_\mu \to \bar{A}_\mu + \chi_\mu$ and require that linear terms in A_μ vanish. This is achieved if we choose χ_μ such that the following equation is satisfied

$$\left[(\partial^2 + m^2)g_{\mu\nu} - \partial_\mu\partial_\nu\right]\chi^\nu + J_\mu = 0. \tag{32}$$

To solve this equation, we re-write it in momentum space. We find

$$\left[(-p^2 + m^2)g_{\mu\nu} + p_\mu p_\nu\right]\chi^\nu(p) = -J_\mu(p). \tag{33}$$

To determine χ^ν, we write an Ansatz

$$\chi_\mu(p) = \left(Ag_{\mu\nu} + Bp_\mu p_\nu\right)J^\nu(p), \tag{34}$$

where A and B are the two coefficients to be determined. We substitute χ from Eq.(34) into Eq.(33) and find

$$A(-p^2 + m^2)g_{\mu\nu} + p_\mu p_\nu\left(A + Bm^2\right) = -g_{\mu\nu}. \tag{35}$$

The two independent tensor structures have to match simultaneously; we find

$$A = \frac{1}{p^2 - m^2}, \qquad B = -\frac{A}{m^2}. \tag{36}$$

Hence,

$$\chi_\mu(p) = \frac{1}{p^2 - m^2}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right]J^\nu(p). \tag{37}$$

The current conservation $\partial_\mu J^\mu = 0$ implies $p_\mu J^\mu(p) = 0$. Hence, the above equation can be simplified

$$\chi_\mu(p) = \frac{1}{p^2 - m^2}J_\mu(p). \tag{38}$$

The pole at $p^2 = m^2$ is regularized by $m^2 \to m^2 - i0$, so that the analytic structure of the propagator in Eq.(38) is identical to the scalar field case.

A simple computation (HW) that follows what has been done in the scalar field case, gives

$$Z[J] = e^{\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y J_\mu(x)D_F(x-y)J^\mu(y)}. \tag{39}$$

We note that, compared to the scalar case, the sign in the exponent has changed. Therefore, if we take $J^0 = \delta^{(3)}(\vec{x} - \vec{x}_1) + \delta^{(3)}(\vec{x} - \vec{x}_2)$ and $J^{1,2,3} = 0$, the interaction energy between two sources will have an additional minus sign relative to Eq.(27). Hence, the interaction between two identical "charge" densities facilitated by vector particles is *repulsive.*

An interesting case to discuss is gravity, where the interaction between two equal "charges" is actually attractive. Since we do not want to get into a discussion of quantum gravity, we will try to construct empirical arguments. To this end, we go back to the case of a vector particle exchange and write Eq.(37) by introducing an appropriate Green's function

$$\chi^\mu(p) = i\,D_F^{\mu\nu}(p)J_\nu(p), \qquad D_F^{\mu\nu}(p) = \frac{i}{p^2 - m^2 + i0}\left[-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right]. \tag{40}$$

We would like to understand the numerator of this Green's function

$$G^{\mu\nu}(p) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}. \tag{41}$$

To achieve this, we take the four-momentum p to be on the mass-shell, i.e. $p^2 = m^2$. Such momentum is a possible four-momentum for real, physical particles with the correct relation between energy E and three-momentum $\vec{p}$, $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. We begin by considering a particle at rest, i.e. $p = (m, \vec{0})$. The (spin-one) vector particle at rest has three polarization states which we write, for simplicity, as

$$\epsilon_1^\mu = (0,1,0,0), \quad \epsilon_2^\mu = (0,0,1,0), \quad \epsilon_3^\mu = (0,0,0,1). \tag{42}$$

It is easy to see that

$$\sum_{a=1}^{3} \epsilon_a^\mu \epsilon_a^\nu = \begin{cases} 0, & \mu \neq \nu \\ 0, & \mu = \nu = 0 \\ 1, & \mu = \nu = i. \end{cases} \tag{43}$$

We can write this tensor as

$$\sum_{a=1}^{3} \epsilon_a^\mu \epsilon_a^\nu = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}, \quad \text{for} \quad p^\mu = (m, \vec{0}). \tag{44}$$

In fact, it is easy to show that this relation holds for an arbitrary vector p, not only the ones that describe particles at rest, provided of course that $p^2 = m^2$ (HW). We conclude that the function $G^{\mu\nu}(p)$, the residue of the Green's function $D_F^{\mu\nu}(p)$ at $p^2 = m^2$, can be reconstructed from the sum over physical polarizations of a spin-one particle, as shown in Eq.(44) and then extended to all values of p.

We now try to construct the Green's function that describes the exchange of spin-2 particles. This will show us how gravity works since excitations of gravitational fields – the *gravitons* - are spin-two particles. A massive spin-two particle is characterized by $(2J+1) = (2\cdot 2+1) = 5$ independent polarization states. To construct them, we consider a rank-two tensor $\epsilon_{\mu\nu}^{(a)}(p)$. A generic rank-two tensor has 16 independent components. To project a rank-two tensor on the spin-two part, we require that this tensor is symmetric, transverse and traceless

$$\epsilon_{\mu\nu} = \epsilon_{\nu\mu}, \quad \epsilon_{\mu\nu}^{(a)}(p)\, p^\mu = 0, \quad \epsilon_{\mu\nu} g^{\mu\nu} = 0. \tag{45}$$

This reduces the number of independent components to $16 \to 10 - 4 - 1 = 5$ thanks to symmetry, transversality and tracelessness, respectively. The numerator of the Feynman propagator for spin-two particle is

$$W_{\mu\nu,\lambda\rho}(p) = \sum_{a=1}^{5} \epsilon_{\mu\nu}^{(a)}(p) \epsilon_{\lambda\rho}^{(a)}(p). \tag{46}$$

Thanks to its definition and the properties of the polarization "tensors", tensor $W_{\mu\nu,\lambda\rho}$ is symmetric with respect to the permutations of (μ, ν) and (λ, ρ), symmetric in $\mu \leftrightarrow \nu$, symmetric in $\lambda \leftrightarrow \rho$, transversal

$$p^\mu W_{\mu\nu,\lambda\rho}(p) = 0, \quad p^\lambda W_{\mu\nu,\lambda\rho}(p) = 0, \tag{47}$$

and traceless

$$g^{\mu\nu} W_{\mu\nu,\lambda\rho} = 0, \quad g^{\lambda\rho} W_{\mu\nu,\lambda\rho} = 0. \tag{48}$$

We will use the tensor $G_{\mu\nu} = -g_{\mu\nu} + p_\mu p_\nu / m^2$ to write an Ansatz for $W_{\mu\nu,\lambda\rho}$ since it automatically satisfies the transversality conditions. The Ansatz reads

$$W_{\mu\nu,\lambda\rho} = x_1 G_{\mu\nu} G_{\lambda\rho} + x_2 \left(G_{\mu\lambda} G_{\nu\rho} + G_{\mu\rho} G_{\nu\lambda} \right). \tag{49}$$

The tracelessness condition needs to be imposed by contracting $W_{\mu\nu,\lambda\rho}$ with $g^{\mu\nu}$ or $g^{\lambda\rho}$. We find that

$$g^{\mu\nu} W_{\mu\nu,\lambda\rho} = 0 \quad \text{if} \quad x_1 = -\frac{2}{3} x_2. \tag{50}$$

Hence,

$$W_{\mu\nu,\lambda\rho} = x_2 \left(G_{\mu\lambda} G_{\nu\rho} + G_{\mu\rho} G_{\nu\lambda} - \frac{2}{3} G_{\mu\nu} G_{\lambda\rho} \right). \tag{51}$$

It remains to fix x_2. To do so, we note that polarization tensors are normalized to one. Since

$$W_{\mu\nu,\lambda\rho}(p) = \sum_{a=1}^{5} \epsilon^{(a)}_{\mu\nu}(p) \epsilon^{(a)}_{\lambda\rho}(p), \tag{52}$$

we require

$$g^{\mu\lambda} g^{\nu\rho} \, W_{\mu\nu,\lambda\rho}(p) = 5. \tag{53}$$

This implies $x_2 = 1/2$. Putting everything together, we deduce the Green's function for the spin-two particle exchange

$$D^F_{\mu\nu,\lambda\rho}(p) = \frac{i}{2} \frac{G_{\mu\lambda} G_{\nu\rho} + G_{\mu\rho} G_{\nu\lambda} - \frac{2}{3} G_{\mu\nu} G_{\lambda\rho}}{p^2 - m^2 + i0}. \tag{54}$$

We are now in a position to determine the change in the vacuum energy due to interactions caused by exchanges of spin-two particle between two sources. Recall that spin-0 and spin-1 exchanges give

$$\begin{aligned} Z[J] &= e^{-1/2 \int \mathrm{d}^4x \mathrm{d}^4y J(x) D_F(x-y) J(y)}, \\ Z[J] &= e^{-1/2 \int \mathrm{d}^4x \mathrm{d}^4y J_\mu(x) D_F^{\mu\nu}(x-y) J_\nu(y)}, \end{aligned} \tag{55}$$

where $D_F^{\mu\nu}(x-y) = D_F(x-y)\,(-g^{\mu\nu} + ..)$ and $-g^{\mu\nu} + ... = \sum \epsilon^\mu \epsilon^\nu$. Hence, we can imagine that virtual gravitons lead to the following result for generating functional

$$Z[J] = e^{-1/2 \int \mathrm{d}^4x \mathrm{d}^4y J_{\mu\nu}(x) D_F^{\mu\nu,\lambda\rho}(x-y) J_{\lambda\rho}(y)}, \tag{56}$$

Since the currents $J_{\mu\nu}$ will be taken to be conserved, we can discard all terms with the four-momentum p in Eq.(54) and replace $G_{\mu\nu}$ with the metric tensor. Moreover, we only need a component with $\mu = \nu = \rho = \lambda = 0$, since we will be interested in interactions of two energy densities described by the J^{00} components of the currents. Then

$$D^F_{\mu\nu,\lambda\rho}(p) \to \frac{1}{2}\left(1 + 1 - \frac{2}{3}\right) D_F(p) = \frac{1}{3} D_F(p). \tag{57}$$

It follows that, for spin-two exchanges, the sign of E_{vac} is the same as in the case of spin-zero exchanges. Hence, the exchange of spin-two particles (gravity), leads to an attractive force.