Lecture 14

Green's functions

Typically, if we want to solve a quantum mechanical problem, we study the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\Psi = H\Psi, \tag{1}$$

to find the wave-function ψ of a quantum-mechanical system. Once we know the wave function, we have complete information about a quantum system, to the extent possible in quantum mechanics.

However, we almost never attempt to solve the Schrödinger equation in quantum field theory, so the primary mathematical object we work with changes completely. To see how it comes about, let us recall that relativistic quantum field theory appeared as an attempt to understand quantum physics of relativistic particles. A standard way to explore properties of relativistic particles is to collide them and study results of such collisions. To describe this process, we imagine that at time $t = -\infty$ the two particles are far apart and head towards each other; we describe this initial state with a ket-vector $|i\rangle$. At $t = +\infty$, the outcome of the collision is the final state $|f\rangle$. The probability amplitude for this transition to happen is given by the matrix element

$$S_{fi} = \langle f|i\rangle. \tag{2}$$

Typically, $|i\rangle$ and $|f\rangle$ are constructed out of creation and annihilation operators, however these operators are not quite the same. Let us imagine that at $t = -\infty$, we describe particles with creation operators $a^\dagger_{\vec p}(-\infty)$ and at $t = \infty$ with $a^\dagger_{\vec p}(+\infty)$. To define these operators precisely, we assume that we deal with a scalar theory described by the Lagrangian

$$L = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - V(t,\varphi), \tag{3}$$

where

$$V(t,\varphi) = V(\varphi)\theta(T_0, t). \tag{4}$$

The function $\theta(T_0, t)$ is constant on the interval $-T_0 < t < T_0$ but adiabatically vanishes for smaller and larger values of t. Hence, there exists $T \gg T_0$ such that for $|t| > T$ our theory is, effectively, a free theory. We can now define the Hilbert space of the theory at $t = \pm\infty$ exactly. To this end, we write

$$\begin{aligned}
\varphi(t > T, \vec x) &= \int \frac{\mathrm{d}^3\vec k}{(2\pi)^3\sqrt{2\omega_{\vec k}}}\left(a_{\vec k}(+\infty)e^{-ik_\mu x^\mu} + a^\dagger_{\vec k}(+\infty)e^{ik_\mu x^\mu}\right), \\
\varphi(t < -T, \vec x) &= \int \frac{\mathrm{d}^3\vec k}{(2\pi)^3\sqrt{2\omega_{\vec k}}}\left(a_{\vec k}(-\infty)e^{-ik_\mu x^\mu} + a^\dagger_{\vec k}(-\infty)e^{ik_\mu x^\mu}\right).
\end{aligned} \tag{5}$$

The initial state $|i\rangle$ and the final state $|f\rangle$ are then constructed using operators $a^\dagger_{\vec{k}}(-\infty)$ and $a^\dagger_{\vec{k}}(+\infty)$, respectively. For example, consider a typical scattering process where two particles with momenta $p_{1,2}$ produce n particles with momenta $p_{3,4,...,n}$, i.e.

$$p_1 + p_2 \to p_3 + p_4 + ... + p_n. \tag{6}$$

We assume that $p_i \neq p_j$, for $i \neq j$ and that $p_i^2 = m^2$ for all i's. To describe this process, we require a matrix element

$$S_{fi} = \langle f|i\rangle = \sqrt{2\omega_1\, 2\omega_2\, ...2\omega_n}\langle 0|a_{p_3}(\infty)...a_{p_n}(\infty)\; a^\dagger_{p_1}(-\infty)a^\dagger_{p_2}(-\infty)|0\rangle. \tag{7}$$

We would like to connect this matrix element to a quantity that depends on fields $\varphi(t,x)$ rather than creation and annihilation operators. To do so, we consider the following integral, for $p^2 = m^2$,

$$I = i\int \mathrm{d}^4x\; e^{ip_\mu x^\mu}\left(\partial^2 + m^2\right)\varphi(x) = i\int \mathrm{d}^4x\; e^{ip_\mu x^\mu}\left(\partial_t^2 - \vec{\partial}_x^2 + m^2\right)\varphi(x). \tag{8}$$

We assume that $\varphi(x)$ vanishes if $|\vec{x}| \to \infty$ and integrate by parts in Eq. (8). Then

$$\int \mathrm{d}^4x\; e^{ip_\mu x^\mu}\vec{\partial}\cdot\vec{\partial}\varphi = -\int \mathrm{d}^4x\; e^{ip_\mu x^\mu}(\vec{p})^2\varphi. \tag{9}$$

Hence,

$$I = i\int \mathrm{d}^4x\; e^{ip_\mu x^\mu}\left(\partial^2 + m^2\right)\varphi(x) = i\int \mathrm{d}^4x\; e^{ip_\mu x^\mu}\left(\partial_t^2 + \omega_{\vec{p}}^2\right)\varphi(x), \tag{10}$$

where $\omega_{\vec{p}}^2 = \vec{p}^2 + m^2 = p_0^2$.

To proceed further, we note that the following identity is valid, if $p_0 = \omega_{\vec{p}}$

$$e^{ip_\mu x^\mu}\left(\partial_t^2 + \omega_{\vec{p}}^2\right)\varphi(x) = -i\partial_t\left[e^{ip_\mu x^\mu}\left(i\partial_t + \omega_{\vec{p}}\right)\varphi(x)\right]. \tag{11}$$

To check it, we compute the right-hand side explicitly. We find

$$\begin{aligned} -i\partial_t\left[e^{ip_\mu x^\mu}\left(i\partial_t + \omega_{\vec{p}}\right)\varphi(x)\right] &= e^{ip_\mu x^\mu}\left(\partial_t^2 - i\omega_{\vec{p}}\partial_t\right)\varphi(x) + e^{ip_\mu x^\mu}\omega_{\vec{p}}\left(i\partial_t + \omega_{\vec{p}}\right)\varphi(x) \\ &= e^{ip_\mu x^\mu}\left(\partial_t^2 + \omega_{\vec{p}}^2\right)\varphi. \end{aligned} \tag{12}$$

We use Eq. (11) in Eq. (10) and find

$$I = i\int \mathrm{d}^4x(-i)\partial_t\left[e^{ip_\mu x^\mu}\left(i\partial_t + \omega_{\vec{p}}\right)\varphi(x)\right] = \int \mathrm{d}^3x\; e^{ip_\mu x^\mu}\left(i\partial_t + \omega_{\vec{p}}\right)\varphi(t,x)\,\Big|_{t=-\infty}^{t=+\infty}. \tag{13}$$

At $t = \pm\infty$, $\varphi(t,\vec{x})$ is written using its asymptotic form, Eq. (5). We find

$$\begin{aligned} &\lim_{t\to\pm\infty} e^{ip_\mu x^\mu}(i\partial_t + \omega_{\vec{p}})\varphi(t,\vec{x}) \\ &= \lim_{t\to\pm\infty} e^{ip_\mu x^\mu}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2k_0}}\left\{a_{\vec{k}}(\pm\infty)(k_0 + \omega_{\vec{p}})e^{-ik_\mu x^\mu} + a^\dagger_{\vec{k}}(\pm\infty)\left(\omega_{\vec{p}} - k_0\right)e^{ik_\mu x^\mu}\right\}. \end{aligned} \tag{14}$$

We use this equation in Eq. (13) and integrate over $\vec{x}$. We find

$$
\begin{aligned}
& I = I_+ - I_-, \\
& I_\pm = \frac{1}{\sqrt{2k_0}} \left[a_{\vec{p}}(\pm\infty)(k_0 + \omega_{\vec{p}}) e^{i(p_0 - k_0)x_0} + a^\dagger_{-\vec{p}}(\pm\infty)(\omega_{\vec{p}} - k_0) e^{i(p_0+k_0)x_0} \right]_{k_0 = \omega_{\vec{p}} = p_0},
\end{aligned} \tag{15}
$$

so that

$$
I = \sqrt{2\omega_{\vec{p}}} \left(a_{\vec{p}}(+\infty) - a_{\vec{p}}(-\infty) \right). \tag{16}
$$

Hence,

$$
i \int \mathrm{d}^4 x \; e^{ip_\mu x^\mu} \left(\partial^2 + m^2 \right) \varphi(x) = \sqrt{2\omega_{\vec{p}}} \left(a_{\vec{p}}(+\infty) - a_{\vec{p}}(-\infty) \right), \tag{17}
$$

and similarly,

$$
-i \int \mathrm{d}^4 x \; e^{-ip_\mu x^\mu} \left(\partial^2 + m^2 \right) \varphi(x) = \sqrt{2\omega_{\vec{p}}} \left(a^\dagger_{\vec{p}}(+\infty) - a^\dagger_{\vec{p}}(-\infty) \right). \tag{18}
$$

We would like to use Eqs. (17,18) to construct the matrix element S_{fi} Eq. (7). For example, we can use the following equation

$$
-i \int \mathrm{d}^4 x \; e^{-ip_{1,\mu} x^\mu} \left(\partial^2 + m^2 \right) \varphi(x) = \sqrt{2\omega_{\vec{p}_1}} \left(a^\dagger_{\vec{p}_1}(+\infty) - a^\dagger_{\vec{p}_1}(-\infty) \right), \tag{19}
$$

to express $a^\dagger_{\vec{p}_1}(-\infty)$ through an integral of φ. The problem is that upon doing that, we will also obtain $a^\dagger_{\vec{p}_1}(+\infty)$ in the relation between $a^\dagger_{\vec{p}_1}(-\infty)$ and φ and this is not what is needed in Eq. (7). A trick that is used to get rid of $a^\dagger_{\vec{p}_1}(+\infty)$ and $a_{\vec{p}_{3,..,n}}(-\infty)$ is to employ properties of the vacuum state $|0\rangle$ since $a_{\vec{p}}|0\rangle$ and $\langle 0|a^\dagger_{\vec{p}}$ vanish. What we need to do is to ensure that all "unwanted" creation (annihilation) operators appear to the left (to the right) of all other operators in Eq. (7). To accomplish this, we rewrite Eq. (7) as

$$
S_{fi} = \langle f|i\rangle = \sqrt{2\omega_1 \, 2\omega_2 \, ... 2\omega_n} \langle 0| T \left[a_{p_3}(\infty) ... a_{p_n}(\infty) \; a^\dagger_{p_1}(-\infty) a^\dagger_{p_2}(-\infty) \right] |0\rangle, \tag{20}
$$

where the operator T is the time-ordering operator which is defined as follows

$$
T\left[O_1(t_1) O_2(t_2) \right] = \theta(t_1 - t_2) O_1 O_2 + \theta(t_2 - t_1) O_2 O_1. \tag{21}
$$

The time ordering ensures that operators that depend on the largest time appear to the left of all other operators and operators that depend on the smallest time appear to the right of all other operators. Then, since $a_{\vec{p}}|0\rangle = 0$ and $\langle 0|a^\dagger_{\vec{p}} = 0$, we can replace all the a and $a^\dagger$ operators in the formula for S_{fi} with integrals over fields φ since additional terms $a^\dagger_{\vec{p}}(+\infty)$ and $a_{\vec{p}}(-\infty)$ provide vanishing contributions because of the T-product in Eq. (20). We find

$$
S_{fi} = i^n \int \prod_{i=1}^{n} \mathrm{d}x_i \; e^{i\left(\sum\limits_{j=3}^{n} p_j x_j - p_1 x_1 - p_2 x_2 \right)} \prod_{i=1}^{n} \left(\partial_i^2 + m^2 \right) \langle 0| T \varphi(x_1) ... \varphi(x_n)|0\rangle. \tag{22}
$$

The object that appeared in Eq. (22), $\langle 0|T\varphi(x_1)...\varphi(x_n)|0\rangle$, is *the time-ordered Green's function* of n scalar fields $\varphi(t_1, x_1), \ldots, \varphi(t_n, x_n)$. We see from Eq. (22) that such Green's functions are used in calculations of scattering amplitudes which are important quantities for understanding interactions of elementary particles. This discussion provides some motivation to the study of Green's functions in quantum field theory.

To get a better idea of what Green's functions are, we will first study them in a non-interacting theory. In such a theory, the relation between fields and creation and annihilation operators is known exactly. The field operator reads

$$\varphi(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{-i(\omega_{\vec{k}}t - \vec{k}\vec{x})} + a^\dagger_{\vec{k}} e^{i(\omega_{\vec{k}}t - \vec{k}\vec{x})} \right). \tag{23}$$

The creation and annihilation operators satisfy the following equations $\langle 0|a^\dagger_{\vec{k}} = 0$ and $a_{\vec{k}}|0\rangle = 0$.

Hence, the simplest Green's function reads $\langle 0|\varphi(t,\vec{x})|0\rangle = 0$. The next-to-simplest Green's function is

$$\langle 0|T\varphi(t_1,\vec{x}_1)\varphi(t_2,\vec{x}_2)|0\rangle. \tag{24}$$

To proceed further, we use the definition of the time-ordering operator T and write

$$\langle 0|T\varphi(t_1,\vec{x}_1)\varphi(t_2,\vec{x}_2)|0\rangle = \theta(t_1-t_2)\langle 0|\varphi(t_1,\vec{x}_1)\varphi(t_2,\vec{x}_2)|0\rangle + \theta(t_2-t_1)\langle 0|\varphi(t_2,\vec{x}_2)\varphi(t_1,\vec{x}_1)|0\rangle. \tag{25}$$

To compute the remaining matrix elements, we use Eq. (23) and find

$$\langle 0|T\varphi(t_1,\vec{x}_1)\varphi(t_2,\vec{x}_2)|0\rangle = \int \prod_{i=1}^{2} \frac{\mathrm{d}^3\vec{k}_i}{(2\pi)^3\sqrt{2\omega_{\vec{k}_i}}} \left[\theta(t_1-t_2) e^{-ik_1x_1 + ik_2x_2} \langle 0|a_{\vec{k}_1} a^\dagger_{\vec{k}_2}|0\rangle + (1 \leftrightarrow 2) \right]. \tag{26}$$

Using

$$\langle 0|a_{\vec{k}_1} a^\dagger_{\vec{k}_2}|0\rangle = \langle 0|[a_{\vec{k}_1}, a^\dagger_{\vec{k}_2}]|0\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1 - \vec{k}_2), \tag{27}$$

we easily find

$$\langle 0|T\varphi(t_1,\vec{x}_1)\varphi(t_2,\vec{x}_2)|0\rangle = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3(2\omega_{\vec{k}})} \left[\theta(t_1-t_2) e^{-ik(x_1-x_2)} + \theta(t_2-t_1) e^{ik(x_1-x_2)} \right]. \tag{28}$$

To simplify this expression, it is convenient to compute a Fourier transform of the Green's function. We define

$$D_F(x_1 - x_2) = \langle 0|T\varphi(t_1,\vec{x}_1)\varphi(t_2,\vec{x}_2)|0\rangle \tag{29}$$

and compute

$$D_F(p) = \int \mathrm{d}^4x D_F(x) e^{ip_\mu x^\mu}. \tag{30}$$

To integrate over x^μ, we use Eq. (28) where we replace $t_1 \to x_0, t_2 \to 0$, $\vec{x}_1 \to \vec{x}$ and $\vec{x}_2 \to 0$. The integration over $\vec{x}$ is straightforward. We obtain

$$\begin{aligned} D_F(p) &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \mathrm{d}t\, \mathrm{d}\vec{x} \left[\theta(t)e^{i(p-k)x} + \theta(-t)e^{i(p+k)x}\right] \\ &= \int \frac{\mathrm{d}^3\vec{k}\mathrm{d}t}{2\omega_{\vec{k}}} \left[\theta(t)e^{i(p_0-\omega_{\vec{k}})t}\delta^{(3)}(\vec{p}-\vec{k}) + \theta(-t)e^{i(p_0+\omega_{\vec{k}})t}\delta^{(3)}(\vec{p}+\vec{k})\right] \\ &= \int \frac{\mathrm{d}t}{2\omega_{\vec{p}}} \left[\theta(t)e^{i(p_0-\omega_{\vec{p}})t} + \theta(-t)e^{i(p_0+\omega_{\vec{p}})t}\right], \end{aligned} \tag{31}$$

where $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. To proceed further, it is convenient to introduce a useful representation for the θ-function

$$\theta(t) = -\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{2\pi i} \frac{e^{-i\xi t}}{\xi + i0}, \quad \theta(-t) = -\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{2\pi i} \frac{e^{i\xi t}}{\xi + i0} = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{2\pi i} \frac{e^{-i\xi t}}{\xi - i0}. \tag{32}$$

We use this representation in Eq. (31), integrate over t and then over ξ and find

$$\begin{aligned} D_F(p) &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{(2\pi i)} \frac{\mathrm{d}t}{2\omega_{\vec{p}}} \left[-\frac{1}{\xi + i0} e^{i(p_0-\omega_{\vec{p}}-\xi)t} + \frac{1}{\xi - i0} e^{i(p_0+\omega_{\vec{p}}-\xi)t}\right] \\ &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{2i\omega_{\vec{p}}} \left[-\frac{1}{\xi + i0}\delta(p_0 - \omega_{\vec{p}} - \xi) + \frac{1}{\xi - i0}\delta(p_0 + \omega_{\vec{p}} - \xi)\right] \\ &= \frac{1}{2i\omega_{\vec{p}}} \left[-\frac{1}{p_0 - \omega_{\vec{p}} + i0} + \frac{1}{p_0 + \omega_{\vec{p}} - i0}\right] = \frac{1}{2i\omega_{\vec{p}}} \frac{-2\omega_{\vec{p}}}{\left[p_0^2 - (\omega_{\vec{p}} - i0)^2\right]} \end{aligned} \tag{33}$$

Upon substituting $\omega_{\vec{p}} = \sqrt{p^2 + m^2}$ and using the fact that $\omega_{\vec{p}} > 0$, so that $(\omega_{\vec{p}} - i0)^2 \approx \omega_{\vec{p}}^2 - i0$, we obtain the final result for the Green's function of the two scalar fields in momentum space

$$D_F(p) = \frac{i}{p^2 - m^2 + i0}. \tag{34}$$

Note the appearance of $+i0$ in the denominator; this infinitesimal complex number appears in this way because we compute the time-ordered Green's function. We can use Eq. (34) to write

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i0} e^{-ip_\mu(x^\mu - y^\mu)}. \tag{35}$$

Eq. (35) shows that the $+i0$ term provides a definite prescription of how a would-be singularity at $p^2 = m^2$ should be treated in the process of integration over p in Eq. (35). Without such a prescription, integral Eq. (35) is poorly defined.